Non-Hermitian quantum dynamics around an exceptional point

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January 5, 2015

Abstract

We propose an exactly solvable model of the two-state non-Hermitian quantum system which encircles an exceptional point. Using the model we analytically demonstrate the breaking of the adiabatic flip. Gilary *et al.* have shown only numerically that the adiabatic flip breaks if the motion of parameters is non-adiabatic. We use the Hashimoto form and the Trotter decomposition in order to construct the exactly solvable model. We analytically observe the non-adiabatic effect of the time-evolution operator and find that any initial states decay into one of the flipping states after the system non-adiabatically evolves around the exceptional point, which is consistent with the numerical results given by Gilary *et al.* We exactly evaluate the transition rate of the state at $|p|^2 \approx 0.189591$ for the cycle period $T = 1$ and $|p|^2 \to 1$ in the limit $T \to \infty$, which is the adiabatic limit.

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1 Introduction

Today the quantum dynamics of non-Hermitian Hamiltonian system attracts much interest theoretically and experimentally [1–16]. Non-Hermitian Hamiltonians are often used for representing decaying open quantum systems [1, 2] and decaying Schrödinger systems [3, 16]. A recent new development is the theoretical and experimental studies on $\mathcal{PT}$-symmetric Hamiltonians [4–9], which is non-Hermitian in general.

One of the important features of non-Hermitian systems is the existence of the exceptional point, the point in the parameter space at which some of the eigenenergies and the corresponding eigenvectors coalesce into one [17]. The exceptional point is a new type of singular point of the Hamiltonian, which specifically appears in non-Hermitian systems.

The geometric phase around the exceptional points recently has great interest [10–15]. For Hermitian systems, previous studies indicate that the origin of the geometric phase is the diabolic points, another singular point in the parameter space at which the energy spectrum is degenerate [18].

When a non-Hermitian system adiabatically moves around an exceptional point in a cycle, the two eigenstates that would coalesce at the exceptional point flips to each other. This phenomenon is called the adiabatic flip, or simply the flip [10, 13]. Many studies on adiabatic processes in Hermitian

quantum systems [10–15, 18–20] have been carried out for investigating, for example, the many-body ground state in condensed–matter physics [19, 21] and the geometric phase [20].

There have been studies on the experimental observability of the adiabatic flip [13–15]. They pointed out the possible importance of non-adiabatic effects which emerge when the system non-adiabatically moves around an exceptional point in a cycle. Uzdin *et al.* [13] and Gilary *et al.* [14,15] discussed these effects numerically for two-by-two non-Hermitian systems. Their results implied that any initial state approaches a specific superposition of eigenstates. We can obtain two specific states by encircling an exceptional point clockwise or anticlockwise. However, these studies were done analytically only for adiabatic processes or only numerically for non-adiabatic processes.

In this paper, we analytically observe a non-adiabatic effect to the adiabatic flip in a non-adiabatic cycle around an exceptional point. We exactly calculate the time-evolution operator of the time-dependent non-Hermitian Hamiltonian of a two–state system. The two eigenstates which would coalesce into one at an exceptional point are flipped to each other if we move the system around the point adiabatically. We investigate the non-adiabatic time-evolution of the initial state and find that any states evolve to one state after one cycle with incomplete flip. This incompleteness is a result from the

non-adiabatic effects.

In order to calculate the time-evolution operator analytically, we rewrite it by employing the Hashimoto form of non-Hermitian Hamiltonians [16]. The reason of using the Hashimoto form is that we expect it to be more useful than the diagonalization for the analysis of the evolution around exceptional points. Combining it with the Trotter decomposition, we obtain a compact effective Hamiltonian for further use. Demanding it to be a lower triangular matrix, we come up with a model system that encircles an exceptional point for which we can analytically calculate the operator of the non-adiabatic time evolution.

The present paper is organized into three sections: In Section 2, we introduce a non-Hermitian Hamiltonian with parameters and its exceptional points. We discuss the possibility of diagonalization in the neighborhood of an exceptional point and introduce the Hashimoto form. Next, we consider the adiabatic process in quantum systems in Section 3. We present the adiabatic process around an exceptional point. After the consideration of the adiabatic process, we, in Section 4, review the numerical research on non-adiabatic process around an exceptional point by Gilary *et al.* [14, 15], and then present our exact calculation of a non-adiabatic process around an exceptional point. We use the Trotter decomposition and the Hashimoto form so as to rewrite the time-evolution operator of the system, and thereby derive

an effective Hamiltonian. We then introduce an exactly calculable model so that the effective Hamiltonian may be a lower triangular matrix. Finally, we exactly evaluate the transition rate from one eigenstate and compare it with the result Gilary *et al.*

2 The Effects of Exceptional Points

2.1 Exceptional point

A non-Hermitian matrix is generally not diagonalizable. In reality, however, it is rather a rare case to be unable to diagonalize non-Hermitian matrices. We explain this in this subsection.

Let us consider an $N \times N$ Hamiltonian with a complex perturbation parameter λ:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda \hat{\mathcal{H}}_1, \tag{1}$$

where $\hat{\mathcal{H}}_0$ and $\hat{\mathcal{H}}_1$ are both Hermitian. The eigenvalues are given by the secular equation

$$\det\left(E\,\hat{1}_N - \hat{\mathcal{H}}\right) = 0, \tag{2}$$

where $\hat{1}_N$ denotes the N-dimensional identity matrix. The coefficient of the term E^N in Eq. (2) is unity. Therefore Eq. (2) must be equivalent to the

following equation:

$$(E - E_1)(E - E_2) \cdots (E - E_N) = 0, \tag{3}$$

where E_k are eigenvalues and either analytic functions of λ or branches [17]. For example, let us consider a 3×3 Hamiltonian

$$\hat{\mathcal{H}} = \begin{pmatrix} a\lambda & 0 & 0 \\ 0 & b\lambda & c \\ 0 & c & -b\lambda \end{pmatrix} \tag{4}$$

with a, b and c real. The secular equation is

$$(E - a\lambda)[(E + b)(E - b) - c^2] = 0, \tag{5}$$

which gives the eigenvalues

$$E_1(\lambda) = a\lambda, \tag{6}$$

$$E_2(\lambda) = \sqrt{b^2\lambda^2 + c^2}, \tag{7}$$

$$E_3(\lambda) = -\sqrt{b^2\lambda^2 + c^2}. \tag{8}$$

Among them, E_1 is an analytic function of λ, while E_2 and E_3 are a pair of branches. As the above example indicates, the branches are generally represented by

$$E_k(\lambda) = \exp\left(i\frac{2\pi n}{s}\right) \sqrt[s]{f(\lambda)} + g(\lambda) \tag{9}$$

with analytic functions f and g and a positive integer s, where $n = 0, 1, \cdots, s-1$. All the s branches must appear as the eigenvalues if one of them appears. For a set of branches, there are points in the parameter space given by $f(\lambda) = 0$, where all branches in the set meet each other. These points are called the *exceptional points* [17]. We hereafter denote an exceptional point by $\lambda_{\rm EP}$. This is where the matrix becomes non-diagonalizable.

Note that the number of exceptional points is at most finite. In this sense, it is rare that non-Hermitian matrices are not diagonalizable. Let us consider the parameter space of the Hamiltonian (1). It is well known that the number of roots of equation like Eq. (2) are at most finite [17]. Then the number of exceptional points is also at most finite too.

The eigenvectors also coalesce at an exceptional point, corresponding to the coalescing eigenvalues; this is the essential difference from the Hermitian degeneracy. Indeed, in the example of Eq. (4), the right eigenvectors

corresponding to Eq. (8) are

$$\mathbf{v}_1^R(\lambda) = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{v}_2^R(\lambda) = C_2 \begin{pmatrix} 0 \\ -c \\ E_2(\lambda) - b\lambda \end{pmatrix},$$

$$\mathbf{v}_3^R(\lambda) = C_3 \begin{pmatrix} 0 \\ -c \\ E_3(\lambda) - b\lambda \end{pmatrix}, \tag{10}$$

where C_1, C_2 and C_3 are arbitrary complex constants. At the exceptional point λ_{EP}, the coalescence $E_2(\lambda_{\mathrm{EP}}) = E_3(\lambda_{\mathrm{EP}})$ happens. This coalescence of the eigenvalues is accompanied by the coalescence of the corresponding eigenvectors $\mathbf{v}_2^R$ and $\mathbf{v}_3^R$. In other words, the sum of dimensions of the eigenvector spaces decreases only at the exceptional points:

$$\sum_k \dim \mathcal{E}_k(\lambda) < N \quad \text{iff } \lambda = \lambda_{\mathrm{EP}}, \tag{11}$$

where N is the dimensionality of the system and $\mathcal{E}_k(\lambda)$ is the eigenvector space corresponding to $E_k(\lambda)$. Equation (11) says that the Hamiltonian is not diagonalizable only at the exceptional points.

We can prove it as follows. If the Hamiltonian is diagonalizable, there exists a regular matrix $\hat{S}$ such that

$$\hat{S}^{-1}\hat{\mathcal{H}}\hat{S} = \mathrm{diag}\left(E_1, E_2, \cdots, E_N\right), \tag{12}$$

where $\mathrm{diag}(E_1, E_2, \cdots, E_N)$ is a diagonal matrix whose diagonal elements are $E_1, E_2, \cdots, E_N$. Since both $\hat{S}$ and $\hat{S}^{-1}$ are $N \times N$ matrices, we can find two sets of mutually linearly independent N pieces of N-dimensional vectors $\{\mathbf{v}_j\}_{j=1}^{N}$ and $\{\tilde{\mathbf{v}}_j\}_{j=1}^{N}$ satisfying

$$\hat{S} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_N \end{pmatrix} \tag{13}$$

and

$$\hat{S}^{-1} = \begin{pmatrix} {}^{\mathrm{T}}\tilde{\mathbf{v}}_1 \\ {}^{\mathrm{T}}\tilde{\mathbf{v}}_2 \\ \vdots \\ {}^{\mathrm{T}}\tilde{\mathbf{v}}_N \end{pmatrix}. \tag{14}$$

Since $\hat{S}^{-1}\hat{S} = \hat{1}_N$, we have

$$\tilde{\mathbf{v}}_i \cdot \mathbf{v}_j = \delta_{ij}, \tag{15}$$

where δ_{ij} is Kronecker's delta defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{16}$$

Using $\{\mathbf{v}_j\}_{j=1}^N$ and $\{\tilde{\mathbf{v}}_j\}_{j=1}^N$, we can represent Eq. (12) as

$$\tilde{\mathbf{v}}_i \cdot \left(\hat{\mathcal{H}}\mathbf{v}_j\right) = \delta_{ij} E_j \tag{17}$$

for $i, j = 1, 2, \cdots, N$. Then we have $\hat{\mathcal{H}}\mathbf{v}_i = E_i \mathbf{v}_i$ for $i = 1, 2, \cdots, N$, which says $\mathbf{v}_i$ is the eigenvector corresponding to E_i. Since $\{\mathbf{v}_j\}_{j=1}^N$ are linearly independent of each other, we have $\sum_k \dim \mathcal{E}_k = N$.

On the other hand, if $\sum_k \dim \mathcal{E}_k = N$, we can find the N pieces of eigenvectors $\{\mathbf{v}_j\}_{j=1}^N$ which are linearly independent of each other. It is clear that $\hat{S}$ defined by Eq. (13) diagonalizes $\hat{\mathcal{H}}$. This proves that the Hamiltonian is not diagonalizable only at the exceptional points.

Features of exceptional points

Exceptional points have a lot of features. We present some of them which are important in the present paper.

1. At the exceptional points, the energy spectrum becomes degenerate and the corresponding eigenvectors coalesce into each other. The coalescence of the eigenvectors is a unique phenomenon of non-Hermitian systems, which makes the Hamiltonian not diagonalizable. This fact is important because we cannot discuss the perturbation around an exceptional point in a usual way.

2. The non-diagonalizability implies that the Hamiltonian is non-Hermitian when the parameter is at an exceptional point.

3. In the parameter space, there are at most a finite number of exceptional points. In other words, the Hamiltonian perturbed to be non-Hermitian is actually diagonalizable at almost every point in the parameter space.

4. The exceptional point is a branch point of some of the eigenenergies. In other words, the exceptional point is a singularity of order of $1/s$. This leads to the flip of the eigenvectors, which is mainly discussed in Section 3.

2.2 Normalization of a non-Hermitian Hamiltonian near an exceptional point

In the present subsection, we discuss the normalization of a non-Hermitian Hamiltonian, especially near an exceptional point. The term "normalization" means a similarity transformation of a matrix into a normal form, for example, the diagonalized form, the Jordan normal form, *etc.* According to the previous subsection, a non-Hermitian Hamiltonian is not diagonalizable iff $\lambda = \lambda_{\mathrm{EP}}$, which is caused by the degeneration of eigenspace. To understand this degeneration, we present a simple example.

For simplicity, let us consider a 2×2 non-Hermitian parametrized Hamil-

toninan

$$\hat{\mathcal{H}} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} + \frac{\lambda}{2} \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}, \tag{18}$$

where ϵ_1, ϵ_2 and V are non-zero real parameters. The eigenvalues are

$$E_1 = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\sqrt{(\epsilon_1 - \epsilon_2)^2 + \lambda^2 V^2}}{2}, \tag{19}$$

$$E_2 = \frac{\epsilon_1 + \epsilon_2}{2} - \frac{\sqrt{(\epsilon_1 - \epsilon_2)^2 + \lambda^2 V^2}}{2}. \tag{20}$$

The corresponding right eigenvectors are

$$\mathbf{v}_1^R = \frac{1}{\sqrt{2}(\epsilon_1 - \epsilon_2)} \begin{pmatrix} c_1 \\ \lambda V \end{pmatrix}, \tag{21}$$

$$\mathbf{v}_2^R = \frac{1}{\sqrt{2}(\epsilon_1 - \epsilon_2)} \begin{pmatrix} c_2 \\ \lambda V \end{pmatrix} \tag{22}$$

and the left eigenvectors are

$$\mathbf{v}_1^L = \frac{1}{\sqrt{2}(\epsilon_1 - \epsilon_2)} \begin{pmatrix} c_1 & \lambda V \end{pmatrix}, \tag{23}$$

$$\mathbf{v}_2^L = \frac{1}{\sqrt{2}(\epsilon_1 - \epsilon_2)} \begin{pmatrix} c_2 & \lambda V \end{pmatrix}, \tag{24}$$

where

$$c_1 = (\epsilon_1 - \epsilon_2) + \sqrt{(\epsilon_1 - \epsilon_2)^2 + \lambda^2 V^2}, \tag{25}$$

$$c_2 = (\epsilon_1 - \epsilon_2) - \sqrt{(\epsilon_1 - \epsilon_2)^2 + \lambda^2 V^2}. \tag{26}$$

Note that normalization constants are omitted for simplicity and the factor $1/[\sqrt{2}(\epsilon_1-\epsilon_2)]$ is instead introduced for later convenience. From Eqs. (19) and (20), we have the exceptional points

$$\lambda_{\mathrm{EP}} = \pm i\frac{\epsilon_1 - \epsilon_2}{V}. \tag{27}$$

It is clear that the right eigenvectors (21) and (22) coalesce into each other iff $\lambda = \lambda_{\mathrm{EP}}$ and the left eigenvectors (23) and (24) do so too. Therefore the Hamiltonian is diagonalizable iff $\lambda \neq \lambda_{\mathrm{EP}}$.

Hashimoto *et al.* [16] showed that it is possible to transform the Hamiltonian into a generalized Jordan form around the exceptional points. The remarkable point is that the generalized Jordan form and the similarity transformation continuously converge to the standard Jordan form and its similarity transformation at the exceptional point. In the present paper, we call the generalized Jordan form the Hashimoto form. We hereafter explain its construction. We use the Hamiltonian (18) in the following discussion. The discussion is greatly based on the work by Hashimoto *et al.* [16].

Definition of *co-eigenvector* when $\lambda = \lambda_{\mathbf{EP}}$

Let us first overview the Jordan form. At the exceptional points (27) the Hamiltonian (18) reduces to

$$\hat{\mathcal{H}} = \begin{pmatrix} \epsilon_1 & \pm i\delta \\ \pm i\delta & \epsilon_2 \end{pmatrix} \tag{28}$$

with

$$\delta = \frac{\epsilon_1 - \epsilon_2}{2}, \tag{29}$$

the eigenvalues (19) and (20) to

$$E_{\mathrm{EP}} = \frac{\epsilon_1 + \epsilon_2}{2}, \tag{30}$$

and the right and left eigenvectors (21)–(24) respectively to

$$\mathbf{v}^R_{\mathrm{EP}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}, \tag{31}$$

$$\mathbf{v}^L_{\mathrm{EP}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \pm i \end{pmatrix}. \tag{32}$$

For the standard Jordan form, we take another set of vectors

$$\mathbf{w}^R_{\mathrm{EP}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp i \end{pmatrix}, \tag{33}$$

$$\mathbf{w}^L_{\mathrm{EP}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mp i \end{pmatrix}, \tag{34}$$

which constitute orthogonal systems $\{\mathbf{v}^R_{\mathrm{EP}}, \mathbf{w}^R_{\mathrm{EP}}\}$ and $\{\mathbf{w}^L_{\mathrm{EP}}, \mathbf{v}^L_{\mathrm{EP}}\}$ respectively. In the present paper, we call these vectors *co-eigenvectors*, or more

simply *co-vectors* whenever it is not ambiguous. The co-vectors have the following properties:

$$\mathbf{w}^L_{\rm EP}\cdot\mathbf{v}^R_{\rm EP}=1,\quad \mathbf{w}^L_{\rm EP}\cdot\mathbf{w}^R_{\rm EP}=0,$$

$$\mathbf{v}^L_{\rm EP}\cdot\mathbf{v}^R_{\rm EP}=0,\quad \mathbf{v}^L_{\rm EP}\cdot\mathbf{w}^R_{\rm EP}=1. \tag{35}$$

With the eigenvectors and co-vectors, we construct the matrices

$$\hat{S}_{\rm EP}=\begin{pmatrix}\mathbf{v}^R_{\rm EP} & \mathbf{w}^R_{\rm EP}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ \pm i & \mp i\end{pmatrix}, \tag{36}$$

$$\hat{S}^{-1}_{\rm EP}=\begin{pmatrix}\mathbf{w}^L_{\rm EP}\\ \mathbf{v}^L_{\rm EP}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mp i\\ 1 & \pm i\end{pmatrix}, \tag{37}$$

which constitute a similarity transformation which brings the Hamiltonian into the Jordan form:

$$\hat{S}^{-1}_{\rm EP}\hat{\mathcal{H}}_{\rm EP}\hat{S}_{\rm EP}=\begin{pmatrix}E_{\rm EP} & 2\delta\\ 0 & E_{\rm EP}\end{pmatrix}. \tag{38}$$

In other words, $\mathbf{w}^L_{\rm EP}$ and $\mathbf{w}^R_{\rm EP}$ satisifies

$$\mathbf{w}^L_{\rm EP}\cdot\mathbf{v}^R_1=1, \tag{39}$$

$$\mathbf{w}^L_{\rm EP}\hat{\mathcal{H}}_{\rm EP}=E_{\rm EP}\mathbf{w}^L_{\rm EP}+2\delta\mathbf{v}^L_{\rm EP} \tag{40}$$

and

$$\mathbf{v}^L_{\rm EP}\cdot\mathbf{w}^R_{\rm EP}=1, \tag{41}$$

$$\hat{\mathcal{H}}_{\rm EP}\mathbf{w}^R_{\rm EP}=2\delta\mathbf{v}^R_{\rm EP}+E_{\rm EP}\mathbf{w}^R_{\rm EP}. \tag{42}$$

Definition of *co-eigenvector* when $\lambda \neq \lambda_{\mathrm{EP}}$

Next, we extend the definition of co-eigenvectors to the case around an exceptional point. The important point is that this extension is continuous at λ_{EP}. The similarity transformation given by the eigenvectors and their generalized co-vectors bring the Hamiltonian into a generalized Jordan form even when $\lambda \neq \lambda_{\mathrm{EP}}$.

We define the generalized co-eigenvectors $\mathbf{w}_1^R$ and $\mathbf{w}_2^L$ for $\lambda \neq \lambda_{\mathrm{EP}}$ by [16]

$$\mathbf{w}_1^L \cdot \mathbf{v}_1^R = 1, \tag{43}$$

$$\mathbf{w}_1^L \hat{\mathcal{H}} = E_1 \mathbf{w}_1^L + 2\delta \mathbf{v}_2^L \tag{44}$$

and

$$\mathbf{v}_2^L \cdot \mathbf{w}_2^R = 1, \tag{45}$$

$$\hat{\mathcal{H}} \mathbf{w}_2^R = 2\delta \mathbf{v}_1^R + E_2 \mathbf{w}_2^R. \tag{46}$$

It should be emphasized that this definition decides $\mathbf{w}_1^R$ and $\mathbf{w}_2^L$ uniquely. Indeed, from Eqs. (45) and (46) we uniquely have

$$\mathbf{w}_1^L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \dfrac{c_2}{V} \end{pmatrix} \tag{47}$$

and from Eqs. (43) and (44)

$$\mathbf{w}_2^R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \\ \dfrac{c_1}{V} \end{pmatrix}. \tag{48}$$

The solutions of Eq. (45) has the arbitrariness of a constant times $\mathbf{v}_2^L$, but Eq. (46) fixes the constant, and hence the uniqueness.

According to the definition, we can find properties similar to Eq. (35):

$$\begin{aligned} \mathbf{w}_1^L \cdot \mathbf{v}_1^R = 1, \quad \mathbf{w}_1^L \cdot \mathbf{w}_2^R = 0, \\ \mathbf{v}_2^L \cdot \mathbf{v}_1^R = 0, \quad \mathbf{v}_2^L \cdot \mathbf{w}_2^R = 1. \end{aligned} \tag{49}$$

Indeed, Eq. (44) gives

$$\mathbf{w}_1^L \hat{\mathcal{H}} \mathbf{w}_2^R = E_1 \mathbf{w}_1^L \cdot \mathbf{w}_2^R + 2\delta, \tag{50}$$

and with Eq. (46) gives

$$\mathbf{w}_1^L \hat{\mathcal{H}} \mathbf{w}_2^R = 2\delta + E_2 \mathbf{w}_1^L \cdot \mathbf{w}_2^R. \tag{51}$$

Since $E_1 \neq E_2$ for $\lambda \neq \lambda_{\mathrm{EP}}$, we have $\mathbf{w}_1^L \cdot \mathbf{w}_2^R = 0$. The other properties of Eq. (49) are obtained by definition.

The generalized co-vectors $\mathbf{w}_1^L$ and $\mathbf{w}_2^R$ respectively converge to $\mathbf{w}_{\mathrm{EP}}^R$ and $\mathbf{w}_{\mathrm{EP}}^L$ as $\lambda \to \lambda_{\mathrm{EP}}$. In the limit, the definitions (43)–(46) reduce to

$$\mathbf{w}_{1,EP}^L \cdot \mathbf{v}_{\mathrm{EP}}^R = 1, \tag{52}$$

$$\mathbf{w}_{1,EP}^L \hat{\mathcal{H}}_{\mathrm{EP}} = E_{\mathrm{EP}} \mathbf{w}_{1,EP}^L + 2\delta \mathbf{v}_{\mathrm{EP}}^L \tag{53}$$

and

$$\mathbf{v}_{\mathrm{EP}}^L \cdot \mathbf{w}_{2,EP}^R = 1, \tag{54}$$

$$\hat{\mathcal{H}}_{\mathrm{EP}} \mathbf{w}_{2,EP}^R = 2\delta \mathbf{v}_{\mathrm{EP}}^R + E_{\mathrm{EP}} \mathbf{w}_{2,EP}^R, \tag{55}$$

where $\mathbf{w}_{1,EP}^L$ and $\mathbf{w}_{2,EP}^R$ are the limits of $\mathbf{w}_1^L$ and $\mathbf{w}_2^R$, respectively. Note that $E_{\rm EP}$ is the limit of both E_1 and E_2 as indicated in the beginning of this subsection. From Eqs. (52)–(55), we can find

$$\mathbf{w}_{1,EP}^L \cdot \mathbf{v}_{\rm EP}^R = 1, \quad \mathbf{w}_{1,EP}^L \cdot \mathbf{w}_{2,EP}^R = 0,$$
$$\mathbf{v}_{\rm EP}^L \cdot \mathbf{v}_{\rm EP}^R = 0, \quad \mathbf{v}_{\rm EP}^L \cdot \mathbf{w}_{2,EP}^R = 0. \tag{56}$$

Comparing Eqs. (35) and (56), we can represent $\mathbf{w}_{1,EP}^L$ and $\mathbf{w}_{2,EP}^R$ as

$$\mathbf{w}_{1,EP}^L = \mathbf{w}_{\rm EP}^L + \alpha \mathbf{v}_{\rm EP}^L, \tag{57}$$

$$\mathbf{w}_{2,EP}^R = \beta \mathbf{v}_{\rm EP}^R + \mathbf{w}_{\rm EP}^R, \tag{58}$$

where α and β are complex constants. By applying Eqs. (40) and (42) to Eqs. (57) and (58), we have

$$\mathbf{w}_{1,EP}^L \hat{\mathcal{H}}_{\rm EP} = E_{\rm EP} \mathbf{w}_{\rm EP}^L + (2\delta + \alpha E_{\rm EP}) \mathbf{v}_{\rm EP}^L, \tag{59}$$

$$\hat{\mathcal{H}}_{\rm EP} \mathbf{w}_{2,EP}^R = (\beta E_{\rm EP} + 2\delta) \mathbf{v}_{\rm EP}^R + E_{\rm EP} \mathbf{w}_{\rm EP}^R. \tag{60}$$

Comparing Eqs. (59) and (60) with Eqs. (53) and (55), we have $\alpha = \beta = 0$. We thereby conclude that

$$\lim_{\lambda \to \lambda_{\rm EP}} \mathbf{w}_1^L = \mathbf{w}_{\rm EP}^L, \tag{61}$$

$$\lim_{\lambda \to \lambda_{\rm EP}} \mathbf{w}_2^R = \mathbf{w}_{\rm EP}^R. \tag{62}$$

Hashimoto form of Hamiltonian

Let us consider the symmetric matrices defined by the generalized co-vectors:

$$\hat{S} = \begin{pmatrix} \mathbf{v}_1^R & \mathbf{w}_2^R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \dfrac{c_1}{2\delta} & 1 \\ \dfrac{\lambda V}{2\delta} & \dfrac{c_2}{\lambda V} \end{pmatrix}, \tag{63}$$

$$\hat{S}^{-1} = \begin{pmatrix} \mathbf{w}_1^L \\ \mathbf{v}_2^L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \dfrac{c_1}{\lambda V} \\ \dfrac{c_2}{2\delta} & \dfrac{\lambda V}{2\delta} \end{pmatrix}. \tag{64}$$

It normalizes the Hamiltonian into the generalized Jordan form

$$\hat{S}^{-1}\hat{\mathcal{H}}\hat{S} = \hat{N} \equiv \begin{pmatrix} E_1 & 2\delta \\ 0 & E_2 \end{pmatrix}. \tag{65}$$

Because of the continuity of $\mathbf{v}_1^R$, $\mathbf{w}_2^R$, $\mathbf{w}_1^L$ and $\mathbf{v}_2^L$ at the exceptional points, $\hat{S}$ and $\hat{S}^{-1}$ are also continuous there. This leads to the fact that the normal form $\hat{N}$ in Eq. (65) is also continuous. The normal form (65) is for the first time given by Hashimoto *et al.* [16], and hence we refer to it as the Hashimoto form.

The Hashimoto form has the following advantages. First, it is well-defined at the exceptional points, while the diagonalized form is not. This can make the calculation with the eigenstates problematic around the exceptional points. The Hashimoto form is free from the problem.

For example, it is impossible to use the diagonalized form for pertur-

bative calculation starting from an exceptional point. This is because the eigenvectors of non-perturbed Hamiltonian are not enough to form a complete system, which is necessary to expand the eigenstates of the perturbed Hamiltonian. However, the set of the co-vectors and the eigenvectors at an exceptional point forms a complete system; its perturbation naturally produces the Hashimoto form.

In fact, there are other choices of vectors to add in order to form a complete system. The present choice of the co-vectors have two merits. One is the simplicity of the behavior of the Hamiltonian applied to the co-vectors, as shown in Eqs. (43)–(46). The other is the continuity of the co-vectors; we can discuss the perturbation around the exceptional points and away from them by the same method.

The second advantage of the Hashimoto form is its similarity to the diagonalized form. For example, the nth power of $\hat{\mathcal{H}}$ is given by

$$\hat{\mathcal{H}}^n = \left(\hat{S}\hat{N}\hat{S}^{-1}\right)^n = \hat{S}\hat{N}^n\hat{S}^{-1} \tag{66}$$

with

$$\hat{N}^n = \begin{pmatrix} E_1^n & \dfrac{2\delta(E_1^n - E_2^n)}{E_1 - E_2} \\ 0 & E_2^n \end{pmatrix}, \tag{67}$$

which resembles the nth power of the diagonalized matrix. We have

$$\mathbf{w}_1^L \exp\left[-\frac{i}{\hbar}\hat{\mathcal{H}}t\right]\mathbf{v}_1^R = \exp\left(-\frac{i}{\hbar}E_1 t\right), \tag{68}$$

$$\mathbf{v}_2^L \exp\left[-\frac{i}{\hbar}\hat{\mathcal{H}}t\right]\mathbf{w}_2^R = \exp\left(-\frac{i}{\hbar}E_2 t\right). \tag{69}$$

Equation (69) indicates that the $\mathbf{v}_2^L$ component of $\mathbf{w}_2^R(t)$ is $\exp(-i/\hbar E_2 t)$. Thus, the time evolution of $\mathbf{w}_2^R$ is similar to that of $\mathbf{v}_2^R$.

3 Adiabatic process moving around exceptional points

3.1 Adiabatic process in quantum mechanics

There have been many studies on the dynamical effect of quantum systems induced by the motion of parameters of Hamiltonians. These problems can be divided into two, namely, adiabatic processes and non-adiabatic ones.

The adiabatic process of a quantum system is very useful in investigating the structure of the energy spectrum of the system. In condensed-matter physics, for example, the ground state of a system with many-body interactions is sometimes assumed to be well approximated by that of the system without the interactions. This assumption is based on the adiabatic continuation from the system without interactions to that with interactions [19, 21].

The geometric phase [20] is another instance of using the adiabatic process. The method of geometric phase is expected to be a good way to know the structure of singularities of the spectrum.

Many problems concerning the geometric phase have been mainly studied for Hermitian Hamiltonians. This is because in quantum mechanics the Hamiltonian is often assumed to be Hermitian. Recently, however, there appeared studies of the geometric phase for non-Hermitian Hamiltonians, which is of both theoretical and experimental interest [10–12, 22].

3.2 Exceptional point – non-Hermitian case

Let us come back to the Hamiltonian (18) and express the eigenvalues (19) and (20) as

$$E_1 = E_{\mathrm{EP}} + \Delta, \tag{70}$$

$$E_2 = E_{\mathrm{EP}} - \Delta, \tag{71}$$

where

$$\Delta = \frac{\sqrt{(\epsilon_1 - \epsilon_2)^2 + \lambda^2 V^2}}{2}$$

vanishes at $\lambda = \lambda_{\rm EP}$. The parameters in Eqs. (25) and (26) are then given by

$$c_1 = (\epsilon_1 - \epsilon_2) + 2\Delta, \tag{72}$$

$$c_2 = (\epsilon_1 - \epsilon_2) - 2\Delta. \tag{73}$$

We now show that when λ moves around an exceptional point, the eigenvectors (21) and (22) interchange with each other.

Let us assume that λ depends on time t and use the notation $\lambda(t)$. We also assume that $\lambda(t)$ is periodic with $\lambda(T) = \lambda(0)$. Then there are two possibilities that the phase difference between $\Delta(0)$ and $\Delta(T)$ are $2n\pi$ or $(2n+1)\pi$ with integer n. For each case, we have $\Delta(T) = \Delta(0)$ and $\Delta(T) = -\Delta(0)$. Especially in the latter case, c_1 and c_2 are flipped to each other:

$$c_1(T) = c_2(0), \tag{74}$$

$$c_2(T) = c_1(0). \tag{75}$$

This leads to the flips $E_1 \leftrightarrow E_2$ and $\mathbf{v}_1^R \leftrightarrow \mathbf{v}_2^R$.

When the parameter $\lambda(t)$ changes slowly enough in the latter case, each eigenstate of $\mathbf{v}_1^R$ and $\mathbf{v}_2^R$ evolves into the other. This flip is caused by the singularity of $\Delta(\lambda)$, which is an exceptional point. This is a unique effect of encircling an exceptional point and is called *the flip* or *the adiabatic flip* [10, 13].

The geometric phase with encircling an exceptional point is experimentally measured in a microwave-cavity system [11]. However, it is always difficult to achieve the complete adiabaticity in experiments [13]. It is not only interesting but also necessary to investigate non-adiabatic effects in encircling an exceptional point.

4 Non-adiabatic process around an exceptional point

Suppose that we move the parameter λ around an exceptional point in the parameter space. As we explained in the previous section, the eigenstates are swapped when we move the parameter λ adiabatically. However, Gilary *et al.* [13–15] numerically demonstrated that any states collapsed to one of the eigenstates when we move it non-adiabatically.

In the present section, we discuss this non-adiabatic process. We theoretically show the state collapse, exactly solving the non-adiabatic time-evolution of a model system.

4.1 Numerical discussion

Uzdin *et al.* [13] and Gilary *et al.* [14, 15] discussed non-adiabatic effects by varying the parameter λ around an exceptional point. When the parameter moves slowly enough, we have

$$|\psi_j(t)\rangle = e^{i\theta_j(t)} v_j(t), \quad |\psi_j(0)\rangle = v_j(0) \tag{76}$$

under the adiabatic approximation, which is good for Hermitian systems. However, if we consider the non-adiabatic coupling, the solution of the time-dependent Schrödinger equation is generally given by

$$|\psi_j(t)\rangle = \alpha(t) v_1(t) + \beta(t) v_2(t), \quad |\psi_j(0)\rangle = v_j(0), \tag{77}$$

which Uzdin *et al.* and Gilary *et al.* calculated $|\psi_j(t)\rangle$ numerically.

In their papers, they showed two points about non-adiabatic effects. First, the wave function was always the same for any initial wave functions after λ encircled an exceptional point. Second, the wave function after one period converged to one of the eigenvectors in the adiabatic limit.

They considered the Hamiltonian

$$\hat{\mathcal{H}} = \begin{pmatrix} E_1 + \hbar\omega + \frac{i\Delta\Gamma}{2} & \frac{\epsilon_0 d_{12}}{2} \\ \frac{\epsilon_0 d_{12}}{2} & E_2 - \frac{i\Delta\Gamma}{2} \end{pmatrix} - i\frac{\Gamma_1 + \Gamma_2}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{78}$$

where $\Delta\Gamma = \Gamma_2 - \Gamma_1$. This non-Hermitian Hamiltonian is an effective model of the ninth and tenth resonance states of the H_2^+ molecule in a laser

field [14, 15, 23]. For this setting, Γ_1 and Γ_2 are the decay rates of the ninth and tenth resonance states, respectively. The parameter d_{12} is a complex dipole transition-matrix element, while ϵ_0 and ω are the amplitude and the frequency of the laser field, respectively. The pair of these two parameters corresponds to the complex parameter λ.

For the external parameter (ϵ_0, ω), we can find an exceptional point as follows [15]:

$$(\epsilon_0^{\rm EP}, \omega^{\rm EP}) = \left(\frac{\Delta\Gamma}{{\rm Re}\, d_{12}}, \frac{E_2 - E_1 - {\rm Im}\, d_{12}\epsilon_0^{\rm EP}}{\hbar} \right). \tag{79}$$

We can encircle this exceptional point, for example, as in Fig. 1. When the parameter encircles the exceptional point along the red broken line, the quantities $|\alpha|/(|\alpha|^2 + |\beta|^2)$ and $|\beta|/(|\alpha|^2 + |\beta|^2)$ evolve as shown in Fig. 2. This result suggests that any states almost converge to ninth state if the parameter moves clockwise while to the tenth state if the parameter moves anti-clockwise. In this paper, we call this effect the collectivity.

Moreover, when the period $T_{\rm loop}$ of the parameter change becomes longer, one of the absolute value of the coefficients α and β converges to zero; see Fig. 3. The rate $R = |\alpha|/|\beta|$ converges to either infinity or zero. These mean in the adiabatic limit, $\alpha \to 0$ or $\beta \to 0$.

The results shown in Figs. 2 and 3 imply that the adiabatic approximation may be no longer a good one. This is one of the important features of non-

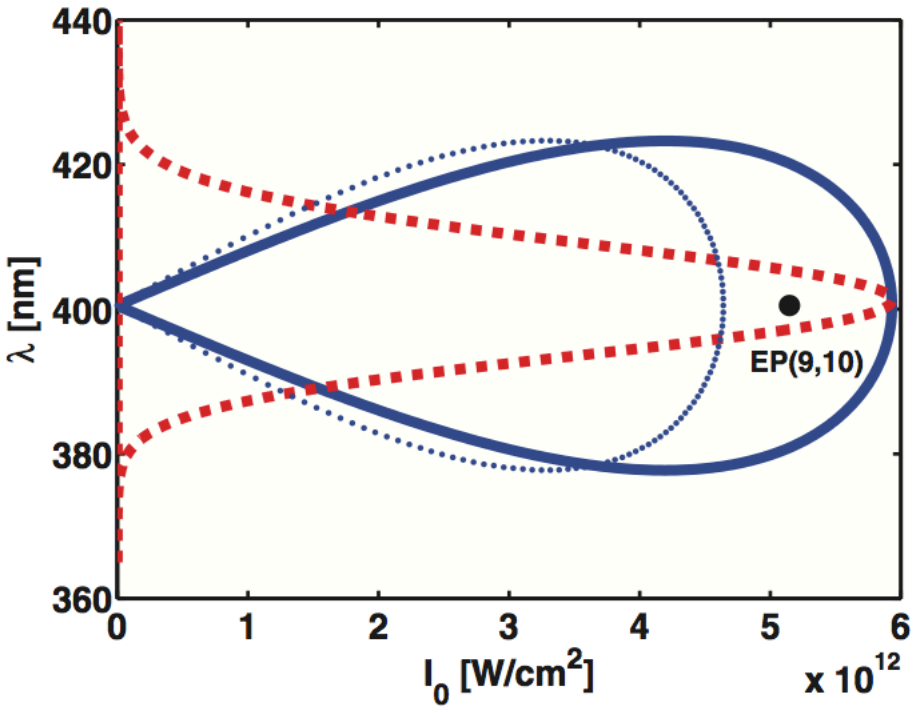


Figure 1: Loops in the parameter space of the laser intensity and wavelength [15]. The horizontal axis is the laser intensity I_0 and the vertical axis is the wavelength λ, which correspond to ϵ_0 and ω, respectively.

Hermitian systems.

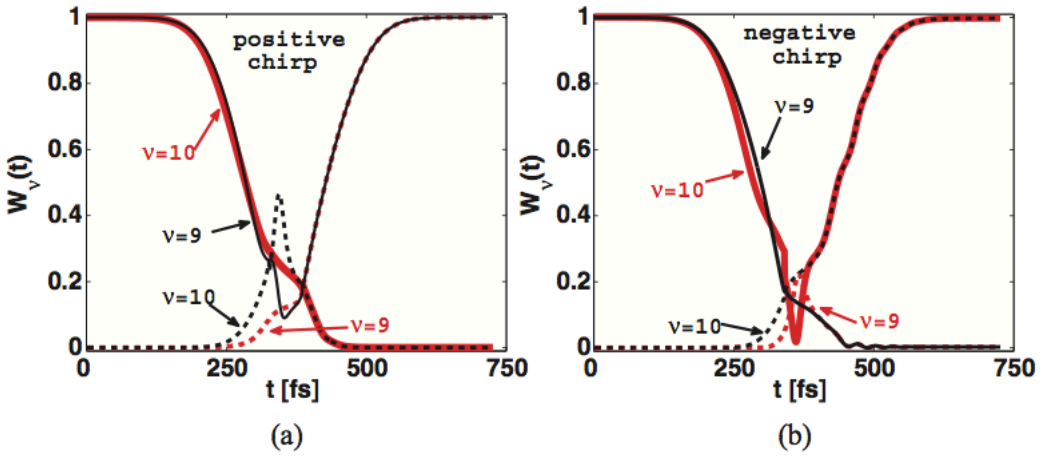


Figure 2: The time evolution of $|\alpha|/(|\alpha|^2+|\beta|^2)^{1/2}$ and $|\beta|/(|\alpha|^2+|\beta|^2)$ [15]. The trajectory of the parameter is the red broken line in Fig. 1. (a) the case when the parameter moves clockwise and (b) when it moves anti-clockwise. The thin black line and broken black line indicate $|\alpha|/(|\alpha|^2+|\beta|^2)^{1/2}$ and $|\beta|/(|\alpha|^2+|\beta|^2)^{1/2}$, respectively, where initially $\alpha=1$ and $\beta=0$. The thick red line and broken red line indicate $|\beta|/(|\alpha|^2+|\beta|^2)^{1/2}$ and $|\alpha|/(|\alpha|^2+|\beta|^2)^{1/2}$, where initially $\beta=1$ and $\alpha=0$.

4.2 Normalized representation of a general time-evolution operator

The solution of the Schrödingier equation with a time-dependent Hamiltonian,

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle=\hat{\mathcal{H}}(\lambda(t))\left|\psi(t)\right\rangle, \tag{80}$$

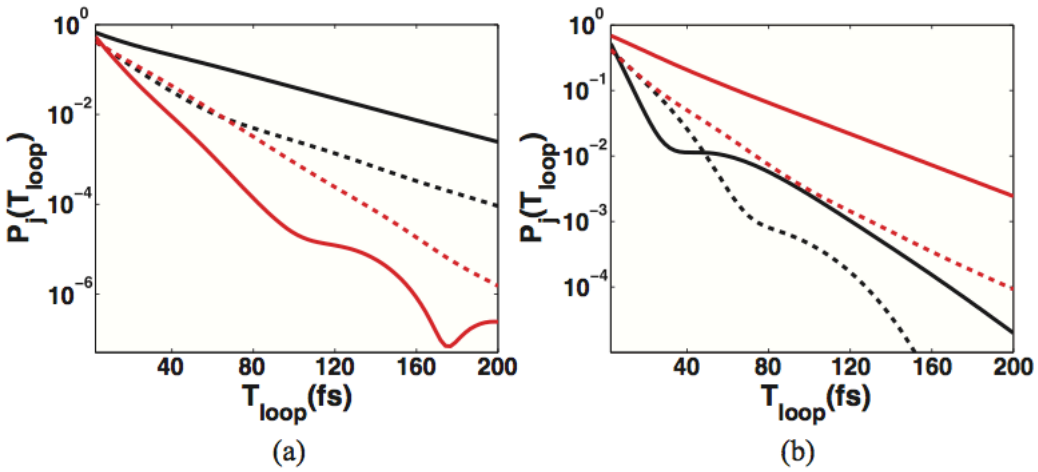


Figure 3: The population P_j of the ninth (black) and tenth (red) resonance states [15]. The trajectories are the blue solid line and the red dashed line in Fig. 1. (a) the case when the parameter moves clockwise and (b) when it moves anti-clockwise. The solid line have corresponds to the solid loop in Fig. 2 and the broken line have corresponds to the broken loop there.

is given by

$$|\psi(t)\rangle = \hat{\mathcal{U}}_T \, |\psi(t)\rangle \tag{81}$$

with the time-evolution operator

$$\hat{\mathcal{U}}_T = \mathcal{T} \exp\left[-\frac{i}{\hbar}\int_0^T dt \; \hat{\mathcal{H}}(\lambda(t))\right], \tag{82}$$

where $\mathcal{T}$ is the time-ordering operator defined as

$$\mathcal{T}\hat{A}(t_1)\hat{A}(t_2)\cdots\hat{A}(t_n) \equiv \hat{A}(t_{N_1})\hat{A}(t_{N_2})\cdots\hat{A}(t_{N_n}) \tag{83}$$

with $t_{N_1} > t_{N_2} > \cdots > t_{N_n}$. Specifically, let us consider a two-by-two Hamiltonian of the form

$$\hat{\mathcal{H}}(\lambda) = \begin{pmatrix} E_0(\lambda) + ia(\lambda) & b(\lambda) \\ b(\lambda) & E_0(\lambda) - ia(\lambda) \end{pmatrix}, \tag{84}$$

where a, b and E_0 are complex functions of a complex variable λ. When a, b and E_0 are real, this Hamiltonian has the $\mathcal{PT}$ symmetry [9].

The secular equation is given by

$$0 = \left| E\,\hat{1}_2 - \hat{\mathcal{H}} \right| = (E - E_0)^2 + a^2 - b^2, \tag{85}$$

where $\hat{1}_2$ is the two-by-two identity matrix. The eigenenergies are

$$E_1 = E_0 + \sqrt{b^2 - a^2}, \tag{86}$$

$$E_2 = E_0 - \sqrt{b^2 - a^2}. \tag{87}$$

The right eigenvectors defined by

$$\hat{\mathcal{H}}\ \mathbf{v}_n^R = E_n \mathbf{v}_n^R \tag{88}$$

for $n = 1$ and 2 are given by

$$\mathbf{v}_1^R = \frac{C_1}{\sqrt{2}} \begin{pmatrix} -\dfrac{b}{a} \\ \\ \dfrac{c_1}{a} \end{pmatrix}, \tag{89}$$

$$\mathbf{v}_2^R = \frac{C_2}{\sqrt{2}} \begin{pmatrix} -\dfrac{b}{a} \\ \\ \dfrac{c_2}{a} \end{pmatrix}, \tag{90}$$

where

$$c_1 = ia - \Delta, \tag{91}$$

$$c_2 = ia + \Delta \tag{92}$$

with

$$\Delta = b^2 - a^2. \tag{93}$$

Conventionally, the normalization constants C_1 and C_2 in Eqs. (89) and (90) are chosen so as to obey the following conditions:

$${}^{\mathrm{T}}\mathbf{v}_1^R \mathbf{v}_1^R = 1, \tag{94}$$

$${}^{\mathrm{T}}\mathbf{v}_2^R \mathbf{v}_2^R = 1. \tag{95}$$

These conditions decide the constants as

$$C_1 = \sqrt{\frac{a^2}{-c_1\Delta}}, \tag{96}$$

$$C_2 = \sqrt{\frac{a^2}{c_2\Delta}}. \tag{97}$$

However, in this thesis we use

$$C_1 = 1, \tag{98}$$

$$C_2 = 1 \tag{99}$$

instead, because the constants (96) and (97) diverge at the exceptional points. In order to construct the Hashimoto form, we need finite vectors of $\mathbf{v}_1^R$, $\mathbf{v}_2^R$, $\mathbf{v}_1^L$ and $\mathbf{v}_2^L$.

We can find the left eigenvectors easily. Since the Hamiltonian (84) is symmetric, the transpose of Eq. (88) gives us

$$ {}^{\mathrm{T}}\mathbf{v}_n^R \, \hat{\mathcal{H}} = E_n \, {}^{\mathrm{T}}\mathbf{v}_n^R \tag{100} $$

for $n = 1$ and 2, which yields the left eigenvectors as

$$ \mathbf{v}_1^L = {}^{\mathrm{T}}\mathbf{v}_1^R = \frac{1}{\sqrt{2}} \begin{pmatrix} -\dfrac{b}{a} & \dfrac{c_1}{a} \end{pmatrix}, \tag{101} $$

$$ \mathbf{v}_2^L = {}^{\mathrm{T}}\mathbf{v}_2^R = \frac{1}{\sqrt{2}} \begin{pmatrix} -\dfrac{b}{a} & \dfrac{c_2}{a} \end{pmatrix}. \tag{102} $$

As discussed previously in Section 2, the exceptional points λ_{EP} are defined by the points in the parameter space where the two eigenvectors $\mathbf{v}_1^R$ and $\mathbf{v}_2^R$ coalesce. In other words, λ_{EP} is given by

$$ \Delta(\lambda_{\mathrm{EP}}) = 0, \tag{103} $$

at which we have

$$ \Delta(\lambda_{\mathrm{EP}}) \equiv b(\lambda_{\mathrm{EP}})^2 - a(\lambda_{\mathrm{EP}})^2 = 0. \tag{104} $$

In order to obtain the exact representation of the time-evolution operator, we utilize the Hashimoto form of the Hamiltonian [16]. Let us first derive the Jordan form at the exceptional point. For $\lambda = \lambda_{\mathrm{EP}}$, we have $a = \pm b$ because of Eq. (104). In the following discussion, we focus on the exceptional point with $a = b$ for simplicity. The solution for $a = -b$ is obtained parallel by.

We then have

$$\hat{\mathcal{H}}_{\rm EP} = \begin{pmatrix} E_0 + ia & a \\ a & E_0 - ia \end{pmatrix} \tag{105}$$

and the two eigenvectors coalesce into one:

$$\mathbf{v}_1^R = \mathbf{v}_2^R = \mathbf{v}_{\rm EP}^R \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ i \end{pmatrix}, \tag{106}$$

$$\mathbf{v}_1^L = \mathbf{v}_2^L = \mathbf{v}_{\rm EP}^L \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i \end{pmatrix}, \tag{107}$$

for which we define the co-vector

$$\mathbf{w}_{\rm EP}^R = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -i \end{pmatrix}, \tag{108}$$

$$\mathbf{w}_{\rm EP}^L = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -i \end{pmatrix}. \tag{109}$$

Indeed, $\mathbf{v}_{\rm EP}^R$ and $\mathbf{w}_{\rm EP}^R$ satisfy (35). The similarity transformation,

$$\hat{S}_{\rm EP} = \begin{pmatrix} \mathbf{v}_{\rm EP}^R & \mathbf{w}_{\rm EP}^R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ i & -i \end{pmatrix}, \tag{110}$$

$$\hat{S}_{\rm EP}^{-1} = \begin{pmatrix} \mathbf{w}_{\rm EP}^L \\ \mathbf{v}_{\rm EP}^L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -i \\ -1 & i \end{pmatrix}, \tag{111}$$

brings the Hamiltonian (105) to the Jordan form:

$$\hat{N}_{\rm EP} = \hat{S}_{\rm EP}^{-1} \hat{\mathcal{H}}_{\rm EP} \hat{S}_{\rm EP} = \begin{pmatrix} E_0 & 2ia \\ 0 & E_0 \end{pmatrix}. \tag{112}$$

Let us now derive the Hashimoto form [16] around the exceptional point. As we discussed in Section 2, we defined the generalized co-vectors as

$$\begin{aligned} &\mathbf{w}_1^L \cdot \mathbf{v}_1^R = 1, \\ &\mathbf{w}_1^L \hat{\mathcal{H}} = E_1 \mathbf{w}_1^L + 2ia \mathbf{v}_2^L \end{aligned} \tag{113}$$

and

$$\begin{aligned} &\mathbf{v}_2^L \cdot \mathbf{w}_2^R = 1, \\ &\hat{\mathcal{H}} \mathbf{w}_2^R = 2ia \mathbf{v}_1^R + E_2 \mathbf{w}_2^R. \end{aligned} \tag{114}$$

We can indeed find the following vectors uniquely:

$$\mathbf{w}_1^L = \begin{pmatrix} \frac{ic_2}{b} & -i \end{pmatrix}, \tag{115}$$

$$\mathbf{w}_2^R = \begin{pmatrix} \frac{ic_1}{b} \\ \\ -i \end{pmatrix}. \tag{116}$$

The orthogonal matrices

$$\hat{S} = \begin{pmatrix} \mathbf{v}_1^R & \mathbf{w}_2^R \end{pmatrix} = \begin{pmatrix} -\frac{b}{a} & \frac{ic_1}{b} \\ \\ \frac{c_1}{a} & -i \end{pmatrix}, \tag{117}$$

$$\hat{S}^{-1} = \begin{pmatrix} \mathbf{w}_1^L \\ \\ \mathbf{v}_2^L \end{pmatrix} = \begin{pmatrix} \frac{ic_2}{b} & -i \\ \\ -\frac{b}{a} & \frac{c_2}{a} \end{pmatrix} \tag{118}$$

constitute a similarity transformation which brings the Hamiltonian (84) to the Hashimoto form

$$\hat{N} = \hat{S}^{-1}\hat{\mathcal{H}}\hat{S} = \begin{pmatrix} E_1 & 2ia \\ 0 & E_2 \end{pmatrix}. \tag{119}$$

As discussed in Section 2, both $\hat{S}$ and $\hat{N}$ are continuous around an exceptional point. We thereby expect that the Hashimoto form is more useful than the diagonalization for the analysis of the evolution around the exceptional point.

Let us assume a cyclic dynamics with $\lambda(T) = \lambda(0)$. The time-evolution operator (82) is then given by

$$\hat{\mathcal{U}}_T = \mathcal{T} \exp\left[-\frac{i}{\hbar}\int_0^T dt\ \hat{S}(\lambda(t))\,\hat{N}(\lambda(t))\,\hat{S}^{-1}(\lambda(t))\right]. \tag{120}$$

Its Trotter decomposition produces

$$\begin{aligned}\hat{\mathcal{U}}_T = \lim_{N\to\infty} \Bigg\{ & \exp\left[-\frac{i}{\hbar}\,\Delta t\,\hat{S}_N\hat{N}_N\hat{S}_N^{-1}\right] \\ & \times \exp\left[-\frac{i}{\hbar}\,\Delta t\,\hat{S}_{N-1}\hat{N}_{N-1}\hat{S}_{N-1}^{-1}\right]\ \cdots\ \exp\left[-\frac{i}{\hbar}\,\Delta t\,\hat{S}_1\hat{N}_1\hat{S}_1^{-1}\right]\Bigg\} \\ = \lim_{N\to\infty} \Bigg\{ & \hat{S}_N\, e^{-\frac{i}{\hbar}\Delta t\,\hat{N}_N}\,\hat{S}_N^{-1} \\ & \times \hat{S}_{N-1}\, e^{-\frac{i}{\hbar}\Delta t\,\hat{N}_{N-1}}\,\hat{S}_{N-1}^{-1} \times \cdots \times \hat{S}_1\, e^{-\frac{i}{\hbar}\Delta t\,\hat{N}_1}\,\hat{S}_1^{-1}\Bigg\},\end{aligned} \tag{121}$$

where $\Delta t = T/N$, $t_n = n\Delta t$, $\hat{N}_n \equiv \hat{N}(\lambda(t_n))$, and $\hat{S}_n \equiv \hat{S}(\lambda(t_n))$. To continue this calculation, we need to simplify $\hat{S}_n^{-1}\hat{S}_{n-1}$:

$$\hat{S}_n^{-1}\hat{S}_{n-1} \simeq \hat{S}_n^{-1}\left(\hat{S}_n - \frac{d\hat{S}_n}{d\lambda}\frac{d\lambda}{dt}\Delta t\right) = \hat{1}_2 - \hat{S}_n^{-1}\frac{d\hat{S}_n}{dt}\Delta t \tag{122}$$

with

$$\frac{d\hat{S}_n}{dt} = \frac{d\hat{S}_n}{d\lambda}\frac{d\lambda}{dt}. \tag{123}$$

We can achieve the equality of Eq. (122) in the limit $\Delta t \to 0$. We thereby have

$$\begin{aligned}
\hat{\mathcal{U}}_T = \lim_{N\to\infty} \mathcal{T}\Bigg\{ & \hat{S}_N\, e^{-\frac{i}{\hbar}\Delta t\, \hat{N}_N} \left(\hat{1} - \hat{S}_N^{-1}\frac{d\hat{S}_N}{dt}\Delta t\right) \\
& \times e^{-\frac{i}{\hbar}\Delta t\, \hat{N}_{N-1}} \left(\hat{1} - \hat{S}_{N-1}^{-1}\frac{d\hat{S}_{N-1}}{dt}\Delta t\right) \times \cdots \\
& \times e^{-\frac{i}{\hbar}\Delta t\, \hat{N}_1} \left(\hat{1} - \hat{S}_1^{-1}\frac{d\hat{S}_1}{dt}\Delta t\right) \hat{S}_0^{-1} \Bigg\} \\
= \hat{S}\left(\lambda(T)\right) \lim_{N\to\infty} \mathcal{T}\Bigg\{ & \exp\left[-\frac{i}{\hbar}\Delta t \left(\hat{N}_N - i\hbar \hat{S}_N^{-1}\frac{d\hat{S}_N}{dt}\right)\right] \\
& \times \exp\left[-\frac{i}{\hbar}\Delta t \left(\hat{N}_{N-1} - i\hbar \hat{S}_{N-1}^{-1}\frac{d\hat{S}_{N-1}}{dt}\right)\right] \times \cdots \\
& \times \exp\left[-\frac{i}{\hbar}\Delta t \left(\hat{N}_1 - i\hbar \hat{S}_1^{-1}\frac{d\hat{S}_1}{dt}\right)\right] \Bigg\} \hat{S}^{-1}\left(\lambda(0)\right),
\end{aligned} \tag{124}$$

or formally, with $\hat{S}_{\rm f} = \hat{S}(T)$ and $\hat{S}_{\rm i} = \hat{S}(0)$,

$$\begin{aligned}
\hat{\mathcal{U}}_T &= \hat{S}_{\rm f}\mathcal{T} \exp\left[-\frac{i}{\hbar}\int_0^T dt \left(\hat{N}(\lambda(t)) - i\hbar \hat{S}^{-1}(\lambda(t))\frac{d\hat{S}}{dt}(\lambda(t))\right)\right] \hat{S}_{\rm i}^{-1} \\
&= \hat{S}_{\rm f}\mathcal{T} \exp\left[-\frac{i}{\hbar}\int_0^T dt \hat{\mathcal{H}}'(t)\right] \hat{S}_{\rm i}^{-1},
\end{aligned} \tag{125}$$

where

$$\hat{\mathcal{H}}'(t) = \hat{N}(\lambda(t)) - i\hbar \hat{S}^{-1}(\lambda(t))\frac{d\hat{S}}{dt}(\lambda(t)). \tag{126}$$

We refer to this as the effective Hamiltonian hereafter.

4.3 An exactly solvable model

In this subsection we construct a nontrivial model for which we can exactly calculate the time-evolution operator and compare the non-adiabatic time-evolution with the adiabatic one. The discussion in the present subsection is based on Eq. (125).

Before deriving our model, we remark that we hereafter use the notations $a(t)$, $b(t)$, $\hat{S}(t)$ and so on instead of $a(\lambda(t))$, $b(\lambda(t))$, $\hat{S}(\lambda(t))$ and so on, respectively.

Our method consists of three steps:

1. We derive an exact form of the time-evolution operator in terms of $a(t)$, $b(t)$ and $E_0(t)$.

2. We find $a(t)$, $b(t)$ and $E_0(t)$ so that the time-evolution operator may be easy to calculate and at the same time the flip of the eigenstates may occur.

3. We finally find $a(\lambda)$, $b(\lambda)$ and $E_0(\lambda)$ as the functions of λ by solving the following equations about $a(\lambda)$, $b(\lambda)$, $E_0(\lambda)$ and $\lambda(t)$:

$$a(\lambda(t)) = a(t), \tag{127}$$

$$b(\lambda(t)) = b(t), \tag{128}$$

$$E_0(\lambda(t)) = E_0(t). \tag{129}$$

Since the step 3 is difficult to treat generally, we assume that there exist the solutions of Eqs. (127)–(129). In other words, we concentrate on deriving $a(t)$, $b(t)$ and $E_0(t)$ in the present thesis.

To derive our model, we first compute the effective Hamiltonian (126) explicitly. For simplicity, we set $E_0(t) \equiv 0$. Using the fact

$$\frac{d}{dt}(\mathbf{w}_1^L \cdot \mathbf{v}_1^R) = 0, \tag{130}$$

we obtain

$$\begin{aligned}\hat{S}^{-1}(t)\dot{\hat{S}}(t) &= \begin{pmatrix} \mathbf{w}_1^L(t)\cdot\dot{\mathbf{v}}_1^R(t) & \mathbf{w}_1^L(t)\cdot\dot{\mathbf{w}}_2^R(t) \\ \mathbf{v}_2^L(t)\cdot\dot{\mathbf{v}}_1^R(t) & \mathbf{v}_2^L(t)\cdot\dot{\mathbf{w}}_2^R(t) \end{pmatrix} \\ &= \begin{pmatrix} -\dot{\mathbf{w}}_1^L(t)\cdot\mathbf{v}_1^R(t) & \mathbf{w}_1^L(t)\cdot\dot{\mathbf{w}}_2^R(t) \\ \mathbf{v}_2^L(t)\cdot\dot{\mathbf{v}}_1^R(t) & \mathbf{v}_2^L(t)\cdot\dot{\mathbf{w}}_2^R(t) \end{pmatrix}. \end{aligned} \tag{131}$$

The time derivatives are

$$\begin{aligned}\dot{\mathbf{w}}_1^L(t) &= \frac{1}{\sqrt{2}}\begin{pmatrix} \dfrac{ic_2(t)}{b(t)}\left(\dfrac{\dot{c}_2(t)}{c_2(t)} - \dfrac{\dot{b}(t)}{b(t)}\right) & 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} -\left(\dfrac{1}{b(t)} + \dfrac{ia(t)}{b(t)\Delta(t)}\right)\dot{a}(t) + \left(\dfrac{i}{\Delta(t)} - \dfrac{ic_2(t)}{b(t)^2}\right)\dot{b}(t) & 0 \end{pmatrix}, \end{aligned} \tag{132}$$

$$\dot{\mathbf{w}}_2^R(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \dfrac{ic_1(t)}{b(t)} \left(\dfrac{\dot{c}_1(t)}{c_1(t)} - \dfrac{\dot{b}(t)}{b(t)} \right) \\ \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -\left(\dfrac{1}{b(t)} - \dfrac{ia(t)}{b(t)\Delta(t)} \right) \dot{a}(t) + \left(-\dfrac{i}{\Delta(t)} + \dfrac{ic_1(t)}{b(t)^2} \right) \dot{b}(t) \\ \\ 0 \end{pmatrix}, \tag{133}$$

and

$$\dot{\mathbf{v}}_1^R(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\dfrac{b(t)}{a(t)} \left(\dfrac{\dot{b}(t)}{b(t)} - \dfrac{\dot{a}(t)}{a(t)} \right) \\ \\ \dfrac{c_1(t)}{a(t)} \left(\dfrac{\dot{c}_1(t)}{c_1(t)} - \dfrac{\dot{a}(t)}{a(t)} \right) \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -\dfrac{b(t)}{a(t)} \left(\dfrac{\dot{b}(t)}{b(t)} - \dfrac{\dot{a}(t)}{a(t)} \right) \\ \\ \left(\dfrac{1}{\Delta(t)} + \dfrac{\Delta(t)}{a(t)^2} \right) \dot{a}(t) - \dfrac{b(t)}{a(t)\Delta(t)} \dot{b}(t) \end{pmatrix}. \tag{134}$$

Because $c_1(t)\ c_2(t) = -b(t)^2$, we have

$$-\dot{\mathbf{w}}_1^L(t) \cdot \mathbf{v}_1^R(t) = \frac{1}{2} \Big[-\left(\frac{1}{a(t)} + \frac{i}{\Delta(t)} \right) \dot{a}(t) \\ + i \left(\frac{b(t)}{a(t)\Delta(t)} + \frac{ic_2(t)}{a(t)b(t)} \right) \dot{b}(t) \Big]$$

$$= \frac{b(t)\dot{a}(t) - \dot{b}(t)a(t)}{2\ b(t)\Delta(t)}\ \frac{c_2(t)}{a(t)}, \tag{135}$$

$$\mathbf{w}_1^L(t)\cdot\dot{\mathbf{w}}_2^R(t)=\frac{1}{2}\Bigg[-\left(\frac{ic_2(t)}{b(t)^2}+\frac{ac_2(t)}{b(t)^2\,\Delta(t)}\right)\dot{a}(t)$$

$$+\left(\frac{c_2(t)}{b(t)\Delta(t)}+\frac{1}{b(t)}\right)\dot{b}(t)\Bigg]$$

$$=\frac{b(t)\dot{a}(t)-\dot{b}(t)a(t)}{2\;b(t)\Delta(t)}\;(-i), \tag{136}$$

$$\mathbf{v}_2^L(t)\cdot\dot{\mathbf{v}}_1^R(t)=\frac{1}{2}\Bigg[\left(\frac{b(t)}{a(t)}\right)^2\left(\frac{\dot{b}(t)}{b(t)}-\frac{\dot{a}(t)}{a(t)}\right)$$

$$+\left(\frac{c_2(t)}{a(t)\Delta(t)}+\frac{c_2(t)\Delta(t)}{a(t)^3}\right)\dot{a}(t)-\frac{c_2(t)b(t)}{a(t)^2\Delta(t)}\dot{b}(t)\Bigg]$$

$$=\frac{b(t)\dot{a}(t)-\dot{b}(t)a(t)}{2\;b(t)\Delta(t)}\;i\left(\frac{b(t)}{a(t)}\right)^2, \tag{137}$$

$$\mathbf{v}_2^L(t)\cdot\dot{\mathbf{w}}_2^R(t)=\frac{1}{2}\Bigg[-\left(\frac{1}{a(t)}-\frac{i}{\Delta(t)}\right)\dot{a}(t)$$

$$-i\left(\frac{b(t)}{a(t)\Delta(t)}+\frac{ic_1(t)}{a(t)b(t)}\right)\dot{b}(t)\Bigg]$$

$$=-\frac{b(t)\dot{a}(t)-\dot{b}(t)a(t)}{2\;b(t)\Delta(t)}\;\frac{c_1(t)}{a(t)}. \tag{138}$$

Inserting Eqs. (135)–(138) into Eq. (131), we arrive at

$$\hat{S}^{-1}(t)\dot{\hat{S}}(t)=\frac{b(t)\dot{a}(t)-\dot{b}(t)a(t)}{2\;b(t)\Delta(t)}\begin{pmatrix}\dfrac{c_2(t)}{a(t)} & -i\\[2ex] i\left(\dfrac{b(t)}{a(t)}\right)^2 & -\dfrac{c_1(t)}{a(t)}\end{pmatrix}, \tag{139}$$

which is followed by the effective Hamiltonian

$$\hat{\mathcal{H}}'=\begin{pmatrix}\Delta & 2ia\\ 0 & -\Delta\end{pmatrix}-i\hbar\frac{b\dot{a}-\dot{b}a}{2b\Delta}\begin{pmatrix}\dfrac{c_2}{a} & -i\\[2ex] i\left(\dfrac{b}{a}\right)^2 & -\dfrac{c_1}{a}\end{pmatrix}. \tag{140}$$

In the following, we look for an exactly solvable non-adiabatic and yet nontrivial model. For the purpose, we set the (1,2) element of Eq. (140) to zero so that $\hat{\mathcal{H}}'$ may be a lower triangular matrix. The condition is given by

$$\frac{b\dot{a} - \dot{b}a}{ab} = -\frac{i}{\hbar}4\Delta. \tag{141}$$

Because

$$\frac{d}{dt}\frac{a}{b} = \frac{a}{b}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right) = \frac{a}{b}\frac{b\dot{a} - \dot{b}a}{ab}, \tag{142}$$

Eq. (141) reduces to

$$\frac{\dot{r}}{r} = -\frac{i}{\hbar}4\Delta \tag{143}$$

with $r = a/b$. By solving this, we have

$$r(t) = r_0 \exp\left[-\frac{i}{\hbar}4\int_0^t dt'\Delta(t')\right]. \tag{144}$$

This result indicates that $r(t)$ is given by $\Delta(t)$ in our condition. We can also express $a(t)$ and $b(t)$ in terms of $\Delta(t)$. Since

$$\Delta^2 = b^2 - a^2 = b^2\left(1 - r^2\right), \tag{145}$$

we have

$$b(t) = \frac{\Delta(t)}{\sqrt{1 - r(t)^2}} \tag{146}$$

and then

$$a(t) = r(t)b(t). \tag{147}$$

Let us find a function $\Delta(t)$ that achieves our aim. Our task here is to see the breaking of the adiabatic flip in our non-adiabatic process. We therefore set $\lambda(T) = \lambda(0)$ and $\Delta(T) = -\Delta(0)$. The former condition means that the Hamiltonian is periodic with period T and the latter means that the system must exhibit the flip if the motion of λ is adiabatic. Under these extra conditions,

$$a(T) = a(0), \tag{148}$$

$$b(T) = b(0), \tag{149}$$

$$r(T) = r(0), \tag{150}$$

$$\Delta(T) = -\Delta(0) \tag{151}$$

should be satisfied.

Equations (148)–(150) are obviously consistent with Eq. (147). Comparing Eqs. (146), (149) and (151), on the other hand, we realize that $\sqrt{1 - r(t)^2}$ must be anti-periodic as

$$\sqrt{1 - r(T)^2} = -\sqrt{1 - r(0)^2}, \tag{152}$$

although $r(t)$ itself must be periodic. Therefore $\Delta(t)$ must be given so that $r(t)^2$ may go around the origin in the complex plane even times and around the point $r^2 = 1$ in the complex plane odd times. Because of Eq. (144), the

periodicity of $r(t)$ gives

$$\int_0^T dt' \Delta(t') = \frac{\hbar}{4} 2n\pi \tag{153}$$

with integer n. For simplicity, let us choose $n = 0$. Equation (144) dictates that the real part of Δ contributes to the phase of $r(t)$ and the imaginary part does to the amplitude. Thus the conditions $\mathrm{Im}\,\Delta(T) = \mathrm{Im}\,\Delta(0) = 0$, $\mathrm{Re}\,\Delta(T) \neq 0$ and $\mathrm{Re}\,\Delta(0) \neq 0$ are reasonable. Taken together, the conditions for our model are

$$\mathrm{Re}\,\Delta(T) \neq 0, \tag{154}$$

$$\mathrm{Re}\,\Delta(0) \neq 0, \tag{155}$$

$$\int_0^T dt'\, \mathrm{Re}\,\Delta(t') = 0, \tag{156}$$

$$\mathrm{Im}\,\Delta(T) = \mathrm{Im}\,\Delta(0) = 0, \tag{157}$$

$$\int_0^T dt'\, \mathrm{Im}\,\Delta(t') = 0. \tag{158}$$

Equations (154)–(156) are easily satisfied by a cosine function. Equations (157) and (158) are satisfied by a sine function but it does not produce $r(t)^2$ that goes around the point $r^2 = 1$. We here propose the following model:

$$\Delta(t) = \hbar\omega \left\{ \cos\left(\frac{\pi t}{T}\right) + i\frac{\alpha}{T^4}\left[-\left(t - \frac{T}{2}\right)^4 + \frac{3T^2}{10}\left(t - \frac{T}{2}\right)^2 - \frac{T^4}{80} \right] \right\}, \tag{159}$$

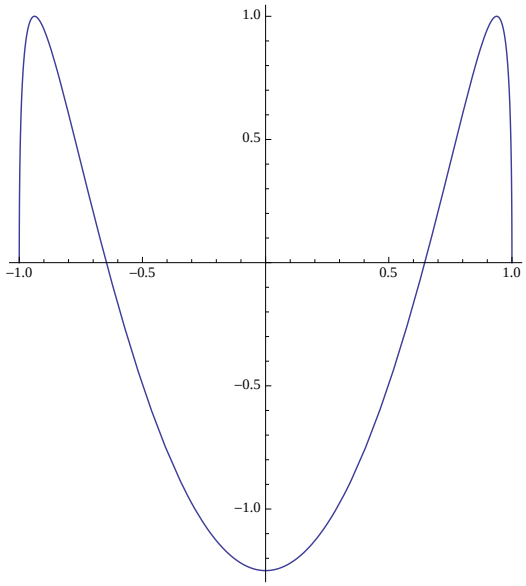
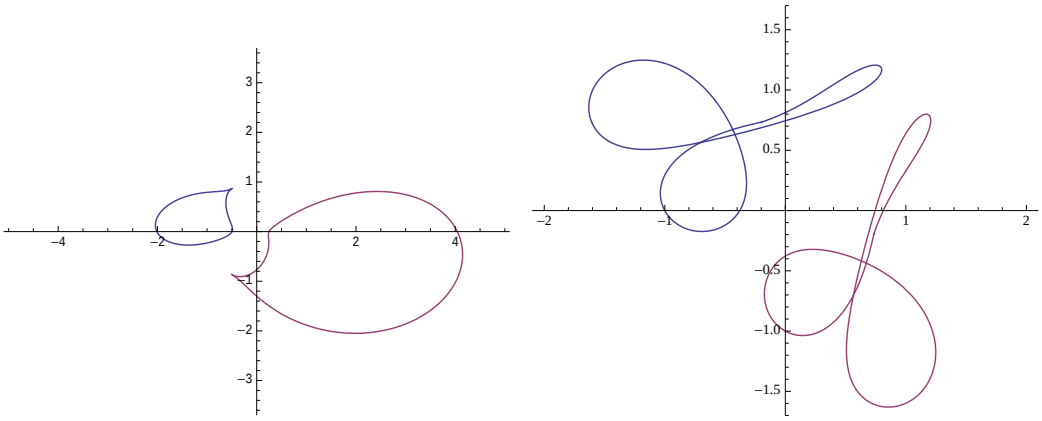


Figure 4: The time dependence of $\Delta(t)$ for our model (159) plotted in the complex plane. Parameters are set to $\alpha = 100$ and $\hbar\omega = 1$.

where ω and α are real constants. Figure 4 shows the plot of $\Delta(t)$ with parameters $\alpha = 100$ and $\hbar\omega = 1$. Equation (144) then gives

$$r(t) = r_0 \exp\left\{-T\omega\left[\frac{4\alpha}{5}\left(\frac{t}{T}\right)^2\left(\frac{t}{T}-1\right)^2\left(\frac{t}{T}-\frac{1}{2}\right) + i\frac{4}{\pi}\sin\left(\frac{\pi}{T}t\right)\right]\right\}. \tag{160}$$

We also have $b(t)$ and $a(t)$ from Eqs. (146) and (147). We plot $r(t)$, $b(t)$ and $a(t)$ in Fig. 5 with the parameters $\alpha = 100$, $\omega = 1$, $r_0 = \exp(i2\pi/3)$ and assuming $\hbar = T = 1$ We can see that $r(t)^2$ goes around the point $r^2 = 1$ once.



(a) (b)

Figure 5: The time dependence of (a) $r(t)$ (blue curve at the left of (a)) and $r(t)^2$ (red curve at the right of (a)) and (b) $b(t)$ (blue curve at the left-top of (b)) and $b(t)$ (red curve at the right-bottom of (b)) for our model (159) plotted in the complex plane. The parameters are set to $\alpha = 100$, $\omega = 1$, $r_0 = \exp(i2\pi/3)$ and $\hbar = T = 1$. The exceptional point exists at 1 in the panel (a).

Calculation of the time-evolution and comparing with the adiabatic result

In order to calculate the time-evolution operator of our model, we first simplify $\hat{\mathcal{H}}'$ by applying Eq. (141) to $\hat{S}^{-1}\dot{\hat{S}}$. We have

$$\begin{aligned}\hat{\mathcal{H}}' &= \begin{pmatrix} \Delta & 2ia \\ 0 & -\Delta \end{pmatrix} + 2a \begin{pmatrix} \frac{c_2}{a} & -i \\ i\left(\frac{b}{a}\right)^2 & -\frac{c_1}{a} \end{pmatrix} \\ &= \begin{pmatrix} 2i+\Delta & 0 \\ 2ia\left(\frac{b}{a}\right)^2 & -2i-3\Delta \end{pmatrix} \\ &= \begin{pmatrix} 2ia+E_1 & 0 \\ 2ia\left(\frac{b}{a}\right)^2 & -2ia+3E_2 \end{pmatrix}, \end{aligned} \tag{161}$$

which is followed by

$$\exp\left(-\frac{i}{\hbar}\hat{\mathcal{H}}'\Delta t\right) = \begin{pmatrix} e^{\frac{2}{\hbar}a\Delta t}\, e^{-\frac{i}{\hbar}E_1\Delta t} & 0 \\ \beta(t;\Delta t) & e^{-\frac{2}{\hbar}a\Delta t}\, e^{-\frac{i}{\hbar}3E_2\Delta t} \end{pmatrix} \tag{162}$$

with

$$\beta(t;\Delta t) \equiv \frac{b}{4\,(ia+\Delta)}\left(e^{-\frac{i}{\hbar}(2\,i\,a+E_1)\Delta t} - e^{-\frac{i}{\hbar}(-2\,i\,a+3E_2)\Delta t}\right). \tag{163}$$

The Trotter decomposition (121) therefore yields

$$\hat{\mathcal{U}}_T = \hat{S}_{\rm f} \begin{pmatrix} e^{\frac{2}{\hbar}\int_0^T dt\, a(t)}\ e^{-\frac{i}{\hbar}\int_0^T dt\, E_1(t)} & 0 \\ B(T) & e^{-\frac{2}{\hbar}\int_0^T dt\, a(t)}\ e^{-\frac{i}{\hbar}3\int_0^T dt\, E_2(t)} \end{pmatrix} \hat{S}_{\rm i}^{-1}$$

$$= \hat{S}_{\rm f} \begin{pmatrix} e^{\frac{2}{\hbar}\int_0^T dt\, a(t)} & 0 \\ B(T) & e^{-\frac{2}{\hbar}\int_0^T dt\, a(t)} \end{pmatrix} \hat{S}_{\rm i}^{-1} \tag{164}$$

with

$$B(T) = \lim_{N\to\infty} \sum_{m=1}^{N} b_N b_{N-1} \cdots b_{m+1} c_m a_{m-1} a_{m-2} \cdots a_1, \tag{165}$$

where

$$a_n = e^{\frac{2}{\hbar}a(t_n)\Delta t}\ e^{-\frac{i}{\hbar}E_1(t_n)\Delta t}, \tag{166}$$

$$b_n = e^{-\frac{2}{\hbar}a(t_n)\Delta t}\ e^{-\frac{i}{\hbar}3E_2(t_n)\Delta t}, \tag{167}$$

$$c_n = \beta(t_n;\Delta t) = \frac{b(t_n)}{4(ia(t_n)+\Delta(t_n))}\left(a_n - b_n\right). \tag{168}$$

Note that we used the conditions (156) and (158) here.

Equation (164) is the propagator from $t = 0$ to $t = T$ in a representation given by $\hat{S}_{\rm i}^{-1}$ at $t = 0$ and $\hat{S}_{\rm f}$ at $t = T$. Let p and q denote the (1, 1) and (2, 1) elements of Eq. (164), respectively. For example, suppose that the state $|\psi(t)\rangle$ is initially the eigenstate of the Hamiltonian:

$$|\psi(t=0)\rangle = \mathbf{v}_1^R(0). \tag{169}$$

Since

$$\hat{S}_{\rm i}^{-1} = \begin{pmatrix} \mathbf{w}_1^L(0) \\ \\ \mathbf{v}_2^L(0) \end{pmatrix} \tag{170}$$

and

$$\mathbf{w}_1^L(0) \cdot \mathbf{v}_1^R(0) = 1, \tag{171}$$

$$\mathbf{v}_2^L(0) \cdot \mathbf{v}_1^R(0) = 0, \tag{172}$$

we have

$$\hat{S}_{\rm i}^{-1} \left|\psi(t=0)\right\rangle = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}, \tag{173}$$

and hence our state at $t = T$ is represented by

$$\left|\psi(t=T)\right\rangle = \hat{S}_{\rm f} \begin{pmatrix} p \\ \\ q \end{pmatrix} = p\mathbf{v}_1^R(T) + q\mathbf{w}_2^R(T) = p\mathbf{v}_2^R(0) + q\mathbf{w}_1^R(0), \tag{174}$$

where

$$p = e^{\frac{2}{\hbar}\int_0^T dt\, a(t)}, \tag{175}$$

$$q = B(T), \tag{176}$$

We can also calculate $B(T)$. Using Eqs. (165-168), we have

$$\begin{aligned}
B(T) &= \lim_{N\to\infty} \sum_{m=1}^{N} b_N b_{N-1} \cdots b_{m+1} a_{m-1} a_{m-2} \cdots a_1 \\
&\qquad \times \frac{b(t_m)}{4c_2(t_m)} \frac{4}{\hbar} \left(a(t_m) - i\Delta(t_m)\right) \Delta t \\
&= \lim_{N\to\infty} \sum_{m=1}^{N} \exp\left[\sum_{n=m+1}^{N} \left(\frac{2}{\hbar} a(t_n) - \frac{i}{\hbar} \Delta(t_m) \right) \Delta t \right. \\
&\qquad \left. - \sum_{n=1}^{m} \left(\frac{2}{\hbar} a(t_n) - 3\Delta(t_n) \right) \Delta t \right] \times \frac{-i}{\hbar} 4b(t_m) \Delta t \\
&= \int_0^T dt \exp\left[-\frac{4}{\hbar} \int_0^t dt' \left(a(t') - i\Delta(t')\right) \right] b(t) \\
&= \int_0^T dt \frac{\exp\left[-i\frac{4}{\hbar} \int_0^t dt' \Delta(t') \right] a(t)}{\exp\left[\frac{4}{\hbar} \int_0^t dt' a(t') \right] r(t)}.
\end{aligned} \tag{177}$$

$\mathbf{v}_2^R(0) = {}^{\mathrm{T}}\mathbf{v}_2^L(0)$, and $\mathbf{w}_1^R(0) = {}^{\mathrm{T}}\mathbf{w}_1^L(0)$. We thereby conclude that $|p|^2$ and $|q|^2$ are the rate of propagation to the states $\mathbf{v}_2^R$ and $\mathbf{w}_1^R$, respectively.

The eigenstate $\mathbf{v}_1^R$ therefore flips to the other eigenstate $\mathbf{v}_2^R$ with the rate

$$|p|^2 = \exp\left[\frac{4}{\hbar} \mathrm{Re} \int_0^T dt\, a(t) \right]. \tag{178}$$

For example, numerically exact calculation with $\alpha = 100$, $\omega = 1$, $r_0 = \exp(i2\pi/3)$ and $\hbar = T = 1$ gives

$$|p|^2 \approx 0.189914 \tag{179}$$

and

$$|q|^2 \approx 456.195\ . \tag{180}$$

This result means that if the process is non-adiabatic, we find that make the flipping of the eigenstate incomplete.

We now show that $|p|^2$ in Eq. (178) converges to 1 in the limit $T \to \infty$. For the purpose, we will hereafter show

$$\theta_T \equiv \mathrm{Re} \int_0^T dt\, a(t) \to 0 \tag{181}$$

as $T \to \infty$. Using the definitions of our model, we obtain

$$\begin{aligned} \theta_T &= \mathrm{Re} \int_0^T \Delta(t) \sqrt{\frac{r(t)^2}{1-r(t)^2}} dt \\ &= \mathrm{Re}\, T \int_0^1 \Delta(Ts) \sqrt{\frac{r(Ts)^2}{1-r(Ts)^2}} ds, \end{aligned} \tag{182}$$

where we made the transformation $t = Ts$. Because $\Delta(t) = f(t/T)$ with a function f independent of T, $\Delta(Ts)$ is independent of T. We therefore have

$$\theta_T = \mathrm{Re} \int_0^1 f(s) K(s;T) ds, \tag{183}$$

where

$$K(s;T) = T \sqrt{\frac{r(Ts)^2}{1-r(Ts)^2}}. \tag{184}$$

Using the conditions (156) and (158), we can find

$$\int_0^1 f(s) ds = 0. \tag{185}$$

The central problem is therefore the evaluation of $K(s;T)$ in the limit $T \to$

∞. We can rewrite it in the form

$$[K(s;T)]^2 = T^2 \frac{r_0^2 \exp[2TCg(s)]}{1 - r_0^2 \exp[2TCg(s)]} \tag{186}$$

$$= T^2 \frac{1}{r_0^{-2} \exp[-2TCg(s)] - 1}, \tag{187}$$

where

$$C = \frac{8}{\hbar}, \quad g(s) = -i \int_0^s ds'\, f(s'). \tag{188}$$

If $\operatorname{Re} g(s) > 0$, the term $r_0^{-2} e^{-2Cg(s)T}$ in the denominator of (187) is negligible for sufficiently large T and hence

$$[K(s;T)]^2 \simeq -T^2. \tag{189}$$

If $\operatorname{Re} g(s) < 0$, the factor $r_0^2 e^{2TCg(s)}$ in (186) is negligible for sufficiently large T and hence

$$[K(s;T)]^2 \simeq 0. \tag{190}$$

We therefore conclude that

$$K(s;T) \simeq \pm iT\Theta(\operatorname{Re} g(s)) \tag{191}$$

for sufficient large T, where Θ is the Heaviside step function:

$$\Theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ \\ 0 & \text{if } x < 0. \end{cases} \tag{192}$$

This result gives

$$\begin{aligned}\theta_T &= \int_0^1 ds\,[\mathrm{Re}\,f(s)\,\mathrm{Re}\,K(s;T) - \mathrm{Im}\,f(s)\,\mathrm{Im}\,K(s;T)] \\ &\simeq \mp T\int_0^1 \mathrm{Im}\,f(s)\Theta(\mathrm{Re}\,g(s))ds. \end{aligned} \tag{193}$$

Note that

$$\begin{aligned}\mathrm{Re}\,g(s) &= \mathrm{Re}\left(-i\int_0^s ds' f(s')\right) \\ &= I(s), \end{aligned} \tag{194}$$

where

$$I(s) = \mathrm{Im}\int_0^s ds' f(s'). \tag{195}$$

Equation (193) thereby reduces to

$$\theta_T \simeq \mp T\int_0^1 I'(s)\Theta(I(s))ds \tag{196}$$

under the conditions $I(0) = I(1) = 0$, the latter of which is due to Eq. (185).

We finally evaluate Eq. (196). Suppose that $I(s)$ behaves as shown in Fig. 6. We then have

$$\begin{aligned}\theta_T &\simeq \mp\left(\mathrm{Im}\int_{s_1}^{e_1} dsf(s) + \mathrm{Im}\int_{s_2}^{e_2} dsf(s) + \cdots + \mathrm{Im}\int_{s_n}^{e_n} dsf(s)\right) \\ &= \mp\left[(I(e_1) - I(s_1)) + (I(e_2) - I(s_2)) + \cdots + (I(e_n) - I(s_n))\right] \\ &= 0. \end{aligned} \tag{197}$$

We can also have $|B(T)|^2 \to 0$ as $T \to \infty$ with similar techniques.

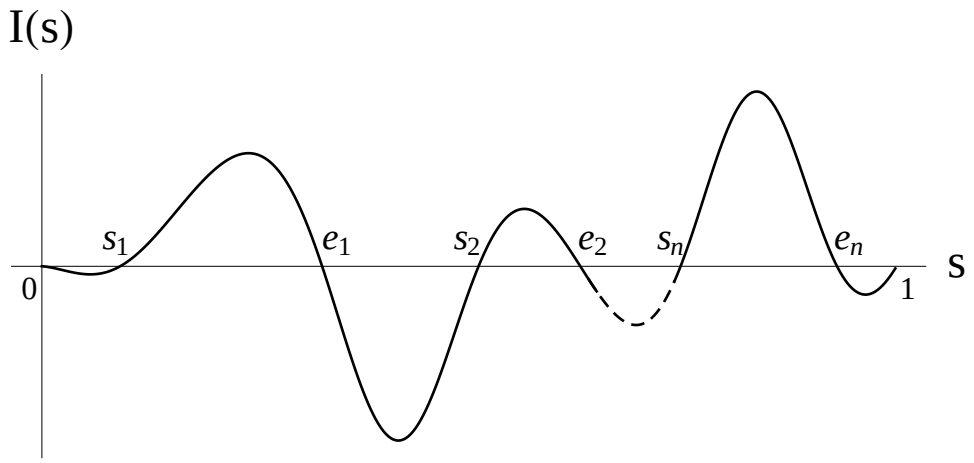


Figure 6: Schematic image of $I(s)$. $I(s)$ is positive for $s_i < s < e_i$ with $i = 1, 2, \ldots, n$.

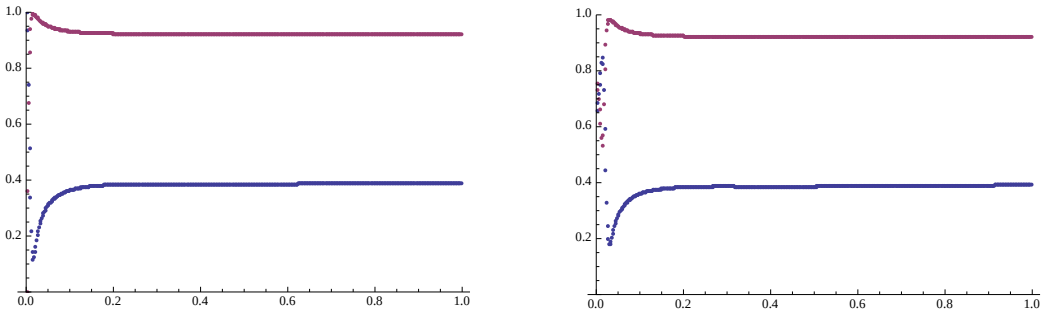
4.4 Compairing our results and the previous results

For our model with the parameters set to $\alpha = 100$, $\omega = 1$, $r_0 = \exp(i2\pi/3)$ and $\hbar = T = 1$, we can have the time evolution of $\alpha_i(t)$ and $\beta_i(t)$ defined by

$$\hat{\mathcal{U}}_t v_j(0) = \alpha_j(t) v_1(t) + \beta_j(t) v_2(t), \quad j = 1, 2. \tag{198}$$

The plot of $\hat{\alpha}_i(t) = \alpha_i(t)/C_i(t)$ and $\hat{\beta}_i(t) = \beta_i(t)/C_i(t)$ with $C_i(t) = \sqrt{\alpha_i(t)^2 + \beta_i(t)^2}$ are indicated in Fig. 7. The figure implies that any initial state evolves to the state having a similar superposition rate. Comparing to Fig. 2, we can find the following:

1. Our results in Fig. 7 are similar to Fig. 2(b). In the both cases, we find $\hat{\beta}_i(T) > \hat{\alpha}_i(T)$. This means that our case in Fig. 7 corresponds to the anti-clockwise case in Gilary *et al.*



(a) (b)

Figure 7: Time evolution of (a) $\hat{\alpha}_1(t)$ and $\hat{\beta}_1(t)$ and (b) $\hat{\alpha}_2(t)$ and $\hat{\beta}_2(t)$. The blue lower curves indicate $\hat{\alpha}_i(t)$ and the upper red curves indicate $\hat{\beta}_i(t)$.

2. In our results, $\hat{\beta}_i(T)$ is less than one and $\hat{\alpha}_i(T)$ is more than zero, in spite of $\hat{\beta}_i(T) \approx 1$ and $\hat{\alpha}_i(T) \approx 0$ in Gilary *et al.*

Based on these points, we find that the second state in our example is flipping, but the flip is far from complete. In other words, a part of the state flips but the rest remains in the initial state.

This incomplete flip is due to the non-adiabatic coupling. In the calculation under the adiabatic approximation, the state initially at a single eigenstate evolves along with the eigenvector. If there is a non-adiabatic coupling, however, a part of the state is transferred to the other eigenstate. Our result is probably due to quicken motion of the parameter λ than in the study by Gilary *et al.*

5 Summary

We have proposed an exactly solvable model of the two-state non-Hermitian quantum system which encircles an exceptional point and have observed the breaking of the adiabatic flip. When the system encircles an exceptional point adiabatically, each of the eigenstates is flipped to the other. In reality, however, it is impossible to vary the parameter slowly enough in experiments. According to the numerical studies by Gilary *et al.* [13–15], the adiabatic flip breaks if the motion of the parameter is non-adiabatic. We have observed the breaking exactly.

The evaluation of the breaking of the adiabatic flip is based on the exact calculation of the time-evolution operator $\hat{\mathcal{U}}_T$ with a time-dependent Hamiltonian $\hat{\mathcal{H}}(t)$. We used the Hashimoto form (119) and the Trotter decomposition (121) for the calculation of the time-ordered product (82). We expect that the Hashimoto form is more useful than the diagonalization for the analysis of the evolution around the exceptional points.

We have an alternative form (126) of $\hat{\mathcal{U}}_T$ with the effective Hamiltonian $\hat{\mathcal{H}}'(t)$ in (125). We then derived an exactly solvable model so that $\hat{\mathcal{H}}'(t)$ may become a lower triangular matrix. This makes the exact calculation of $\hat{\mathcal{U}}_T$ possible.

With our calculation, we exactly find the incompleteness of the flip of the

second state due to the non-adiabatic effect. This is similar to the case of anti-clockwise motion of parameter around an exceptional point in the numerical study given by Gilary *et al.* [13–15]. For example, for $\alpha = 100$, $\omega = 1$, $r_0 = \exp(i2\pi/3)$ and $\hbar = T = 1$ in the model (159), the rates that the initial state is flipped to the other flipping eigenstate is given by $|p|^2 \approx 0.189914$ and $|q|^2 \approx 456.195$. Moreover, in the adiabatic limit $T \to \infty$, we have $|p|^2 \to 1$ and $|q|^2 \to 0$ in our model. We also find that any states evolve to one state after one cycle with incomplete flip. This incompleteness is a result from the non-adiabatic effects.

Acknowledgment

My heartfelt appreciation goes to Assoc. Prof. Hatano whose comments and suggestions were of inestimable value for my study. I also owe a very important debt to the members of the Hatano group in Institute of Industrial Science, the University of Tokyo who provided technical help and sincere encouragement. I would also like to express my gratitude to my family for their moral support and warm encouragements.

Bibliography

[1] J. Dalibard, Y. Castin, and K. Molmer. Wave-function approach to dissipative processes in quantum optics. *Physical Review Letter*, 68:580, 1922.

[2] F. Haake, F. Izrailev, N. Lehmann, D. Saher, and H. Sommers. Statistics of complex levels of random matrices for decaying systems. *Zeitschrift für Physik B Condensed Matter*, 88(3):359–370, 1992.

[3] T. Petrosky and I. Prigogine. *The Liouville Space Extension of Quantum Mechanics*. Advances in Chemical Physics, volume 99, 1997.

[4] S. Klaiman, U. Gunther, and N. Moiseyev. Visualization of branch points in $\mathcal{PT}$-symmetric waveguides. *Physical Review Letter*, 101:080402, 2008.

[5] G. D. Valle and S. Longhi. Spectral and transport properties of time-periodic $\mathcal{PT}$-symmetric tight-binding lattices. *Physical Review A*,

87:022119, 2013.

[6] O. ́Vazquez-Candanedo, J. C. Herńandez-Herrejón, F. M. Izrailev, and D. N. Christodoulides. Gain- or loss-induced localization in one-dimensional $\mathcal{PT}$-symmetric tight-binding models. *Physical Review A*, 89:013832, 2014.

[7] B. Peng, S. K. Ozdemir, F. L. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, C. M. Bender, and L. Yang. Parity–time-symmetric whispering-gallery microcavities. *Nature Physics*, 10:394–398, 2014.

[8] C. M. Bender and S. Boettcher. Real spectra in non-hermitian hamiltonians having $\mathcal{PT}$ symmetry. *Physical Review Letter*, 80:5243–5346, 1998.

[9] C. M. Bender. Introduction to $\mathcal{PT}$-symmetric quantum theory. *Contemporary Physics*, 46:277–292, 2005.

[10] M. V. Berry and R. Uzdin. Slow non-hermitian cycling: exact solutions and the stokes phenomenon. *Journal of Physics A: Mathematical and Theoretical*, 44:435303, 2011.

[11] C. Dembowski, B. Dietz, H.D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, and A. Richter. Encircling an exceptional point. *Physical Review E*, 69(5):056216, 2004.

[12] W.D. Heiss. Phase of wave functions and level repulsion. *The European Physical Journal D*, 7(1):1–4, 1999.

[13] R. Uzdin, A. Maiybaev, and N. Moiseyev. On the observability and asymmetry of adiabatic state flips generated by exceptional points. *Journal of Physics A: Mathematical and Theoretical*, 44:435302, 2011.

[14] I. Gilary and N. Moiseyev. Asymmetric effect of slowly varying chirped laser pulses on the adiabatic state exchange of a molecule. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 45:051002, 2012.

[15] I. Gilary, A. Mailybaev, and N. Moiseyev. Time-asymmetric quantum-state-exchange mechanism. *Physical Review A*, 88:010102(R), 2013.

[16] K. Hashimoto, K. Kanki, H. Hayakawa, and T. Petrosky. Non-divergent representation of non-hermitian operator near the exceptional point with application to a quantum lorentz gas. arXiv: 1409.7453.

[17] T. Kato. *Perturbation Theory of Linear Operators*. Springer, Berlin, 1966.

[18] M. V. Berry and M. Wilkinson. Diabolical points in the spectra of triangles. *Proceedings of the Royal Society A*, 392:15–43, 1984.

[19] J. Harris and R.O. Jones. The surface energy of the bounded electron gas. *Journal of Physics F: Metal Physics*, 4:1170–1186, 1974.

[20] M. V. Berry. Quantal phase factors accompanying adiabatic changes. *Proceedings of the Royal Society A*, 392:45–57, 1984.

[21] A. Altland and B.D. Simons. *Condensed matter field theory.* Cambridge University Press, second edition, 2010.

[22] W. Heiss. Repulsion of resonance states and exceptional points. *Physical Review E*, 61(1):929–932, 2000.

[23] R. Lefebvre". Resonance coalescence in molecular photodissociation. *Physical Review Letters*, 103(12), 2009.