Quantification of multipartite entanglement with the use of LOCC transformation

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Abstract

Entanglement is known to be a promising resource which enables us to execute various quantum tasks. Its quantification has been successful for bipartite pure states. However, the quantification of entanglement for multipartite states has been in a difficult situation. The difficulty is due to the difference of structures between bipartite and multipartite states. In the present thesis, we introduce a new approach for this problem. We give a necessary and sufficient condition of the possibility of a deterministic LOCC transformation of three-qubit pure states. This condition is expressed as a transformation law of six entanglement parameters j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5 and Q_e, where j_{AB}, j_{AC} and j_{BC} are bipartite entanglements, j_{ABC} is a tripartite entanglement, J_5 is a tripartite parameter which means a kind of phase, and Q_e is a new tripartite parameter which means a kind of charge. This fact shows that three-qubit pure states are a partially ordered set parametrized by the six entanglement parameters. The order of the partially ordered set is defined by the possibility of a deterministic LOCC transformation from a state to another state. In this sense, the present condition is an extension of Nielsen's work [14] to three-qubit pure states. We also clarify the rules of transfer and dissipation of entanglement. These rules guarantee that the tripartite entanglement can be transformed into bipartite entanglements, but that the bipartite entanglements cannot be transformed into the tripartite entanglement. This implies that the tripartite entanglement is a higher entity than the bipartite entanglements. With a new combination of the six entanglement parameters, the present condition can be simplified enough to determine easily whether a deterministic LOCC transformation from an arbitrary state $|\psi\rangle$ to another arbitrary state $|\psi'\rangle$ is possible or not. Moreover, the minimum number of times of measurements to reproduce an arbitrary deterministic LOCC transformation is given. This is an extension of Horodecki *et al.*'s work [24] to three-qubit pure states.

Acknowledgements

Chapter 1

Introduction

In recent years, a physical quantity called "entanglement" has become the center of attention in quantum physics. The entanglement is considered as an index of the nonlocality of the system. The entanglement is also known to be a promising resource which enables us to execute various quantum tasks such as quantum computing, teleportation, superdense coding *etc.* [1–4]. Thus, the quantification of the entanglement is a very important subject.

The entanglement describes nonlocal features of the system, and thus the system's response to local operations is important in investigating the entanglement. Hence, the operations called the local operations and classical communications (LOCC) play an important role in the research of the entanglement. The study of LOCC is also important for quantum communication, because any processes of quantum communication is a kind of the LOCC process. The research of the LOCC brought us the quantification of the entanglement for bipartite pure states. Many indices such as the concurrence [5–7] and the negativity [8] have been proposed. Bennett *et al.* [9] has proven that all of them can be expressed by the set of the coefficients of the Schmidt decomposition [10–13]. Based on the properties of this set, the following have been given:

(i) an explicit necessary and sufficient condition to determine whether an arbitrary bipartite state is an entangled state or a separable state;

(ii) an explicit necessary and sufficient condition to determine whether a deterministic LOCC transformation from an arbitrary bipartite pure state to another arbitrary bipartite pure state is executable or not [14];

(iii) an explicit necessary and sufficient condition to determine whether a stochastic LOCC transformation from an arbitrary bipartite pure state to an arbitrary set of bipartite pure states with arbitrary probability is executable or not [15];

(iv) the fact that copies of an arbitrary partially entangled pure state can be distilled to the Bell states by an LOCC transformation, where the ratio between the copies and the Bell states is proportional to the entanglement entropy of the partially entangled state [9];

(v) the fact that copies of an arbitrary partially entangled pure state can be reduced from the Bell states by an LOCC transformation, where the ratio between the copies and

the Bell states is inversely proportional to the entanglement entropy of the partially entangled state [9].

These results suggested the relation among the entanglement, the entropy and information, and thus the research of entanglement became important not only for the field of quantum information, but also for other various fields.

Extension of the above to multipartite states has been vigorously sought, but albeit it is a hard problem. It has been shown that the entanglement of three-qubit pure states is expressed by five parameters [16, 17]. The tangle has been defined [18], which together with the concurrences gives a solution to (i) for three-qubit pure states. The tangle τ_{ABC} has an important property that $C^2_{A(BC)} = C^2_{AB} + C^2_{AC} + \tau_{ABC}$, where C_{AB} is the concurrence between the qubits A and B, C_{AC} is the concurrence between the qubits A and C, and $C_{A(BC)}$ is the concurrence between the qubit A and the set of the qubits B and C. This property implies that the tangle is an index of the tripartite entanglement. The tangle can be expressed by the hyper-determinant [19]. The tangle and the concurrences between the qubits A and B, A and C, and B and C can be expressed by the coefficients of the generalized Schmidt decomposition [20, 21]. These facts imply that for three-qubit states, the concurrences and the tangle play the role which the Schmidt coefficients play for bipartite states. However, the results corresponding to (ii)–(v) have not been provided yet. The reason of the difficulty is as follows: there is a difference between the structure of bipartite pure states and that of multipartite pure states, and thus the approach which was used in quantification of entanglement of bipartite pure states cannot be applied to multipartite.

In the present thesis, we give a new approach for this problem. With this approach, we obtain the following four results. First, a complete solution corresponding to (ii) for three-qubit pure states is given. To be precise, we give an explicit necessary and sufficient condition to determine whether a deterministic LOCC transformation from an arbitrary three-qubit pure state $|\psi\rangle$ to another arbitrary state $|\psi'\rangle$ is executable or not. We express the present condition in terms of the tangle, the concurrence between A and B, the concurrence between A and C, the concurrence between B and C, J_5, which is a kind of phase, and a new parameter Q_e, which means a kind of charge. We thereby clarify the rules of conversion of the entanglement by arbitrary deterministic LOCC transformations. Thus, defining the order between two states $|\psi'\rangle \preccurlyeq |\psi\rangle$ by the existence of an executable deterministic LOCC transformation from $|\psi\rangle$ to $|\psi'\rangle$, we can make the whole set of three-qubit pure states a partially ordered set. To summarize the above, we find that three-qubit pure states are a partially ordered set parametrized by the six entanglement parameters. This is an extension of Nielsen's work [14] to three-qubit pure states.

Second, as we already mentioned above we find a new entanglement parameter Q_e. The new parameter has the following three features:

- . Arbitrary three-qubit pure states are LU-equivalent if and only if their entanglement parameters $(\tau_{ABC}, C_{AB}, C_{AC}, C_{BC}, J_5, Q_e)$ are the same.

- The parameter Q_e has a discrete value: -1, 0 or 1.

- The complex conjugate transformation on $|\psi\rangle$ reverses the sign of Q_e.

Third, we clarify the rules of conversion of the entanglement by deterministic LOCC transformations. We also find that we can interpret the conversion as the transfer and dissipation of the entanglement.

Fourth, we obtain the minimum number of times of measurements to reproduce an arbitrary deterministic LOCC transformation. The minimum number of times depends on the set of the initial and the final states of the deterministic LOCC transformation; we will list up the dependence in Table 3.1 below. We also show that the order of measurements are commutable; we can choose which qubit is measured first, second and third. This is an extension of Horodecki *et al.*'s work [24] to three-qubit pure states.

After completing the present work, we noticed other important results [25–33] which give partial solutions to (ⅱ) for three qubits. In particular, two recent studies [28] and [33] are remarkable. The former [28] gives a necessary and sufficient condition of the possibility of a deterministic LOCC transformation of truly multipartite states whose tensor rank is two; the latter [33] gives a necessary and sufficient condition of the possibility of a deterministic LOCC transformation of W-type states, both using approaches different from the present thesis. However, these studies have not achieved the complete solution to (ⅱ) for three-qubit pure states. Specifically, they cannot determine whether a deterministic LOCC transformation from an arbitrary GHZ-type truly tripartite state to an arbitrary bipartite state is possible or not. Rules of conversion of entanglement have been provided only in implicit forms and explicit forms of the rules are yet to be given.

Let us overview the structure of the present thesis. In section 2.1, we define LOCC protocols. In section 2.2, we review how to perform the quantification of the entanglement for bipartite pure states. In section 2.3, we see what hinders the quantification of the entanglement for multipartite pure states. In section 3.1, we define the parameters which are used in the new approach and examine how an arbitrary measurement changes the parameters. In section 3.2, we present Main Theorems of the present thesis. In section 3.3, we overview the proofs of the Main Theorems. In section 3.4, we prove three Lemmas which are often used in the present thesis. In section 3.5, we prove Main Theorems. In this section, we furthermore obtain rules of the entanglement transfer. The rules appear in sections 3.5, 3.6.1.1, 3.6.1.5 and 3.6.3.2. In section 3.6, we present an explicit protocol of determining whether a deterministic LOCC transformation from an arbitrary three-qubit pure state to an another arbitrary three-qubit pure state is executable or not. In Appendix, we summarize expressions which are often used throughout the present thesis.

Chapter 2

Review of Previous Studies

2.1 LOCC and d-LOCC

The entanglement describes nonlocal features of the system, and thus the system's response to local operations is important in investigating the entanglement. Hence, the operations called local operations and classical communications (LOCC) play an important role in the research of the entanglement. The research of the LOCC is also important for quantum communication, because any processes of quantum communication is a kind of the LOCC process.

Let us show the definition of the LOCC. We consider n sets of a particle and an operator which are separate from each other (Fig.2.1). Each particle has spin $(D-1)/2$, and thus an operation on the total system can be described as a linear transformation of a D^n-dimensional linear space. Each operator can perform the following three kinds of operations:

- Local unitary operations on his/her own particle,

- Local measurements on his/her own particle,

- Telling to other operators what kinds of local unitary operators and local measurements he/she performed on his/her own particle as well as the results of the measurements.

Each operator can*not* perform global unitary transformations nor global measurements.

Let us present an example of the LOCC process (Fig.2.2). For simplicity, we consider a two-qubit system.

Step 1 The operator A performs a measurement $\{M_{(i)}|i=0,1\}$ on the qubit A. If the result of the measurement is $i=0$, proceed to Step 2-0. If the result is $i=1$, proceed to Step 2-1.

Step 2-0 The operator A performs a unitary transformation U_0 on the qubit A. The operator B performs a measurement $\{N_{(i)}|i=0,1\}$ on the qubit B. If the result of the measurement $\{N_{(i)}|i=0,1\}$ is $i=0$, proceed to Step 3-0. If the result is $i=1$, proceed to Step 3-1.

Step 2-1 The operator B performs a unitary transformation U_1 on the qubit B.
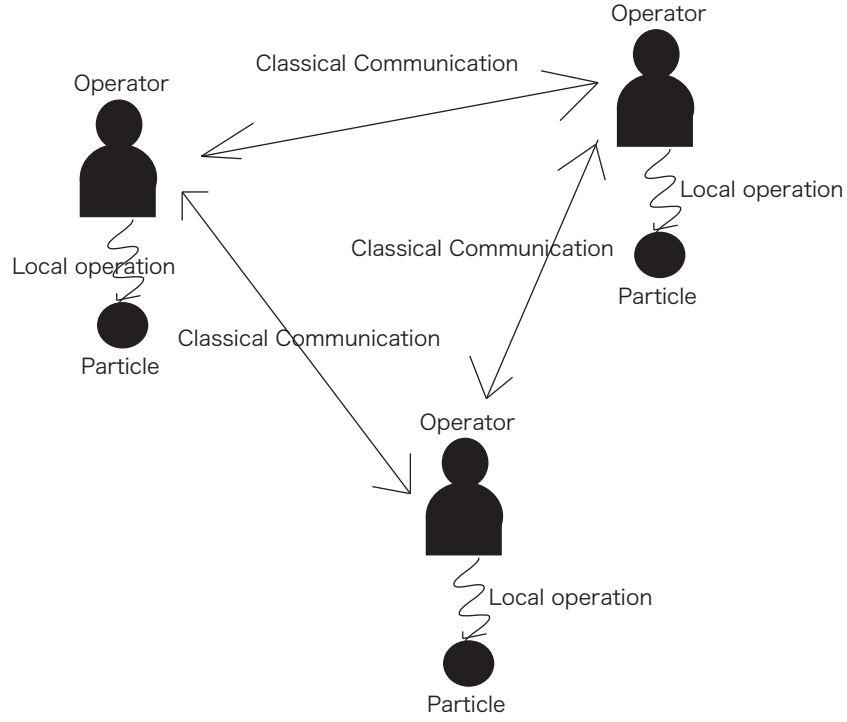
Figure 2.1: A schematic diagram of the LOCC. In this picture, the number of the sets is three.
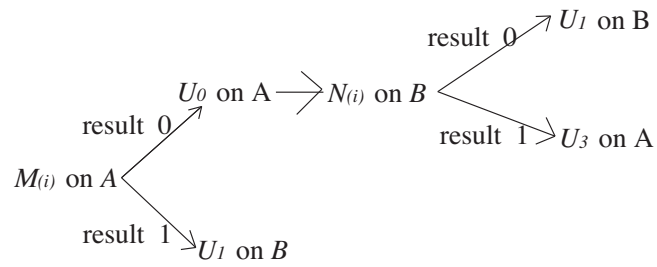

Figure 2.2: A schematic diagram of an example LOCC.

Step 3-0 The operator B performs a unitary transformation U_2 on the qubit B.

Step 3-1 The operator A performs a unitary transformation U_3 on the qubit A.

As in the above example, an LOCC protocol can have branches. The final states of the branches are generally different from each other. However, there are LOCC protocols where the final states of all the branches are the same. Such an LOCC protocol is called a deterministic LOCC (d-LOCC). A necessary and sufficient condition of the possibility of a deterministic LOCC transformation was given by Nielsen [14] for bipartite pure states. The quantification of the bipartite entanglement is performed by using this condition [4]. We review the quantification in the next section.

2.2 Quantification of bipartite entanglement

The present Section is composed of four subsections. In the first subsection, we review the theorem called the quantification of the bipartite entanglement and show the points of the theorem. In the second subsection, we introduce some knowledge of classical information theory which is necessary for the proof of the theorem. In the third subsection, we give a necessary and sufficient condition of the possibility of a d-LOCC of bipartite pure states. In the fourth subsection, we present the proof of the theorem by using the results of the second and third subsections. The proofs in the present Section are based on the proofs in Ref. [4].

2.2.1 Theorem

Theorem 1$_{\mathbf{pre}}$ *Let the notation $|\psi_{AB}\rangle$ and $|\phi_{AB}\rangle$ stand for pure states of two $(D-1)/2$-spin particles. The reduced density matrices $\rho_{\psi A}$ and $\rho_{\phi A}$ are defined as*

$$\rho_{\psi A} \equiv tr_B\{|\psi_{AB}\rangle\langle\psi_{AB}|\}, \quad \rho_{\phi A} \equiv tr_B\{|\phi_{AB}\rangle\langle\phi_{AB}|\}. \tag{2.1}$$

Then the following d-LOCC transformation is executable in the limit of $N, M \to \infty$ with $N/M = S_{\rho_{\phi A}}/S_{\rho_{\psi A}}$:

$$d - LOCC: \ |\psi_{AB}\rangle^{\otimes N} \leftrightarrow |\phi_{AB}\rangle^{\otimes M}, \tag{2.2}$$

where $S_{\rho_{\phi A}}$ and $S_{\rho_{\psi A}}$ are the von Neumann entropies of $\rho_{\phi A}$ and $\rho_{\psi A}$, respectively.

This theorem has the two significant points:

Entanglement quantification: The entanglement of a bipartite pure state can be measured by the von Neumann entropy of the reduced density matrix of one particle. Under the measure, we can treat the entanglement as an extensive quantity; we can gather the entanglement from many states to a few states or distribute the entanglement to many states from a few states, without loss of the total entanglement (Fig.2.3). Note that the dimensionality D is arbitrary. In fact, this equivalence between the entanglement and the entropy suggests the following two statements:

- An arbitrary system's entropy is equal to the entanglement between the system and the environment of the system. This statement follows from the fact that we can regard the system as one particle and the environment as the other particle.
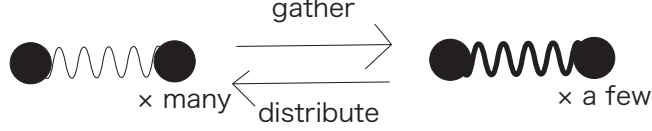
Figure 2.3: A schematic diagram of gathering and distributing of entanglement.

- The entanglement is quantification of the information. This statement follows from the fact that we can regard one particle as the memory of the other particle.

Restoration of entanglement Theorem 1_{pre} also gives a method of making a few maximally entangled states from many non-maximal states. This contributes much to the field of quantum communication. In a quantum communication protocol, we often send out one particle of an entangled pair and the entanglement of the pair is often damaged by noise. The method enables us to restore the damaged entanglement from the other particle.

2.2.2 Theorem of typical sequences

In the present subsection, we introduce a theorem of classical information theory called the theorem of typical sequences. This theorem is necessary for the proof of Theorem 1_{pre}. First, we define the ϵ-typical sequences.

ϵ-**typical sequence:** Consider a sequence of random variables $\{X_1, ..., X_n\}$. Each random variable is equal to i with probability p_i, where i is a natural number from 1 to d. The random variables are independent of each other, and thus the following equation holds:

$$p(\{X_1, ..., X_n\} = \{x_1, ..., x_n\}) = p(X_1 = x_1)p(X_2 = x_2)...p(X_n = x_n). \qquad (2.3)$$

Then, if a sequence of natural numbers $\{x_1, ..., x_n\}$ satisfies the following inequalities, we refer to the sequence $x_1, ..., x_n$ as an ϵ-typical sequence:

$$e^{-n(H\{p_i\}+\epsilon)} \leq p(\{X_1, ..., X_n\} = \{x_1, ..., x_n\}) \leq e^{-n(H\{p_i\}-\epsilon)}, \qquad (2.4)$$

where $H\{p_i\} \equiv -\sum_{i=1}^{d} p_i \log p_i$.

The following theorem, which is called the theorem of typical sequences, holds for the ϵ-typical sequences.

Theorem 2_{pre} (i) *Take two arbitrary real numbers $\epsilon > 0$ and $0 < \delta \leq 1$. Then, there is a natural number N for which*

$$n > N \Rightarrow P_\epsilon(\{x_1, ..., x_n\}) \geq 1 - \delta \qquad (2.5)$$

holds, where $P_\epsilon(\{x_1, ..., x_n\})$ is the probability that the sequence $\{x_1, ..., x_n\}$ is an ϵ-typical sequence.

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(ii) *Take two arbitrary real numbers $\epsilon > 0$ and $0 < \delta \le 1$. Then,*

$$(1 - \delta)e^{n(H\{p_i\}-\epsilon)} \le |T(n, \epsilon)| \le e^{n(H\{p_i\}+\epsilon)} \tag{2.6}$$

holds for $n > N$, where $T(n, \epsilon)$ is the set of the ϵ-typical sequences $\{x_1, ..., x_n\}$ and $|T(n, \epsilon)|$ is the number of elements of $T(n, \epsilon)$.

Proof

First we prove (i). Because of the law of large numbers,

$$\lim_{n\to\infty} \text{Prob}\left(\left|\frac{\sum_{i=1}^{n} \log p(X_i)}{n} - \langle \log p(X_1) \rangle\right| > \epsilon\right) = 0 \tag{2.7}$$

holds, where $\langle ... \rangle$ denotes the average with respect to the random variables. Because of $\langle \log p(X_1) \rangle = H\{p_i\}$ and $\prod_{i=1}^{n} p(X_i) = p(X_1, ..., X_n)$, we can reduce the equation (2.7) to

$$\lim_{n\to\infty} \text{Prob}\left(\left|\frac{\log p(X_1, ..., X_n)}{n} - H\{p_i\}\right| > \epsilon\right) = 0. \tag{2.8}$$

Thus, for arbitrary $\epsilon > 0$ and $0 < \delta \le 1$, there is a natural number N which satisfies

$$n > N \Rightarrow \text{Prob}\left(\left|\frac{\log p(X_1, ..., X_n)}{n} - H\{p_i\}\right| > \epsilon\right) < \delta. \tag{2.9}$$

Note that

$$P_\epsilon(\{x_1, ..., x_n\}) = 1 - \text{Prob}\left(\left|\frac{\log p(X_1 = x_1, ..., X_n = x_n)}{n} - H\{p_i\}\right| > \epsilon\right). \tag{2.10}$$

Thus, Eq. (2.9) is equivalent to (2.5).

Second, we prove (ii). Because of (2.5),

$$1 - \delta \le P_\epsilon(\{x_1, ..., x_n\}) = \sum_{\{x_1,...,x_n\}\in T(n,\epsilon)} p(X_1 = x_1, ..., X_n = x_n) \le 1 \tag{2.11}$$

holds for $n > N$. Because of the definition of ϵ-typical sequences,

$$|T(n, \epsilon)|e^{-n(H\{p_i\}+\epsilon)} \le \sum_{\{x_1,...,x_n\}\in T(n,\epsilon)} p(X_1 = x_1, ..., X_n = x_n) \le |T(n, \epsilon)|e^{-n(H\{p_i\}-\epsilon)}$$

$$\tag{2.12}$$

holds for $n > N$. Because of (2.11) and (2.12), the inequalities (2.6) holds.□

2.2.3 A necessary and sufficient condition of the possibility of d-LOCC of bipartite pure states

In the present subsection, we give a necessary and sufficient condition of the possibility of d-LOCC of bipartite pure states. First, we prove three lemmas which are necessary to obtain the condition.

Lemma 1pre *For an arbitrary matrix G, there is a unitary matrix U which satisfies*

$$G = U\sqrt{G^\dagger G} = \sqrt{GG^\dagger}U. \tag{2.13}$$

Proof

We prove this lemma by writing down the matrix U. Because the matrix $\sqrt{G^\dagger G}$ is a positive matrix, we can diagonalize the matrix $\sqrt{G^\dagger G}$ to obtain nonnegative eigenvalues. Let the notations $\{\lambda_i | i = 1, ..., n\}$ and $\{|i\rangle | i = 1, ..., n\}$ stand for the nonnegative eigenvalues and the corresponding eigenvectors of $\sqrt{G^\dagger G}$, respectively. Without losing generality, we can assume that $\{\lambda_i\}$ is in a non-increasing order. Let us assume that the eigenvalues for $i \leq n_+$ are positive and the rest are zero. If all of $\{\lambda_i\}$ are positive, the number n_+ is equal to n. We then give the matrix U in terms of G, $\{\lambda_i | i = 1, ..., n\}$ and $\{|i\rangle | i = 1, ..., n\}$ as :

$$U = \sum_{i=1}^{n_+} \frac{1}{\lambda_i} G |i\rangle \langle i| + \sum_{i=n_++1}^{n} |i\rangle \langle i| = \sum_{i=1}^{n} |e_i\rangle \langle i|, \tag{2.14}$$

where

$$|e_i\rangle \equiv \begin{cases} \frac{1}{\lambda_i} G |i\rangle & (i \leq n_+), \\ |i\rangle & (i \geq n_+ + 1). \end{cases} \tag{2.15}$$

The set of the vectors $\{|e_i\rangle\}$ constitutes a complete orthonormal basis. The orthonormality is explicitly shown as

$$\langle e_j | e_i \rangle = \begin{cases} \langle j| G^\dagger \frac{1}{\lambda_j} \frac{1}{\lambda_i} G |i\rangle = \delta_{ij} & (1 \leq i, j \leq n_+), \\ \langle j| \frac{1}{\lambda_i} G |i\rangle = 0 & (1 \leq i \leq n_+, 1 \leq j \leq n_+ + 1), \\ \langle j| G^\dagger \frac{1}{\lambda_j} |i\rangle = 0 & (n_+ + 1 \leq i \leq n, 1 \leq j \leq n_+), \\ \langle j|i\rangle = \delta_{ij} & (n_+ + 1 \leq i, j \leq n), \end{cases} \tag{2.16}$$

where for $i \geq n_+ + 1$, the equations $G |i\rangle = G^\dagger |i\rangle = 0$ hold because $\langle i| G^\dagger G |i\rangle = \langle i| GG^\dagger |i\rangle = 0$. Since the vectors exhaust the n-dimensional space, the set of $\{|e_i\rangle\}$ is also complete. Then, the matrix U is unitary because

$$UU^\dagger = \sum_{i=1}^{n} \sum_{j=1}^{n} |e_i\rangle \langle i|j\rangle \langle e_j| = \sum_{i=1}^{n} |e_i\rangle \langle e_i| = I, \tag{2.17}$$

$$U^\dagger U = \sum_{i=1}^{n} \sum_{j=1}^{n} |i\rangle \langle e_i|e_j\rangle \langle j| = \sum_{i=1}^{n} |i\rangle \langle i| = I. \tag{2.18}$$

The unitary matrix U then satisfies (2.13) because

$$
\begin{aligned}
U\sqrt{G^\dagger G} &= \sum_{i=1}^{n}\sum_{j=1}^{n} |e_i\rangle\langle i|\lambda_j|j\rangle\langle j| = \sum_{i=1}^{n_+}\sum_{j=1}^{n_+} \frac{1}{\lambda_i} G|i\rangle\langle i|\lambda_j|j\rangle\langle j| \\
&= \sum_{i=1}^{n_+} G|i\rangle\langle i| = \sum_{i=1}^{n} G|i\rangle\langle i| = G, \qquad\qquad (2.19) \\
GU^\dagger GU^\dagger &= \sum_{i=1}^{n}\sum_{j=1}^{n} G|i\rangle\langle e_i|G|j\rangle\langle e_j| = \sum_{i=1}^{n_+}\sum_{j=1}^{n_+} G|i\rangle\langle e_i|G|j\rangle\langle e_j| \\
&= \sum_{i=1}^{n_+}\sum_{j=1}^{n_+} G|i\rangle\langle i|G^\dagger\frac{1}{\lambda_i} G|j\rangle\langle j|G^\dagger\frac{1}{\lambda_j} \\
&= \sum_{i=1}^{n_+} G|i\rangle\langle i|G^\dagger = \sum_{i=1}^{n} G|i\rangle\langle i|G^\dagger = GG^\dagger, \qquad (2.20) \\
GU^\dagger &= \sqrt{GG^\dagger}, \qquad\qquad\qquad\qquad\qquad\qquad (2.21)
\end{aligned}
$$

where we used the fact $G|i\rangle = G^\dagger|i\rangle = 0$ for $i \geq n_+ + 1$. $\square$

Lemma 2$_{pre}$ *For an arbitrary matrix G, there are two unitary matrices U and V which satisfy*

$$
G = UDV, \qquad\qquad (2.22)
$$

where D is a diagonal matrix whose elements are nonnegative. This is called the singular-value decomposition.

Proof

We prove this lemma by writing down the matrices U, V and D. Because the matrix $\sqrt{G^\dagger G}$ is a positive matrix, we can diagonalize the matrix $\sqrt{G^\dagger G}$ to obtain nonnegative eigenvalues. Let the notations $\{\lambda_i | i = 1, ..., n\}$ and $\{|i\rangle | i = 1, ..., n\}$ stand for the non-negative eigenvalues and the corresponding eigenvectors of $\sqrt{G^\dagger G}$, respectively. There is a unitary matrix U_0 which diagonalizes the matrix $\sqrt{G^\dagger G}$ as

$$
U_0\sqrt{G^\dagger G}U_0^\dagger = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}. \qquad (2.23)
$$

We then give the matrices U, V and D in terms of G, $\{\lambda_i | i = 1, ..., n\}$, $\{|i\rangle | i = 1, ..., n\}$ and U_0:

$$
U = \sum_{i=1}^{n} |e_i\rangle\langle i|U_0^\dagger, \qquad\qquad (2.24)
$$

$$
V = U_0, \qquad\qquad (2.25)
$$

$$
D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \qquad\qquad (2.26)
$$

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where $\{|e_i\rangle\}$ are defined in (2.15). We can reduce (2.24)–(2.26) to (2.22) as follows:

$$
\begin{aligned}
UDV &= \sum_{i=1}^{n} |e_i\rangle \langle i| U_0^\dagger \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U_0 \\
&= \sum_{i=1}^{n} |e_i\rangle \langle i| \sqrt{G^\dagger G} \\
&= \sum_{i=1}^{n} \frac{1}{\lambda_i} G |i\rangle \left(\langle i| \sum_{j=1}^{n} \lambda_j |j\rangle \right) \langle j| = \sum_{i=1}^{n} \frac{1}{\lambda_i} G |i\rangle \langle i| = G.
\end{aligned}
\tag{2.27}
$$

$\square$

Lemma 2$_{\text{pre}}$ guarantees that for an arbitrary bipartite pure state there is a unitary transformation U_A on the particle A and a unitary transformation U_B on the particle B which satisfy the following equation:

$$
U_A \otimes U_B |\psi_{AB}\rangle = \sum_{i=1}^{n} \lambda_i |i_A i_B\rangle,
\tag{2.28}
$$

where $|i_A i_B\rangle$ is an abbreviation of $|i_A\rangle \otimes |i_B\rangle$ with $|i_A\rangle$ and $|i_B\rangle$ the eigenstates of the particle A and B, respectively. This is given by Lemma 2$_{\text{pre}}$ and the following equation:

$$
U_A \otimes U_B |\psi_{AB}\rangle = \sum_{i,j} \sum_{k,l} u_{ik}^{(A)} \lambda_{kl} u_{lj}^{(B)} |i_A j_B\rangle,
\tag{2.29}
$$

where

$$
u_{ij}^{(A)} = \langle i_A| U_A |j_A\rangle,
\tag{2.30}
$$

$$
u_{ij}^{(B)} = \langle i_B| U_B |j_B\rangle,
\tag{2.31}
$$

$$
\lambda_{ij} = \langle i_A j_B|\psi_{AB}\rangle.
\tag{2.32}
$$

The decomposition (2.28) is achieved when $u_{ij}^{(A)}$ and $u_{ij}^{(B)}$ are the matrices which transform the matrix λ_{ij} in the singular-value decomposition. The decomposition (2.28) is called the Schmidt decomposition. The coefficients $\{\lambda_i\}$ are called the Schmidt coefficients.

The next lemma performs ordering of Hermitian matrices with majorization theory. Before showing the lemma, we introduce the majorization of vectors and matrices.

Majorization of normalized vectors : Let the notations $\vec{x} = (x_1, ..., x_n)$ and $\vec{y} = (y_1, ..., y_n)$ stand for vectors which satisfy $x_i \geq 0$, $y_i \geq 0$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1$. Let the notation $\vec{x}^\downarrow$ stand for the vector whose components are the same as those of $\vec{x}$ but ordered in the non-increasing order. We say that $\vec{x}$ is majorized by $\vec{y}$ if and only if $\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i$ for all $k = 1, ..., n$.

Majorization of normalized Hermitian matrices : Let the notation H and K stand for Hermitian matrices whose eigenvalues are $\vec{\lambda}^{(H)} = (\lambda_1^{(H)}, ..., \lambda_n^{(H)})$ and $\vec{\lambda}^{(K)} = (\lambda_1^{(K)}, ..., \lambda_n^{(K)})$. We assume that the matrices H and K are normalized so that $\sum_{i=1}^{n} \lambda_i^{(H)} = \sum_{i=1}^{n} \lambda_i^{(K)} = 1$. We say that H is majorized by K if and only if $\vec{\lambda}^{(H)}$ is majorized by $\vec{\lambda}^{(K)}$.

Hereafter, we refer to a Hermitian matrix whose eigenvalues add up to unity as a normalized Hermitian matrix.

Lemma 3$_{\mathbf{pre}}$ *Fix two arbitrary normalized Hermitian matrices H and K. Then H is majorized by K if and only if there are a probability distribution $\{p_\mu | \mu = 1, ..., d\}$ and a set of unitary matrices $\{U_\mu | \mu = 1, ..., d\}$ which satisfy*

$$H = \sum_{\mu=1}^{d} p_\mu U_\mu K U_\mu^\dagger. \tag{2.33}$$

Proof

We prove this lemma with using the following statements:

Statement 1 A normalized vector $\vec{x}$ is majorized by a normalized vector $\vec{y}$ if and only if there are a probability distribution $\{p_\mu | \mu = 1, ..., d\}$ and a set of permutation matrices $\{P_\mu | \mu = 1, ..., d\}$ which satisfy

$$\vec{x} = \sum_{\mu=1}^{d} p_\mu P_\mu \vec{y}. \tag{2.34}$$

Statement 2 (Birkhoff's Theorem) We refer to an n-by-n matrix G whose elements G_{ij} are all nonnegative and satisfy $\sum_{i=1}^{n} G_{ij} = 1$ for any j and $\sum_{j=1}^{n} G_{ij} = 1$ for any i as a doubly stochastic matrix. A matrix G is doubly stochastic if and only if there are a probability distribution $\{p_\mu | \mu = 1, ..., d\}$ and a set of permutation matrices $\{P_\mu | \mu = 1, ..., d\}$ which satisfy

$$G = \sum_{\mu=1}^{d} p_\mu P_\mu. \tag{2.35}$$

We do not prove Birkhoff's Theorem here for brevity.

By using Statements 1 and 2, we can prove Lemma 3$_{\mathrm{pre}}$. First, we prove that H is majorized by K if there are a probability distribution $\{p_\mu | \mu = 1, ..., d\}$ and a set of unitary matrices $\{U_\mu | \mu = 1, ..., d\}$ which satisfy (2.33). Because H and K are Hermitian, there are two unitary matrices U_H and U_K which satisfy

$$\Lambda(H) \equiv \begin{pmatrix} \lambda_1^{(H)} & & \\ & \ddots & \\ & & \lambda_n^{(H)} \end{pmatrix} = U_H H U_H^\dagger, \tag{2.36}$$

$$\Lambda(K) \equiv \begin{pmatrix} \lambda_1^{(K)} & & \\ & \ddots & \\ & & \lambda_n^{(K)} \end{pmatrix} = U_K K U_K^\dagger. \tag{2.37}$$

Because of (2.36) and (2.37), we reduce (2.33) to the following:

$$\Lambda(H) = \sum_{\mu=1}^{d} p_\mu U_H U_\mu U_K^\dagger \Lambda(K) U_K U_\mu^\dagger U_H^\dagger \tag{2.38}$$

$$\lambda_i^{(H)} = \sum_{\mu=1}^{d} \sum_{j=1}^{n} p_\mu |(U_H U_\mu U_K^\dagger)_{ij}|^2 \lambda_j^{(K)}. \tag{2.39}$$

Because the matrix $U_H U_\mu U_K^\dagger$ is unitary, the equations $\sum_{i=1}^n |(U_H^\dagger U_\mu U_K^\dagger)_{ij}|^2 = 1$ for any j and $\sum_{j=1}^n |(U_H U_\mu U_K^\dagger)_{ij}|^2 = 1$ for any i hold. Thus, a matrix G_μ which satisfies

$$(G_\mu)_{ij} = |(U_H U_\mu U_K^\dagger)_{ij}|^2 \tag{2.40}$$

is a doubly stochastic matrix. Because of Birkhoff's Theorem (Statement 2), there are a probability distribution $\{p_\nu^{(\mu)}|\nu = 1, ..., d^{(\mu)}\}$ and a set of permutation matrices $\{P_\nu^{(\mu)}|\nu = 1, ..., d^{(\mu)}\}$ which satisfy

$$G_\mu = \sum_{\nu=1}^{d^{(\mu)}} p_\nu^{(\mu)} P_\nu^{(\mu)} \tag{2.41}$$

for each μ. Because of (2.41), we reduce (2.39) as follows:

$$\vec{\lambda}(H) = \sum_{\mu=1}^d \sum_{\nu=1}^{d^{(\mu)}} p_\mu p_\nu^{(\mu)} P_\nu^{(\mu)} \vec{\lambda}(K), \tag{2.42}$$

where $\vec{\lambda}(H) \equiv (\lambda_1^{(H)}, ..., \lambda_n^{(H)})$ and $\vec{\lambda}(K) \equiv (\lambda_1^{(K)}, ..., \lambda_n^{(K)})$. Note that $\{p_\mu p_\nu^{(\mu)}\}$ and $\{P_\nu^{(\mu)}\}$ are a probability distribution and a set of permutation matrices, respectively. Because of Statement 1 and (2.42), therefore, the matrix H is majorized by the matrix K if there are a probability distribution $\{p_\mu|\mu = 1, ..., d\}$ and a set of unitary matrices $\{U_\mu|\mu = 1, ..., d\}$ which satisfy (2.33). This proves the necessity of Lemma 3$_{\text{pre}}$.

Second, we prove the sufficiency, in other words, that if the matrix H is majorized by the matrix K, then there are a probability distribution $\{p_\mu|\mu = 1, ..., d\}$ and a set of unitary matrices $\{U_\mu|\mu = 1, ..., d\}$ which satisfy (2.33). When the matrix H is majorized by the matrix K, Statement 1 tells us that there are a probability distribution $\{p_\mu|\mu = 1, ..., d\}$ and a set of permutation matrices $\{P_\mu|\mu = 1, ..., d\}$ which satisfy

$$\vec{\lambda}(H) = \sum_{\mu=1}^d p_\mu P_\mu \vec{\lambda}(K). \tag{2.43}$$

Because of (2.36), (2.37) and (2.43), we obtain (2.33) as follows:

$$\Lambda(H) = \sum_{\mu=1}^d p_\mu P_\mu \Lambda(K) P_\mu^\dagger, \tag{2.44}$$

$$H = \sum_{\mu=1}^d p_\mu U_H^\dagger P_\mu U_K K U_K^\dagger P_\mu^\dagger U_H. \tag{2.45}$$

Note that the matrix $U_H^\dagger P_\mu U_K$ is unitary. Therefore, the matrices $\{U_H^\dagger P_\mu U_K\}$ can be regarded as the set of matrices $\{U_\mu\}$ which we want. Hence, if the matrix H is majorized by the matrix K, then there are a probability distribution $\{p_\mu|\mu = 1, ..., d\}$ and a set of unitary matrices $\{U_\mu|\mu = 1, ..., d\}$ which satisfy (2.33). Now we have completed the proof of Lemma 3$_{\text{pre}}$ with the use of Statements 1 and 2.

Thus, we only have to prove Statement 1 in order to prove Lemma 3$_{\text{pre}}$. At first, we prove the necessity; in other words, we prove that $\vec{x}$ is majorized by $\vec{y}$ if there are a

probability distribution $\{p_i | i = 1, ..., d\}$ and a set of permutation matrices $\{P_i | i = 1, ..., d\}$ which satisfy (2.34). For the purpose, we only have to note that the vector $P_i \vec{y}$ is majorized by $\vec{y}$ and that the set of vectors which are majorized by $\vec{y}$ is convex.

Next, we prove the sufficiency: in other words, we prove that if $\vec{x}$ is majorized by $\vec{y}$, there are a probability distribution $\{p_i | i = 1, ..., d\}$ and a set of permutation matrices $\{P_i | i = 1, ..., d\}$ which satisfy (2.34). We perform the proof by mathematical induction with respect to n, which is the dimensionality of $\vec{x}$. Statement 1 clearly holds for $n = 1$. We prove Statement 1 for $n = k + 1$, assuming that the statement 1 is proved whenever $1 \leq n \leq k$. Let us take a natural number n_0 which satisfies

$$y_{n_0-1}^{\downarrow} \geq x_1^{\downarrow} \geq y_{n_0}^{\downarrow}. \tag{2.46}$$

Then, we can take a real number $0 \leq t \leq 1$ which satisfies

$$x_1^{\downarrow} = t y_{n_0}^{\downarrow} + (1 - t) y_1^{\downarrow}. \tag{2.47}$$

Hence, we can take a permutation matrix P_t which permutes y_1 and y_{n_0}:

$$
\begin{aligned}
t P_t \vec{y}^{\downarrow} + (1 - t) I \vec{y}^{\downarrow} &= (t y_{n_0}^{\downarrow} + (1 - t) y_1^{\downarrow}, y_2^{\downarrow}, ..., y_{n_0-1}^{\downarrow}, (1 - t) y_{n_0}^{\downarrow} + t y_1^{\downarrow}, y_{n_0+1}^{\downarrow}, ..., y_n^{\downarrow}) \\
&= (x_1^{\downarrow}, y_2^{\downarrow}, ..., y_{n_0-1}^{\downarrow}, (1 - t) y_{n_0}^{\downarrow} + t y_1^{\downarrow}, y_{n_0+1}^{\downarrow}, ..., y_n^{\downarrow}). \tag{2.48}
\end{aligned}
$$

Note that the vector $(x_2^{\downarrow}, ..., x_n^{\downarrow})$ is majorized by $(y_2^{\downarrow}, ..., (1 - t) y_{n_0}^{\downarrow} + t y_1^{\downarrow}, ..., y_n^{\downarrow})$. Thus, because of the assumption for $n = k$, there are a probability distribution $\{p_i' | i = 1, ..., d\}$ and a set of permutation matrices $\{P_i' | i = 1, ..., d\}$ which satisfy

$$\vec{x}^{\downarrow} = \sum_{i=1}^{d} p_i' P_i' (t P_t \vec{y}^{\downarrow} + (1 - t) I \vec{y}^{\downarrow}). \tag{2.49}$$

Hence, $\{t p_1', ..., t p_d', (1-t) p_1', ..., (1-t) p_d'\}$ and $\{P_t P_1', ..., P_t P_d', P_1', ..., P_d'\}$ are the probability distribution and the set of permutation matrices which we want, respectively. □

By using Lemmas 1$_{\text{pre}}$ and 2$_{\text{pre}}$, we now obtain a necessary and sufficient condition of the possibility of d-LOCC of bipartite pure states.

Theorem 3$_{\text{pre}}$ *Let the notations $|\psi_{AB}\rangle$ and $|\psi_{AB}'\rangle$ stand for bipartite pure states. We refer to the reduced density operators of the particle A of $|\psi_{AB}\rangle$ and $|\psi_{AB}'\rangle$ as ρ_A and ρ_A', respectively. Then, there is an executable d-LOCC transformation from $|\psi_{AB}\rangle$ to $|\psi_{AB}'\rangle$ if and only if ρ_A is majorized by ρ_A'.*

Proof

First, we prove the necessity; in other words, we prove that there is an executable d-LOCC transformation from $|\psi_{AB}\rangle$ to $|\psi_{AB}'\rangle$ if ρ_A is majorized by ρ_A'. For the purpose, we only have to obtain a measurement $\{M_{(i)}\}$ which satisfies

$$\rho_A' = \sum_i M_{(i)} \rho_A M_{(i)}^{\dagger}. \tag{2.50}$$

When ρ_A is majorized by ρ_A', there are a probability distribution $\{p_i | i = 1, ..., d\}$ and a set of unitary matrices $\{U_i | i = 1, ..., d\}$ which satisfy

$$\rho_A = \sum_{i=1}^{d} p_i U_i \rho_A' U_i^{\dagger}. \tag{2.51}$$

From (2.51), we obtain the set of matrices $\{M_{(i)}|i = 1, ..., d\}$ as

$$M_{(i)}\sqrt{\rho_A} = \sqrt{p_i \rho'_A} U_i^{\dagger}. \tag{2.52}$$

If ρ_A is a regular matrix, the set of matrices $\{M_{(i)}|i = 1, ..., d\}$ is determined uniquely. Because of (2.51) and (2.52), the set $\{M_{(i)}|i = 1, ..., d\}$ satisfies $\sum_i M_{(i)}^{\dagger} M_{(i)} = I$, and thus the set $\{M_{(i)}|i = 1, ..., d\}$ is a measurement. If ρ_A is not regular, the set $\{M_{(i)}|i = 1, ..., d\}$ is not unique. Then we choose a set $\{M_{(i)}|i = 1, ..., d\}$ which satisfies $\sum_i M_{(i)}^{\dagger} M_{(i)} = I$. Because of (2.52), the measurement $\{M_{(i)}|i = 1, ..., d\}$ satisfies (2.50).

Second, we prove the sufficiency; in other words, we prove that if there is an executable d-LOCC transformation from $|\psi_{AB}\rangle$ to $|\psi'_{AB}\rangle$, then ρ_A is majorized by ρ'_A. First, we prove that if there is a measurement $\{M_{(i)}|i = 1, ..., d\}$ which satisfies

$$p_i \rho'_A = M_{(i)} \rho_A M_{(i)}^{\dagger}, \tag{2.53}$$

where

$$p_i \equiv \mathrm{tr}[M_{(i)}^{\dagger} M_{(i)}], \tag{2.54}$$

ρ_A is majorized by ρ'_A. When there is a measurement $\{M_{(i)}|i = 1, ..., d\}$ which satisfies (2.53), the matrices $\{U_i|i = 1, ..., d\}$ which satisfy

$$M_{(i)}\sqrt{\rho_A} = \sqrt{M_{(i)}\rho_A M_{(i)}^{\dagger}} U_i \tag{2.55}$$

also satisfy

$$\rho_A = \sum_{i=1}^{d} p_i U_i^{\dagger} \rho'_A U_i. \tag{2.56}$$

The existence of the matrices $\{U_i\}$ is guaranteed by Lemma 1$_{\text{pre}}$. Indeed, $\{U_i\}$ satisfy (2.56) as follows:

$$\sum_{i=1}^{d} p_i U_i^{\dagger} \rho'_A U_i = \sum_{i=1}^{d} U_i^{\dagger} M_{(i)} \rho_A M_{(i)}^{\dagger} U_i \tag{2.57}$$

$$= \sum_{i=1}^{d} U_i^{\dagger} U_i \sqrt{\rho_A} M_{(i)}^{\dagger} M_{(i)} \sqrt{\rho_A} U_i^{\dagger} U_i = \rho_A. \tag{2.58}$$

Hence, if there is a measurement $\{M_{(i)}|i = 1, ..., d\}$ which satisfies (2.53), then ρ_A is majorized by ρ'_A.

Finally, we prove that if there is an executable d-LOCC transformation from $|\psi_{AB}\rangle$ to $|\psi'_{AB}\rangle$, then there is a measurement $\{M_{(i)}|i = 1, ..., d\}$ which satisfies (2.53). For the purpose, we only have to prove that an arbitrary measurement on the particle B can be reproduced by a measurement on the particle A and local unitary transformations. Consider the situation in which we perform a measurement $\{N_i\}$ on the particle B of a bipartite pure state $|\psi_{AB}\rangle$. With the Schmidt basis of the particle B, the measurement N_i is expressed as follows:

$$N_i = \sum_{j,k} (N_i)_{j,k} |j_B\rangle \langle k_B|. \tag{2.59}$$

Then, a measurement $\{M_i\}$ which is expressed by the Schmidt basis of the particle A as

$$M_i = \sum_{j,k}(N_i)_{j,k}\,|j_A\rangle\,\langle k_A| \tag{2.60}$$

reproduce the measurement $\{N_i\}$. In order to prove this, we only have to prove that the measurement operators M_i and N_i transform the Schmidt decomposition of $|\psi_{AB}\rangle$ into states whose Schmidt coefficients are the same. This is indeed done as follows:

$$
\begin{aligned}
N_i \sum_{l=1}^{n}\lambda_l\,|l_A l_B\rangle &= \sum_{j=1}^{n}\sum_{k=1}^{n}(N_i)_{j,k}\sum_{l=1}^{n}|j_B\rangle\,\langle k_B|\,\lambda_l\,|l_A l_B\rangle \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}(N_i)_{j,k}\lambda_k\,|k_A j_B\rangle\,,
\end{aligned}
\tag{2.61}
$$

$$
\begin{aligned}
M_i \sum_{l=1}^{n}\lambda_l\,|l_A l_B\rangle &= \sum_{j=1}^{n}\sum_{k=1}^{n}(N_i)_{j,k}\sum_{l=1}^{n}|j_A\rangle\,\langle k_A|\,\lambda_l\,|l_A l_B\rangle \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}(N_i)_{j,k}\lambda_k\,|j_A k_B\rangle\,.
\end{aligned}
\tag{2.62}
$$

Let the notations U_A and U_B stand for the local unitary transformation performed on the particles A and B, respectively, which transforms the state $\sum_{j=1}^{n}\sum_{k=1}^{n}(N_i)_{j,k}\lambda_k\,|k_A j_B\rangle$ into its Schmidt decomposition. Then, we can transform the state $\sum_{j=1}^{n}\sum_{k=1}^{n}(N_i)_{j,k}\lambda_k\,|j_A k_B\rangle$ into the Schmidt decomposition of $\sum_{j=1}^{n}\sum_{k=1}^{n}(N_i)_{j,k}\lambda_k\,|k_A j_B\rangle$ by performing U_A and U_B on the particles B and A, respectively, because the permutation of A and B applied on the state $\sum_{j=1}^{n}\sum_{k=1}^{n}(N_i)_{j,k}\lambda_k\,|k_A j_B\rangle$ in (2.61) yields the state $\sum_{k=1}^{n}(N_i)_{j,k}\lambda_k\,|j_A k_B\rangle$ in (2.62). $\square$

2.2.4 The proof of Theorem 1$_{\mathrm{pre}}$

In the present subsection, we finally prove Theorem 1$_{\mathrm{pre}}$, which we reproduce here.

Theorem 1$_{\mathrm{pre}}$ *Let the notation $|\psi_{AB}\rangle$ and $|\phi_{AB}\rangle$ stand for pure states of two $(D-1)/2$-spin particles. The reduced density matrices $\rho_{\psi A}$ and $\rho_{\phi A}$ are defined as*

$$\rho_{\psi A} \equiv tr_B\{|\psi_{AB}\rangle\,\langle\psi_{AB}|\}, \quad \rho_{\phi A} \equiv tr_B\{|\phi_{AB}\rangle\,\langle\phi_{AB}|\}. \tag{2.63}$$

Then, the following d-LOCC transformation is executable in the limit of $N, M \to \infty$ with $N/M = S_{\rho_{\phi A}}/S_{\rho_{\psi A}}$:

$$d\text{-}LOCC : |\psi_{AB}\rangle^{\otimes N} \leftrightarrow |\phi_{AB}\rangle^{\otimes M}\,, \tag{2.64}$$

where $S_{\rho_{\phi A}}$ and $S_{\rho_{\psi A}}$ are the von Neumann entropies of $\rho_{\phi A}$ and $\rho_{\psi A}$, respectively.

Proof

In order to prove the present theorem, we only have to prove the following statement:

Statement 1 Let the notation $\left|\phi_{AB}^{+}\right\rangle$ denote the maximally entanglemed pure states of two $(D-1)/2$-spin particles. Then, the following d-LOCC transformation is executable in the limit of $N, M^{+} \to \infty$ with $N/M^{+} = \log D/S_{\rho_{\psi A}}$:

$$\text{d-LOCC} : |\psi_{AB}\rangle^{\otimes N} \leftrightarrow \left|\phi_{AB}^{+}\right\rangle^{\otimes M^{+}}\,, \tag{2.65}$$

where $S_{\rho_{\psi A}}$ is the von Neumann entropy of $\rho_{\psi A}$.

If the Statement 1 is true, Theorem 1_{pre} is also true, because if $N/M = S_{\rho_{\phi A}}/S_{\rho_{\psi A}}$ holds, the equations $N/M^+ = \log D/S_{\rho_{\psi A}}$ and $M/M^+ = \log D/S_{\rho_{\phi A}}$ also hold, and thus the transformation (2.64) can be realized as follows:

$$\text{d-LOCC} : |\psi_{AB}\rangle^{\otimes N} \leftrightarrow \left|\phi_{AB}^+\right\rangle^{\otimes M^+} \leftrightarrow |\phi_{AB}\rangle^{\otimes M} . \tag{2.66}$$

Let us prove Statement 1. At first, we introduce the notation of "ϵ-typical states," which is necessary for the proof of Statement 1.

ϵ-typical states: Let us express the state $|\phi_{AB}\rangle$ in the Schmidt basis:

$$|\phi_{AB}\rangle = \sum_{x=1}^{D} \lambda_x |x_A\rangle |x_B\rangle = \sum_{x=1}^{D} \sqrt{p(x)} |x_A\rangle |x_B\rangle , \tag{2.67}$$

where $p(x) \equiv \lambda_x^2$. Then,

$$|\phi_{AB}\rangle^{\otimes N} = \sum_{x_1,...,x_N} \sqrt{p(x_1)....p(x_N)} |x_{1A}...x_{NA}\rangle |x_{1B}...x_{NB}\rangle \tag{2.68}$$

holds. Then the ϵ-typical state of $|\phi_{AB}\rangle$ is

$$\left|\phi_{AB}^{\text{typ}}\right\rangle = C \sum_{x \in T(N,\epsilon)} \sqrt{p(x_1)....p(x_N)} |x_{1A}...x_{NA}\rangle |x_{1B}...x_{NB}\rangle , \tag{2.69}$$

where $C \equiv \sqrt{1/\sum_{x \in T(N,\epsilon)} p(x_1)....p(x_N)}$ is the normalization constant. Note that

$$\langle \phi_A|^{\otimes N} \left|\phi_{AB}^{\text{typ}}\right\rangle = C \sum_{x \in T(N,\epsilon)} p(x_1)....p(x_N) = \sqrt{\sum_{x \in T(N,\epsilon)} p(x_1)....p(x_N)} \tag{2.70}$$

holds. Because of (i) of Theorem 2_{pre},

$$\lim_{N \to \infty} \sqrt{\sum_{x \in T(N,\epsilon)} p(x_1)....p(x_N)} = 1 \tag{2.71}$$

holds. Thus, the state $\left|\phi_{AB}^{\text{typ}}\right\rangle$ coincides with $|\phi_{AB}\rangle^{\otimes N}$ in the limit of $N \to \infty$.

We prove Statement 1 by using the ϵ-typical states. First, we prove that we can perform the following transformation in the limit of $N, M^+ \to \infty$ with $N/M^+ = \log D/S_{\rho_{\psi A}}$:

$$\text{d-LOCC} : |\psi_{AB}\rangle^{\otimes N} \to \left|\phi_{AB}^+\right\rangle^{\otimes M^+} . \tag{2.72}$$

In the limit of $N \to \infty$, the ϵ-typical state $\left|\psi_{AB}^{\text{typ}}\right\rangle$ coincides with $|\psi_{AB}\rangle^{\otimes N}$, and hence we only have to prove that the following transformation is executable:

$$\text{d-LOCC} : \left|\psi_{AB}^{\text{typ}}\right\rangle \to \left|\phi_{AB}^+\right\rangle^{\otimes M^+} . \tag{2.73}$$

Because $\{x_1, ..., x_n\}$ is an ϵ-typical sequence, Eq. (2.4) yields

$$e^{-N(H\{p(x)\}+\epsilon)} \leq p(x_1)....p(x_N) \leq e^{-N(H\{p(x)\}+\epsilon)} \tag{2.74}$$

Theorem 2$_{\text{pre}}$ (i) also yields

$$1 \leq C \equiv \sqrt{\frac{1}{\sum_{x \in T(N,\epsilon)} p(x_1)....p(x_N)}} = \sqrt{\frac{1}{P_\epsilon(\{x_1, ..., x_n\})}} \leq \sqrt{\frac{1}{1-\delta}}. \tag{2.75}$$

Combining the two, we obtain

$$\sqrt{e^{-N(H\{p(x)\}+\epsilon)}} \leq C\sqrt{p(x_1)....p(x_N)} \leq \sqrt{\frac{e^{-N(H\{p(x)\}-\epsilon)}}{1-\delta}} \tag{2.76}$$

Thus, the upper limit of the Schmidt coefficients of $\left|\psi_{AB}^{\text{typ}}\right\rangle$ is $\sqrt{e^{-N(H\{p(x)\}-\epsilon)}/(1-\delta)}$. All of the Schmidt coefficients of $\left|\phi_{AB}^+\right\rangle^{\otimes M^+}$ are equal to $\sqrt{D^{-M}}$, because $\left|\phi_{AB}^+\right\rangle^{\otimes M^+}$ can be seen as a maximally entangled state of two $(D^M - 1)/2$-spin particles. Therefore, because of Theorem 3$_{\text{pre}}$, the transformation (2.73) is executable if

$$\sqrt{\frac{e^{-N(H\{p(x)\}-\epsilon)}}{1-\delta}} \leq \sqrt{D^{-M}} \tag{2.77}$$

holds. In order to show this, we only have to prove that the vector of the Schmidt coefficients of $\left|\psi_{AB}^{\text{typ}}\right\rangle$ is mejorized by the vector of the Schmidt coefficients of $\left|\phi_{AB}^+\right\rangle^{\otimes M^+}$, because the Schmidt coefficients are equal to the eigenvalues of the reduced dencity matrix. Let us refer to the former and the latter as $\vec{x}$ and $\vec{y}$, respectively. Because of the definition of the majorization of vectors, $\vec{x}$ is majorized by $\vec{y}$ if and only if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \text{ for all } k. \tag{2.78}$$

This condition (2.78) clearly holds because of (2.77). Thus, in the limit of $N, M^+ \to \infty$ with $N/M^+ = \log D/S_{\rho_{\psi A}}$, the transformation (2.73) is executable, and thus the transformation (2.72) is also executable.

Next, we prove that we can perform the following transformation in the limit of $N, M^+ \to \infty$ with $N/M^+ = \log D/S_{\rho_{\psi A}}$:

$$\text{d-LOCC} : |\psi_{AB}\rangle^{\otimes N} \leftarrow \left|\phi_{AB}^+\right\rangle^{\otimes M^+}. \tag{2.79}$$

In the limit of $N \to \infty$, the ϵ-typical state $\left|\psi_{AB}^{\text{typ}}\right\rangle$ coincides with $|\psi_{AB}\rangle^{\otimes N}$, and thus we only have to prove that the following transformation is executable:

$$\text{d-LOCC} : \left|\psi_{AB}^{\text{typ}}\right\rangle \leftarrow \left|\phi_{AB}^+\right\rangle^{\otimes M^+}. \tag{2.80}$$

Because of (2.76), the lower limit of the Schmidt coefficients of $\left|\psi_{AB}^{\text{typ}}\right\rangle$ is $\sqrt{e^{-N(H\{p(x)\}+\epsilon)}}$. On the other hand, all of the Schmidt coefficients of $\left|\phi_{AB}^+\right\rangle^{\otimes M^+}$ are equal to $\sqrt{D^{-M}}$,

because $\left|\phi_{AB}^+\right\rangle^{\otimes M^+}$ is a maximally entangled state. Because of the same argument as above, the transformation (2.80) is executable if

$$\sqrt{e^{-N(H\{p(x)\}+\epsilon)}} \geq \sqrt{D^{-M}} \tag{2.81}$$

holds. Thus, in the limit of $N, M^+ \to \infty$ with $N/M^+ = \log D/S_{\rho_{\psi A}}$, the transformation (2.80) is executable, and hence the transformation (2.79) is also executable.□

2.3   Multipartite entanglement?

As we saw in the previous section, the quantification of the entanglement was successful for bipartite pure states. Extension of the above to multipartite states has been vigorously sought in vain. The difficulty is due to the difference between the structure of bipartite pure states and that of multipartite pure states. In the quantification of the entanglement of bipartite pure states, we used the majorization theory. The reason why we can apply the majorization theory to the bipartite pure states is that there is a vector structure in the bipartite pure states, nemely the Schmidt decomposition. However, the Schmidt decomposition does not exist for multipartite pure states. For example, the state $(|000\rangle + |101\rangle + |110\rangle)/\sqrt{3}$ cannot be expressed in such a form as $\sum_i \lambda_i |iii\rangle$. Of course, there are standard forms of decomposition for multipartite pure states, such as the generalized Schmidt decomposition [20, 21]. However, the sets of the coefficients of such standard forms do not have vector structures, but have tensor structures. Thus, the majorization theory cannot be applied to multipartite pure states. This is the reason why the extension of the quantification of the entanglement to multipartite states was not successful.

The present author made a new approach for the problem. In the new approach, we pay attention to the flow of the entanglement. With this approach, we give a necessary and sufficient condition of the possibility of a deterministic LOCC transformation of three-qubit pure states. In the next chapter, we will show the results of the new approach.

Chapter 3

New Approach

3.1 Preparation

In the present section, we define the parameters that we use in my approach. We consider only three-qubit pure states throughout the present paper. An arbitrary pure state $|\psi\rangle$ of the three qubits A, B and C is expressed in the form of the generalized Schmidt decomposition

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\varphi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle \tag{3.1}$$

with a proper basis set [20]. Each component of these ket vectors represents an eigenstate of the corresponding qubit A, B or C. For example, in the case of $|101\rangle$, which is abbreviation of $|1\rangle \otimes |0\rangle \otimes |1\rangle$, the qubit A is in the eigenstate $|1\rangle$, the qubit B is in $|0\rangle$ and the qubit C is in $|1\rangle$. We will occasionally use the notation $|1_A 0_B 1_C\rangle$ hereafter. The coefficients λ_0, λ_1, λ_2, λ_3 and λ_4 in (3.1) are nonnegative real numbers and satisfy that $\sum_{i=0}^{4} \lambda_i^2 = 1$. Note that the phase φ can take any real value if one of the coefficients $\{\lambda_i | i = 0, ..., 4\}$ is zero, in which case we define the phase φ to be zero in order to remove the ambiguity.

Two different decompositions of the form (3.1) are possible for the same state $|\psi\rangle$, one with $0 \leq \varphi \leq \pi$ and the other with $\pi \leq \varphi \leq 2\pi$. These two decompositions are LU-equivalent; in other words, they can be transformed into each other by local unitary (LU) transformations. Hereafter, we refer to the decomposition (3.1) with $0 \leq \varphi \leq \pi$ as the positive decomposition and the one (3.1) with $\pi < \varphi < 2\pi$ as the negative decomposition. We also refer to the coefficients of the positive and negative decompositions as the positive-decomposition coefficients and the negative-decomposition coefficients, respectively. Therefore, a set of coefficients gives a unique set of states that are LU-equivalent to each other, whereas such a set of states may give two possible sets of coefficients: for $\varphi \neq 0$, a set of positive-decomposition coefficients and a set of negative-decomposition coefficients are possible, while for $\varphi = 0$, only one set of positive-decomposition coefficients is possible. A set of LU-equivalent states and a set of positive-decomposition coefficients have a one-to-one correspondence.

We can use the coefficients of the generalized Schmidt decomposition in order to define five entanglement parameters of $|\psi\rangle$ as follows:

$$j_{AB} = \lambda_0 \lambda_3, \tag{3.2}$$

$$j_{AC} = \lambda_0 \lambda_2, \tag{3.3}$$

$$j_{BC} = |\lambda_1 \lambda_4 e^{i\varphi} - \lambda_2 \lambda_3|, \tag{3.4}$$

$$j_{ABC} = \lambda_0 \lambda_4, \tag{3.5}$$

$$J_5 = \lambda_0^2 (j_{BC}^2 + \lambda_2^2 \lambda_3^2 - \lambda_1^2 \lambda_4^2), \tag{3.6}$$

where we can use either the positive-decomposition coefficients or the negative-decomposition coefficients: both give the same values of (3.2)–(3.6). These parameters are real numbers; the parameters j_{AB}, j_{AC}, j_{BC} and j_{ABC} are nonnegative, whereas the parameter J_5 can take a negative value. The parameters j_{AB}, j_{AC}, j_{BC} and j_{ABC} are square roots of the entanglement parameters J_3, J_2, J_1 and J_4 proposed in Ref. [20], respectively. The parameters j_{AB}, j_{AC}, j_{BC} and j_{ABC}^2 are also related to the concurrences C_{AB}, C_{AC} and C_{BC} [6] and the tangle τ_{ABC} [18] as follows:

$$j_{AB} = \frac{1}{2} C_{AB}, \ \ j_{AC} = \frac{1}{2} C_{AC}, \ \ j_{BC} = \frac{1}{2} C_{BC}, \ \ j_{ABC}^2 = \frac{1}{4} \tau_{ABC}. \tag{3.7}$$

The parameter J_5 is equal to J_5 in Ref. [20]. These five parameters are invariant with respect to local unitary transformations. The two parameters j_{ABC} and J_5 are tripartite parameters; these two parameters are invariant with respect to permutation of the qubits A, B and C [20]. Hereafter, we refer to these five parameters as the J-parameters.

In order to express Main Theorems of the present paper in simpler forms, we define three nonnegative real-valued parameters K_{AB}, K_{AC} and K_{BC} as follows:

$$K_{AB} = j_{AB}^2 + j_{ABC}^2, \tag{3.8}$$

$$K_{AC} = j_{AC}^2 + j_{ABC}^2, \tag{3.9}$$

$$K_{BC} = j_{BC}^2 + j_{ABC}^2. \tag{3.10}$$

Then, the five parameters K_{AB}, K_{AC}, K_{BC}, j_{ABC} and J_5 are independent of each other and are invariant with respect to local unitary operations. We can substitute these five parameters as the entanglement parameters for the J-parameters ($j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5$). Let us refer to the new parameters ($K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5$) as the K-parameters. Note that J-parameters and K-parameters have a one-to-one correspondence.

We also define three parameters in order to simplify expressions which often appear in the present paper:

$$J_{\mathrm{ap}} \equiv j_{AB}^2 j_{AC}^2 j_{BC}^2, \ \ K_{\mathrm{ap}} \equiv K_{AB} K_{AC} K_{BC}, \ \ K_5 \equiv j_{ABC}^2 + J_5, \tag{3.11}$$

where the subscript ap is abbreviation of all pairs. Note that these parameters J_{ap}, K_{ap} and K_5 are *not* included in the J-parameters or the K-parameters; these are only for simplicity. By definition, J_{ap}, K_{ap} and K_5 are invariant with respect to local unitary transformations as well as permutations of A, B and C.

The coefficients of the generalized Schmidt decomposition (3.1) give a unique set of the J-parameters as in (3.2)–(3.6). However, when we specify the J-parameters, the

decomposition coefficients still have the following ambiguity [22, 23]:

$$(\lambda_0^\pm)^2 = \frac{J_5 + j_{ABC}^2 \pm \sqrt{\Delta_J}}{2(j_{BC}^2 + j_{ABC}^2)} = \frac{K_5 \pm \sqrt{\Delta_J}}{2K_{BC}}, \tag{3.12}$$

$$(\lambda_2^\pm)^2 = \frac{j_{AC}^2}{(\lambda_0^\pm)^2}, \tag{3.13}$$

$$(\lambda_3^\pm)^2 = \frac{j_{AB}^2}{(\lambda_0^\pm)^2}, \tag{3.14}$$

$$(\lambda_4^\pm)^2 = \frac{j_{ABC}^2}{(\lambda_0^\pm)^2}, \tag{3.15}$$

$$(\lambda_1^\pm)^2 = 1 - (\lambda_0^\pm)^2 - \frac{j_{AB}^2 + j_{AC}^2 + j_{ABC}^2}{(\lambda_0^\pm)^2}, \tag{3.16}$$

$$\cos\varphi^\pm = \frac{(\lambda_1^\pm)^2(\lambda_4^\pm)^2 + (\lambda_2^\pm)^2(\lambda_3^\pm)^2 - j_{BC}^2}{2\lambda_1^\pm \lambda_2^\pm \lambda_3^\pm \lambda_4^\pm}, \tag{3.17}$$

where

$$\Delta_J \equiv K_5^2 - 4K_{\mathrm{ap}} \geq 0, \tag{3.18}$$

$$0 \leq \varphi^\pm \leq \pi. \tag{3.19}$$

Thus, there are four possible sets of coefficients for one set of J-parameters: the positive-decomposition coefficients $\{\lambda_i^+, \varphi^+ | i = 0, ..., 4\}$ and $\{\lambda_i^-, \varphi^- | i = 0, ..., 4\}$ as well as the negative-decomposition coefficients $\{\lambda_i^+, \tilde{\varphi}^+ | i = 0, ..., 4\}$ and $\{\lambda_i^-, \tilde{\varphi}^- | i = 0, ..., 4\}$, where $\tilde{\varphi}^\pm = 2\pi - \varphi^\pm$ with $\pi \leq \tilde{\varphi}^\pm \leq 2\pi$. A state with $\{\lambda_i^+, \varphi^+ | i = 0, ..., 4\}$ is LU-equivalent to a state with $\{\lambda_i^-, \tilde{\varphi}^- | i = 0, ..., 4\}$, while a state with $\{\lambda_i^-, \varphi^- | i = 0, ..., 4\}$ is LU-equivalent to a state with $\{\lambda_i^+, \tilde{\varphi}^+ | i = 0, ..., 4\}$ [22]. Therefore we can focus on two possible positive-decomposition coefficients $\{\lambda_i^+, \varphi^+ | i = 0, ..., 4\}$ and $\{\lambda_i^-, \varphi^- | i = 0, ..., 4\}$ for a set of the J-parameters.

To eliminate this ambiguity further, we define the following new parameter, which we refer to as the entanglement charge:

$$Q_{\mathrm{e}} = \mathrm{sgn}\left[\sin\varphi\left(\lambda_0^2 - \frac{j_{ABC}^2 + J_5}{2(j_{BC}^2 + j_{ABC}^2)}\right)\right] = \mathrm{sgn}\left[\sin\varphi\left(\lambda_0^2 - \frac{K_5}{2K_{BC}}\right)\right], \tag{3.20}$$

where $\mathrm{sgn}[x]$ is the sign function;

$$\mathrm{sgn}[x] = \begin{cases} x/|x| & (x \neq 0), \\ 0 & (x = 0). \end{cases} \tag{3.21}$$

The entanglement charge Q_{e} is equal for the positive- and the negative-decomposition coefficients of a state. Therefore, the parameter Q_{e} is invariant with respect to local unitary transformations. The complex-conjugate transformation of a state does not change the J-parameters nor λ_0^2, but reverses the sign of $\sin\varphi$. Thus, the complex-conjugate transformation reverses the sign of Q_{e}. As we have seen, the parameter Q_{e} has characters that the electric charge has; hence, we refer to Q_{e} as the entanglement charge.

As we show below, two states are LU-equivalent if and only if the J-parameters and the entanglement charge Q_{e} of the two states are the same. If $Q_{\mathrm{e}} \neq 0$, we can determine one
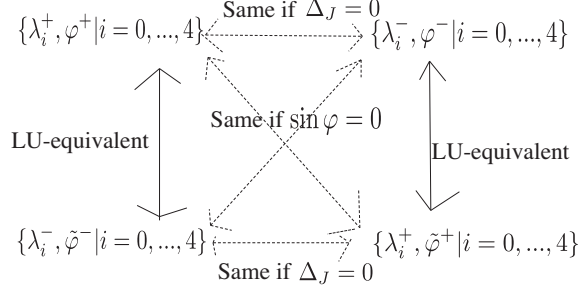
Figure 3.1: The relation among the four sets of coefficients for $Q_e = 0$. The relations indicated by solid lines are always valid, while those indicated by dotted lines are valid if and only if the noted conditions are satisfied.

positive-decomposition coefficients and one negative-decomposition coefficients uniquely from the J-parameters and the entanglement charge Q_e as follows:

$$\lambda_0^2 = \frac{J_5 + j_{ABC}^2 \mp Q_e \sqrt{\Delta_J}}{2(j_{BC}^2 + j_{ABC}^2)} = \frac{K_5 \mp Q_e \sqrt{\Delta_J}}{2K_{BC}}, \tag{3.22}$$

$$\lambda_2^2 = \frac{j_{AC}^2}{\lambda_0^2}, \tag{3.23}$$

$$\lambda_3^2 = \frac{j_{AB}^2}{\lambda_0^2}, \tag{3.24}$$

$$\lambda_4^2 = \frac{j_{ABC}^2}{\lambda_0^2}, \tag{3.25}$$

$$\lambda_1^2 = 1 - \lambda_0^2 - \frac{j_{AB}^2 + j_{AC}^2 + j_{ABC}^2}{\lambda_0^2}, \tag{3.26}$$

$$\cos\varphi = \frac{\lambda_1^2 \lambda_4^2 + \lambda_2^2 \lambda_3^2 - j_{BC}^2}{2\lambda_1 \lambda_2 \lambda_3 \lambda_4}, \tag{3.27}$$

where $\mp$ is $+$ or $-$ when $\{\lambda_i, \varphi | i = 0, ..., 4\}$ is positive-decomposition coefficients or negative-decomposition coefficients, respectively. Thus, if $Q_e \neq 0$, a set of the J-parameters together with the entanglement charge Q_e gives a unique set of LU-equivalent states.

If $Q_e = 0$, at least one of $\sin\varphi$ and Δ_J is zero because of (3.12) and (3.20). If $\sin\varphi$ is zero, $\{\lambda_i^\pm, \varphi^\pm | i = 0, ..., 4\}$ and $\{\lambda_i^\pm, \tilde{\varphi}^\pm | i = 0, ..., 4\}$ are the same. If Δ_J is zero, $\{\lambda_i^+, \varphi^+ | i = 0, ..., 4\}$ and $\{\lambda_i^+, \tilde{\varphi}^+ | i = 0, ..., 4\}$ are the same as $\{\lambda_i^-, \varphi^- | i = 0, ..., 4\}$ and $\{\lambda_i^-, \tilde{\varphi}^- | i = 0, ..., 4\}$, respectively, because of (3.12). Thus, if Q_e is zero, the four sets of coefficients $\{\lambda_i^+, \varphi^+ | i = 0, ..., 4\}$, $\{\lambda_i^-, \varphi^- | i = 0, ..., 4\}$, $\{\lambda_i^+, \tilde{\varphi}^- | i = 0, ..., 4\}$ and $\{\lambda_i^-, \tilde{\varphi}^- | i = 0, ..., 4\}$ are LU-equivalent (Fig. 3.1). Thus, if $Q_e = 0$, a set of the J-parameters gives a unique set of LU-equivalent states. Incidentally, a state is LU-equivalent to its complex conjugate if and only if its entanglement charge Q_e is zero. The complex conjugate transformation of the state only changes the sign of φ. Thus, a state is LU-equivalent to its complex conjugate if and only if $\{\lambda_i^\pm, \varphi^\pm | i = 0, ..., 4\}$ are LU-equivalent to $\{\lambda_i^\pm, \tilde{\varphi}^\pm | i = 0, ..., 4\}$; this LU-equivalence is illustrated in Fig. 3.1.

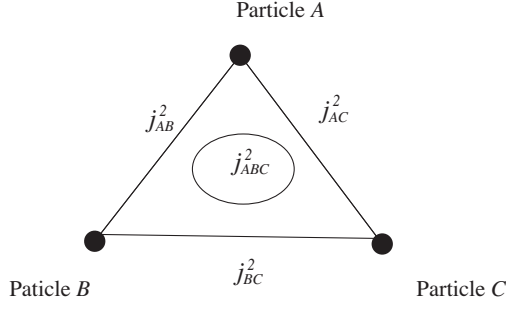For the reasons stated above, a set of the J-parameters together with the entanglement

Figure 3.2: A schematic diagram representing the entanglement parameters j^2_{AB}, j^2_{AC}, j^2_{BC} and j^2_{ABC}.

charge Q_e gives a unique set of LU-equivalent states. In other words, two states are LU-equivalent if and only if the J-parameters and the entanglement charge Q_e of the two states are the same. The J-parameters and the K-parameters have a one-to-one correspondence, and thus it is also true that two states are LU-equivalent if and only if the K-parameters and the entanglement charge Q_e of the two states are the same. Note that if $Q_e = 0$, (3.22) may not hold; in fact, when $\sin \varphi = 0 \wedge \Delta_J \neq 0$ holds, (3.22) is not correct.

We can interpret the entanglement parameters j^2_{AB}, j^2_{AC} and j^2_{BC} as indices of the bipartite entanglements between the qubits A and B, A and C, and B and C, respectively, while the parameter j^2_{ABC} as an index of the tripartite entanglement among the qubits A, B and C (Fig. 3.2) [18]. The entanglement charge Q_e is a tripartite parameter, because Q_e is invariant with respect to the permutation of the qubits A, B and C. This fact is shown in Appendix B.

Then, what does the entanglement parameter J_5 mean? It is not clear what J_5 means in the expression (3.6), but in another expression, we find that the entanglement parameter J_5 is a product of j_{AB}, j_{AC}, j_{BC} and a geometric phase. In order to show this, let us rewrite J_5 in the following form:

$$
\begin{aligned}
J_5 &= 2\lambda_0^2 \lambda_2 \lambda_3 (\lambda_2 \lambda_3 - \lambda_1 \lambda_4 \cos \varphi) \\
&= 2 j_{AB} j_{AC} (\lambda_2 \lambda_3 - \lambda_1 \lambda_4 \cos \varphi) \\
&= 2 j_{AB} j_{AC} j_{BC} \frac{\lambda_2 \lambda_3 - \lambda_1 \lambda_4 \cos \varphi}{|\lambda_2 \lambda_3 - \lambda_1 \lambda_4 e^{i\varphi}|}.
\end{aligned}
\tag{3.28}
$$

Because $\lambda_2 \lambda_3 - \lambda_1 \lambda_4 \cos \varphi = \mathrm{Re}(\lambda_2 \lambda_3 - \lambda_1 \lambda_4 e^{i\varphi})$, the following inequalities hold:

$$
0 \leq \left| \frac{\lambda_2 \lambda_3 - \lambda_1 \lambda_4 \cos \varphi}{|\lambda_2 \lambda_3 - \lambda_1 \lambda_4 e^{i\varphi}|} \right| \leq 1.
\tag{3.29}
$$

Hence, if $j_{AB} j_{AC} j_{BC} = 0$, then $J_5 = 0$. Inversely, if $j_{AB} j_{AC} j_{BC} \neq 0$, then

$$
\frac{J_5}{2 j_{AB} j_{AC} j_{BC}} = \frac{\lambda_2 \lambda_3 - \lambda_1 \lambda_4 \cos \varphi}{|\lambda_2 \lambda_3 - \lambda_1 \lambda_4 e^{i\varphi}|}.
\tag{3.30}
$$

Therefore, if $j_{AB} j_{AC} j_{BC} \neq 0$, we can define a phase $0 \leq \varphi_5 \leq \pi$ as follows:

$$
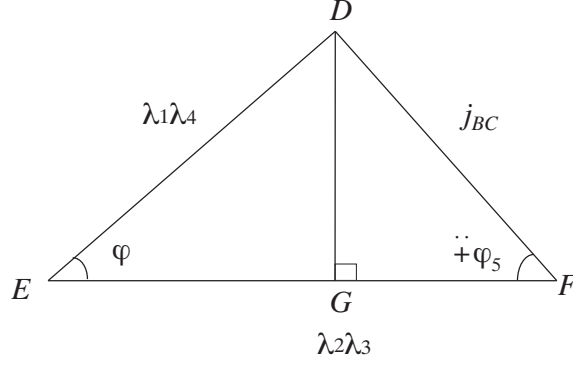\frac{J_5}{2 j_{AB} j_{AC} j_{BC}} = \cos \varphi_5.
\tag{3.31}
$$

25

Figure 3.3: A schematic diagram of an entanglement triangle (ET). The sign $\ddot{+}$ is $+$ or $-$ when $\{\lambda_i, \varphi | i = 0, ..., 4\}$ is positive-decomposition coefficients or negative-decomposition coefficients, respectively.

Let us refer to the phase φ_5 as the entanglement phase (EP).

The entanglement phase φ_5 is invariant with respect to local unitary operations, because both of the parameters J_5 and $j_{AB}j_{AC}j_{BC}$ are. If $j_{AB}j_{AC}j_{BC} = 0$, the entanglement phase φ_5 is indefinite. Hereafter, we refer to a state whose entanglement phase φ_5 is definite as an EP-definite state and to a state whose entanglement phase φ_5 is indefinite as an EP-indefinite state. For an EP-indefinite state, at least one of j_{AB}, j_{AC} and j_{BC} is zero.

The entanglement phase plays an important role in the present paper. We will see that the necessary and sufficient condition of a deterministic LOCC transformation depends on whether the initial and final states of the transformation are EP definite or EP indefinite. The parameters J_5 and $j_{AB}j_{AC}j_{BC}$ are invariant with respect to permutations of A, B and C, and so is the entanglement phase φ_5. This is the reason why we did not define the entanglement phase as $(\lambda_2\lambda_3 - \lambda_1\lambda_4\cos\varphi)|\lambda_2\lambda_3 - \lambda_1\lambda_4 e^{i\varphi}|^{-1}$; actually, $(\lambda_2\lambda_3 - \lambda_1\lambda_4\cos\varphi)|\lambda_2\lambda_3 - \lambda_1\lambda_4 e^{i\varphi}|^{-1}$ is indefinite for $j_{BC} = 0$, but not necessarily so for $j_{AB} = 0$ or $j_{AC} = 0$.

If the state $|\psi\rangle$ is EP indefinite, its entanglement parameters J_5 and Q_e are zero, which we show below. Then, the number of the entanglement parameters of an EP-indefinite state reduces to four; two EP-indefinite states are LU-equivalent if and only if the sets of the four entanglement parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC})$ of the states are equal to each other. Now we show $J_5 = Q_e = 0$ for an EP-indefinite state. If the state $|\psi\rangle$ is EP indefinite, $j_{AB}j_{AC}j_{BC} = 0$ holds. Thus, the entanglement parameter J_5 is also zero because we have $|J_5| \leq 2j_{AB}j_{AC}j_{BC}$ from (3.28) and (3.29). Next we show $Q_e = 0$. Because of $j_{AB}j_{AC}j_{BC} = 0$, at least one of j_{AB}, j_{AC} and j_{BC} is zero. If $j_{AB} = \lambda_0\lambda_3$ or $j_{AC} = \lambda_0\lambda_2$ is zero, there is a zero in $\{\lambda_i^2 | i = 0, ..., 4\}$, and thus the phase φ is zero. If $j_{BC} = |\lambda_1\lambda_4 e^{i\varphi} - \lambda_2\lambda_3|$ is zero, the equation $\varphi = 0$ also has to hold, because $\lambda_1\lambda_4 e^{i\varphi}$ has to be a real number. Therefore, $\varphi = 0$ and hence $Q_e = 0$ as can be seen in (A.19). Thus, if the state $|\psi\rangle$ is EP indefinite, the parameters J_5 and Q_e are zero.

If the entanglement phase φ_5 is definite, we can express the relation among φ_5 and the other parameters λ_1, λ_2, λ_3, λ_4, φ and j_{BC} as a triangle shown in Fig. 3.3. We can derive this relation from (3.4), (3.28) and (3.31). This triangle plays a very important

role in the present paper. Let us refer to this triangle as the entanglement triangle (ET).

The height of the entanglement triangle in Fig. 3.3 gives the following relation between the entanglement phase φ_5 and the phase φ:

$$j_{BC} \sin \varphi_5 = \overset{\cdots}{+} \lambda_1 \lambda_4 \sin \varphi, \tag{3.32}$$

where the sign $\overset{\cdots}{+}$ is $+$ or $-$ when $\{\lambda_i, \varphi | i = 0, ..., 4\}$ is positive-decomposition coefficients or negative-decomposition coefficients, respectively. From (3.32), we can derive the following useful relation:

$$\sin \varphi_5 = 0 \Leftrightarrow \sin \varphi = 0. \tag{3.33}$$

Let us show (3.33). Because now we have assumed that the entanglement phase φ_5 is definite, the J-parameter j_{BC} must not vanish. Thus, (3.32) is followed by the relation $\sin \varphi_5 = 0 \Leftarrow \sin \varphi = 0$. If $\sin \varphi_5 = 0$, on the other hand, Fig. 3.3 implies that at least one of λ_1, λ_4 and $\sin \varphi$ is zero. If λ_1 or λ_4 is zero, then one of $\{\lambda_i | i = 0, ..., 4\}$ is zero, and thus $\sin \varphi$ is also zero. Therefore, the relation (3.33) holds. Incidentally, the length of the segment FG in Fig. 3.3 gives another relation between the entanglement phase φ_5 and φ:

$$j_{BC} \cos \varphi_5 = \lambda_2 \lambda_3 - \lambda_1 \lambda_4 \cos \varphi. \tag{3.34}$$

Next, let us observe how a measurement changes the entanglement parameters. When we perform a transformation which is expressed as

$$M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}, \qquad M_{00}, M_{01}, M_{10}, M_{11} \in \mathbb{C}, \tag{3.35}$$

on the qubit A of a pure state (3.1), the state $|\psi\rangle$ is transformed into

$$
\begin{aligned}
M |\psi\rangle = {} & (\lambda_0 M_{00} |0\rangle + \lambda_0 M_{10} |1\rangle + \lambda_1 e^{i\varphi} M_{01} |0\rangle + \lambda_1 e^{i\varphi} M_{11} |1\rangle) |00\rangle + \lambda_2 (M_{01} |0\rangle + M_{11} |1\rangle) |01\rangle \\
& + \lambda_3 (M_{01} |0\rangle + M_{11} |1\rangle) |10\rangle + \lambda_4 (M_{01} |0\rangle + M_{11} |1\rangle) |11\rangle . \tag{3.36}
\end{aligned}
$$

Let us expand $M |\psi\rangle$ in the form of the generalized Schmidt decomposition (3.1). First, we define two pure states $|1'_A\rangle$ and $|0'_A\rangle$ as

$$|1'_A\rangle = (M_{01} |0\rangle + M_{11} |1\rangle) / \sqrt{|M_{01}|^2 + |M_{11}|^2}, \tag{3.37}$$

$$|0'_A\rangle = (M_{11}^* |0\rangle - M_{01}^* |1\rangle) / \sqrt{|M_{01}|^2 + |M_{11}|^2}, \tag{3.38}$$

where M_{01}^* denotes the complex conjugate of M_{01}, and so on. Then, we have

$$\langle 0'_A | (\lambda_0 M_{00} |0\rangle + \lambda_0 M_{10} |1\rangle + \lambda_1 e^{i\varphi} M_{01} |0\rangle + \lambda_1 e^{i\varphi} M_{11} |1\rangle) = \frac{\lambda_0 (M_{00} M_{11} - M_{01} M_{10})}{\sqrt{|M_{01}|^2 + |M_{11}|^2}}, \tag{3.39}$$

$$
\begin{aligned}
\langle 1'_A | (\lambda_0 M_{00} |0\rangle & + \lambda_0 M_{10} |1\rangle + \lambda_1 e^{i\varphi} M_{01} |0\rangle + \lambda_1 e^{i\varphi} M_{11} |1\rangle) \\
& = \frac{\lambda_0 (M_{00} M_{01}^* + M_{10} M_{11}^*) + \lambda_1 e^{i\varphi} (|M_{01}|^2 + |M_{11}|^2)}{\sqrt{|M_{01}|^2 + |M_{11}|^2}}. \tag{3.40}
\end{aligned}
$$

Hence we obtain the following equation:

$$M\left|\psi\right\rangle = \frac{\lambda_0 \det M}{\sqrt{|M_{01}|^2 + |M_{11}|^2}}\left|0'_A00\right\rangle + \frac{\lambda_0(M_{00}M_{01}^* + M_{10}M_{11}^*) + \lambda_1 e^{i\varphi}(|M_{01}|^2 + |M_{11}|^2)}{\sqrt{|M_{01}|^2 + |M_{11}|^2}}\left|1'_A00\right\rangle$$
$$+ \sqrt{|M_{01}|^2 + |M_{11}|^2}(\lambda_2\left|1'_A01\right\rangle + \lambda_3\left|1'_A10\right\rangle + \lambda_4\left|1'_A11\right\rangle) \quad (3.41)$$

We can still change the phase of $\left|0'_A\right\rangle$. Thus we define $\left|0''_A\right\rangle$ as follows:

$$\frac{\lambda_0 \det M}{\sqrt{|M_{01}|^2 + |M_{11}|^2}}\left|0'_A00\right\rangle = \frac{\lambda_0\sqrt{\det M^\dagger M}}{\sqrt{|M_{01}|^2 + |M_{11}|^2}}\left|0''_A00\right\rangle. \quad (3.42)$$

Because the states $\left|0''_A\right\rangle$ and $\left|1'_A\right\rangle$ are orthogonal to each other, we obtain

$$M\left|\psi\right\rangle = \frac{\lambda_0 \det\sqrt{M^\dagger M}}{\sqrt{|M_{01}|^2 + |M_{11}|^2}}\left|0''_A00\right\rangle + \frac{\lambda_0(M_{00}M_{01}^* + M_{10}M_{11}^*) + \lambda_1 e^{i\varphi}(|M_{01}|^2 + |M_{11}|^2)}{\sqrt{|M_{01}|^2 + |M_{11}|^2}}\left|1'_A00\right\rangle$$
$$+ \sqrt{|M_{01}|^2 + |M_{11}|^2}(\lambda_2\left|1'_A01\right\rangle + \lambda_3\left|1'_A10\right\rangle + \lambda_4\left|1'_A11\right\rangle) \quad (3.43)$$

We thereby achieve the generalized Schmidt decomposition of $M\left|\psi\right\rangle$ except for normalization.

We can express each coefficient of the generalized Schmidt decomposition (3.43) of $M\left|\psi\right\rangle$ above solely by the components of $M^\dagger M$:

$$\begin{aligned}M^\dagger M &= \begin{pmatrix} M_{00}^* & M_{10}^* \\ M_{01}^* & M_{11}^* \end{pmatrix}\begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix} \\ &= \begin{pmatrix} |M_{00}|^2 + |M_{10}|^2 & M_{00}^*M_{01} + M_{10}^*M_{11} \\ M_{01}^*M_{00} + M_{11}^*M_{10} & |M_{01}|^2 + |M_{11}|^2 \end{pmatrix}. \end{aligned} \quad (3.44)$$

Thus we can define real parameters a, b, k and θ, which we refer to the measurement parameters, as

$$\begin{aligned}M^\dagger M &= \begin{pmatrix} a & ke^{-i\theta} \\ ke^{i\theta} & b \end{pmatrix}, \\ ab &- k^2 \geq 0, a \geq 0, b \geq 0, k \geq 0, 0 \leq \theta \leq 2\pi, \end{aligned} \quad (3.45)$$

to express $M\left|\psi\right\rangle$ in (3.43) as

$$M\left|\psi\right\rangle = \frac{\lambda_0\sqrt{ab - k^2}}{\sqrt{b}}\left|0''_A00\right\rangle + \frac{\lambda_0 ke^{i\theta} + \lambda_1 e^{i\varphi}b}{\sqrt{b}}\left|1'_A00\right\rangle$$
$$+ \lambda_2\sqrt{b}\left|1'_A01\right\rangle + \lambda_3\sqrt{b}\left|1'_A10\right\rangle + \lambda_4\sqrt{b}\left|1'_A11\right\rangle. \quad (3.46)$$

Now, let us define a measurement $\{M_{(i)}|i = 1, ..., n\}$ to substitute for the above transformation M, where the subscript (i) denotes the ith measurement result out of n possible results. (Hereafter, all the superscripts and subscripts in parentheses will indicate the order in the observational result.) We can define the measurement parameters $a_{(i)}, b_{(i)}, k_{(i)}$ and $\theta_{(i)}$ for the measurement $M_{(i)}$ as in (3.45) for the measurement M. Besides, the probability $p_{(i)}$ that the result i comes out from the measurement $\{M_{(i)}\}$ is given by

$$\begin{aligned}p_{(i)} &= \left\langle\psi\right| M_{(i)}^\dagger M_{(i)}\left|\psi\right\rangle \\ &= \lambda_0^2 a_{(i)} + (1 - \lambda_0^2)b_{(i)} + 2\lambda_0\lambda_1 k_{(i)}\cos(\theta_{(i)} - \varphi). \end{aligned} \quad (3.47)$$

Therefore, we can define the normalized states $\{|\psi^{(i)}\rangle\}$ as

$$\left|\psi^{(i)}\right\rangle = \frac{M_{(i)}\left|\psi\right\rangle}{\sqrt{p_{(i)}}}. \tag{3.48}$$

Let the notations $\lambda_0^{(i)}$, $\lambda_1^{(i)}$, $\lambda_2^{(i)}$, $\lambda_3^{(i)}$, $\lambda_4^{(i)}$ and $\varphi^{(i)}$ denote the coefficients of the generalized Schmidt decomposition of $\left|\psi^{(i)}\right\rangle$ after the normalization (3.48). From (3.46) and (3.48), we have

$$\lambda_0^{(i)} = \frac{\lambda_0\sqrt{a_{(i)}b_{(i)} - k_{(i)}^2}}{\sqrt{p_{(i)}}\sqrt{b_{(i)}}}, \tag{3.49}$$

$$\lambda_1^{(i)}e^{i\varphi^{(i)}} = \frac{\lambda_0 k_{(i)}e^{i\theta_{(i)}} + \lambda_1 e^{i\varphi}b_{(i)}}{\sqrt{p_{(i)}}\sqrt{b_{(i)}}}, \tag{3.50}$$

$$\lambda_2^{(i)} = \frac{\lambda_2\sqrt{b_{(i)}}}{\sqrt{p_{(i)}}}, \tag{3.51}$$

$$\lambda_3^{(i)} = \frac{\lambda_3\sqrt{b_{(i)}}}{\sqrt{p_{(i)}}}, \tag{3.52}$$

$$\lambda_4^{(i)} = \frac{\lambda_4\sqrt{b_{(i)}}}{\sqrt{p_{(i)}}}. \tag{3.53}$$

The equations (3.49)–(3.53) give how the entanglement parameters j_{AB}, j_{AC}, j_{BC}, j_{ABC} and J_5 change when a measurement $M_{(i)}$ is performed on the qubit A. First, we show the change of the entanglement parameters j_{AB}, j_{AC} and j_{ABC}. They are only multiplied with the same constant when a measurement $M_{(i)}$ is performed on the qubit A:

$$j_{AB}^{(i)} = \alpha^{(i)}j_{AB}, \tag{3.54}$$

$$j_{AC}^{(i)} = \alpha^{(i)}j_{AC}, \tag{3.55}$$

$$j_{ABC}^{(i)} = \alpha^{(i)}j_{ABC}, \tag{3.56}$$

where the multiplication factor $\alpha_{(i)}$ is defined by

$$\alpha^{(i)} = \frac{\sqrt{a_{(i)}b_{(i)} - k_{(i)}^2}}{p_{(i)}}. \tag{3.57}$$

From (3.54)–(3.56), we can also obtain the change of K-parameters

$$K_{AB}^{(i)} = (\alpha^{(i)})^2 K_{AB}, \quad K_{AC}^{(i)} = (\alpha^{(i)})^2 K_{AC}. \tag{3.58}$$

Second, we also express the change of the entanglement parameter j_{BC} as follows:

$$\begin{aligned}
p_{(i)}j_{BC}^{(i)} &= \left|\lambda_0\lambda_4 k_{(i)}e^{i\theta_{(i)}} - (\lambda_2\lambda_3 - \lambda_1\lambda_4 e^{i\varphi})b_{(i)}\right| \\
&= \left|k_{(i)}j_{ABC}e^{i\theta_{(i)}} - j_{BC}b_{(i)}e^{-i\tilde{\varphi}_5}\right| \\
&= \left|k_{(i)}j_{ABC}e^{i(\theta_{(i)}+\tilde{\varphi}_5)} - j_{BC}b_{(i)}\right|,
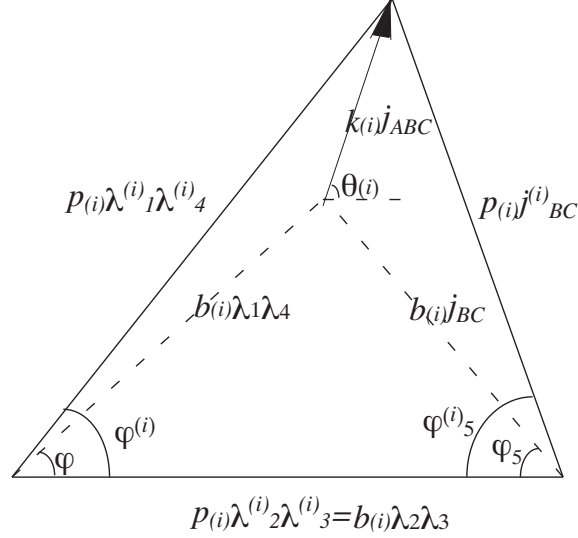\end{aligned} \tag{3.59}$$

Figure 3.4: Entanglement triangle of $M^{(i)} |\psi\rangle$.

where the real number $\tilde{\varphi}_5$ is defined as

$$e^{-i\tilde{\varphi}_5} = \frac{\lambda_2\lambda_3 - \lambda_1\lambda_4 e^{i\varphi}}{|\lambda_2\lambda_3 - \lambda_1\lambda_4 e^{i\varphi}|}. \tag{3.60}$$

The inequalities $0 \le \varphi \le \pi$ give $0 \le \tilde{\varphi}_5 \le \pi$. Thus, if the entanglement phase φ_5 defined in (3.31) is definite, $\varphi_5 = \overset{..}{+}\tilde{\varphi}_5$ holds, where the sign $\overset{..}{+}$ is $+$ or $-$ when $\{\lambda_i, \varphi | i = 0, ..., 4\}$ is positive-decomposition coefficients or negative-decomposition coefficients, respectively.

Finally, we analyze the change of the entanglement parameter J_5. We have shown in (3.28) that the entanglement parameter J_5 can be expressed as $J_5 = 2j_{AB}j_{AC}(\lambda_2\lambda_3 - \lambda_1\lambda_4 \cos \varphi)$. We already know that the entanglement parameters j_{AB} and j_{AC} change as in (3.54) and (3.55). We only have to examine how the quantity $\lambda_2\lambda_3 - \lambda_1\lambda_4 \cos \varphi$ changes. From (3.50)–(3.53), we have the following equation:

$$p_{(i)}(\lambda_2^{(i)}\lambda_3^{(i)} - \lambda_1^{(i)}\lambda_4^{(i)} \cos \varphi^{(i)}) = b_{(i)}(\lambda_2\lambda_3 - \lambda_1\lambda_4 \cos \varphi) - k_{(i)}\lambda_0\lambda_4 \cos \theta_{(i)}. \tag{3.61}$$

If the state $|\psi\rangle$ is EP definite, the equations (3.5) and (3.34) enable us to rewrite (3.61) as the transformation of the entanglement phase φ_5:

$$p_{(i)}j_{BC}^{(i)} \cos \varphi_5^{(i)} = b_{(i)}j_{BC} \cos \varphi_5 - k_{(i)}j_{ABC} \cos \theta_{(i)}. \tag{3.62}$$

We thereby find how the entanglement parameter J_5 changes.

We can summarize the equations (3.50)–(3.52), (3.59) and (3.62) as a transformation of the entanglement triangle (Fig. 3.4). From this figure, we find that $k_{(i)}e^{i\theta_{(i)}}j_{ABC}$ can be interpreted as a vector. Hereafter, the notation $\vec{k}_{(i)}$ stands for $k_{(i)}e^{i\theta_{(i)}}$.

We can also express the change of the average of $j_{BC} \cos \varphi_5$. Because $\{M_{(i)} | i = 1, ..., n\}$ is a measurement, we have

$$\sum_{i=1}^{n} M_{(i)}^{\dagger} M_{(i)} = I, \tag{3.63}$$

where I is the identity matrix. From (3.63), we have equations that the measurement parameters $\vec{k}_{(i)}$, $a_{(i)}$ and $b_{(i)}$ must satisfy:

$$\sum_{i=1}^{n} \vec{k}_{(i)} = \vec{0}, \tag{3.64}$$

$$\sum_{i=1}^{n} a_{(i)} = \sum_{i=1}^{n} b_{(i)} = 1. \tag{3.65}$$

From (3.61), (3.64) and (3.65), we obtain

$$\sum_{i=1}^{n} p_{(i)} (\lambda_2^{(i)} \lambda_3^{(i)} - \lambda_1^{(i)} \lambda_4^{(i)} \cos \varphi^{(i)}) = \lambda_2 \lambda_3 - \lambda_1 \lambda_4 \cos \varphi. \tag{3.66}$$

If the state $|\psi\rangle$ is EP definite, from (3.34) we obtain

$$\sum_{i=1}^{n} p_{(i)} j_{BC}^{(i)} \cos \varphi_5^{(i)} = j_{BC} \cos \varphi_5. \tag{3.67}$$

The equation (3.64) gives another useful equation:

$$\sum_{i=1}^{n} p_{(i)} \lambda_1^{(i)} \lambda_4^{(i)} \sin \varphi^{(i)} = \sum_{i=1}^{n} b_{(i)} \lambda_1^{(i)} \lambda_4^{(i)} \sin \varphi^{(i)} + k_{(i)} \lambda_0^{(i)} \lambda_4^{(i)} \sin \theta_{(i)} = \lambda_1 \lambda_4 \sin \varphi. \tag{3.68}$$

Incidentally, since each of $M_{(i)}^{\dagger} M_{(i)}$ is a positive operator, the measurement parameters $a_{(i)}, b_{(i)}, \vec{k}_{(i)}$ must satisfy

$$a_{(i)} b_{(i)} - \left| \vec{k}_{(i)} \right|^2 \geq 0 \tag{3.69}$$

and

$$b_{(i)} \geq 0. \tag{3.70}$$

Note that $M_{(i)}^{\dagger} M_{(i)}$ are positive operators if and only if (3.69) and (3.70) holds. In order to show this, we only have to see that we can reduce (3.69) and (3.70) into $a_{(i)} \geq 0$ and that the inequalities $a_{(i)} \geq 0$, (3.69) and (3.70) hold if and only if the eigenvalues of $M_{(i)}^{\dagger} M_{(i)}$ are positive, namely $M_{(i)}^{\dagger} M_{(i)}$ is a positive operator.

Finally, we define the names of types of states. We refer to a state whose j_{ABC} is nonzero or whose j_{AB}, j_{AC} and j_{BC} are all nonzero as a truly tripartite state. We refer to a state which has only a single kind of the bipartite entanglement as a biseparable state (Fig. 3.5(a)). Note that there is no state which has only two kinds of the bipartite entanglement (Fig. 3.5(b)). If there were such a state as in Fig. 3.5(b), the coefficients $\{\lambda_i, \varphi | i = 0, ..., 4\}$ of the state would satisfy

$$\lambda_0 \lambda_2 \neq 0, \quad \lambda_0 \lambda_3 \neq 0, \quad \lambda_0 \lambda_4 = 0, \quad |\lambda_1 \lambda_4 e^{i\varphi} - \lambda_2 \lambda_3|, \tag{3.71}$$

but (3.71) is impossible. An EP-indefinite state with $j_{ABC} \neq 0$ and an EP-definite state are truly tripartite states. A truly tripartite state is an EP-indefinite state with $j_{ABC} \neq 0$ or an EP-definite state. A biseparable state is EP indefinite with $j_{ABC} = 0$. An EP-indefinite state with $j_{ABC} = 0$ is a biseparable state.
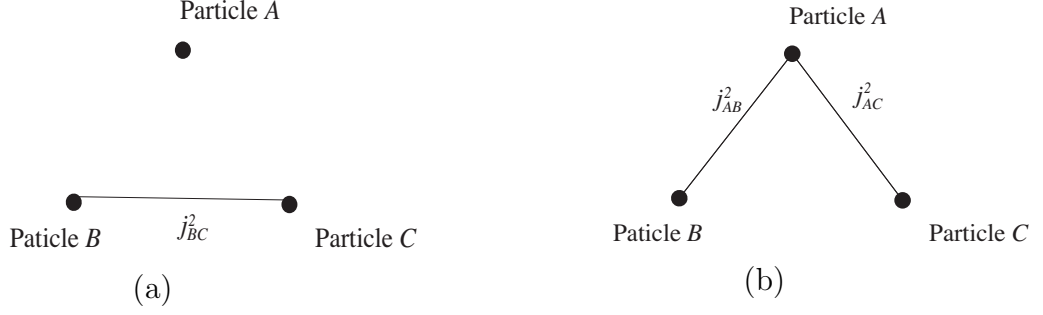
Figure 3.5: The concept of (a) a biseparable state. There is no state such as (b).

3.2 Main Theorems

There are two Main Theorems in the present paper.

First, we define terms which are often used in the present paper. Hereafter, we refer to a measurement which produces n possible results as an n-choice measurement. We refer to an LOCC transformation whose results can be transformed into a unique state by local unitary operations without exception, as a deterministic LOCC transformation. Similarly, we refer to a local measurement $\{M_{(i)}\}$ whose results can be transformed into a unique state by local unitary operations without exception, as a deterministic measurement (DM). We refer to a transformation from a state to another state with the probability one by a single DM on a single qubit followed by local unitary transformations, as a deterministic measurement transformation (DMT). We refer to a DMT whose DM is a two-choice measurement, as a two-choice DMT. Let us label DMTs by the qubit on which the corresponding DM is performed. For example, we refer to a DMT whose DM is performed on the qubit A, as an A-DMT. Moreover, we refer to a transformation which transforms a state to another state with the probability one by sequential operation of two-choice DMTs, as a constraint LOCC transformation (C-LOCC transformation).

Next, we introduce Main Theorems in the present section. Main Theorem 1 is written in terms of the K-parameters and Q_e as follows:

Main Theorem 1 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the K-parameters of $|\psi\rangle$ and $|\psi'\rangle$ as $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5, Q_e)$ and $(K'_{AB}, K'_{AC}, K'_{BC}, j'_{ABC}, J'_5, Q'_e)$, respectively. Then, a necessary and sufficient condition of the possibility of a deterministic LOCC transformation from $|\psi\rangle$ to $|\psi'\rangle$ is that the following two conditions are satisfied:*

Condition 1: There are real numbers $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$, $0 \leq \zeta_C \leq 1$ and $\zeta_{lower} \leq \zeta \leq 1$ which satisfy the following equation:

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta \begin{pmatrix} \zeta_A\zeta_B & & & & \\ & \zeta_A\zeta_C & & & \\ & & \zeta_B\zeta_C & & \\ & & & \zeta_A\zeta_B\zeta_C & \\ & & & & \zeta_A\zeta_B\zeta_C \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}, \qquad (3.72)
$$

where

$$
\zeta_{lower} = \frac{J_{ap}}{(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}, \qquad (3.73)
$$

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and we refer to ζ, ζ_A, ζ_B and ζ_C as the sub parameter and the main parameters of A, B and C, respectively.

Condition 2: If the state $|\psi'\rangle$ is EP definite, we check whether

$$(\Delta_J = 0) \wedge (4J_{ap} - J_5^2 = 0) \tag{3.74}$$

holds or not. When (3.74) does not hold, the condition is

$$Q_e = Q'_e \ \ and \ \ \zeta = \tilde{\zeta}, \tag{3.75}$$

where

$$\tilde{\zeta} \equiv \frac{K_{ap}(4J_{ap} - J_5^2) + \Delta_J J_{ap}}{K_{ap}(4J_{ap} - J_5^2) + \Delta_J(K_{AB} - \zeta_C j_{ABC}^2)(K_{AC} - \zeta_B j_{ABC}^2)(K_{BC} - \zeta_C j_{ABC}^2)}. \tag{3.76}$$

We refer to the parameter $\tilde{\zeta}$ as the ζ-specifying parameter. When (3.74) holds, the condition is

$$|Q'_e| = sgn[(1 - \zeta)(\zeta - \zeta_{lower})], \tag{3.77}$$

or in other expressions,

$$Q'_e \begin{cases} = 0 & (\zeta = 1 \ \ or \ \ \zeta = \zeta_{lower}), \\ \neq 0 & (otherwise). \end{cases} \tag{3.78}$$

Hereafter, we refer to a state which satisfies (3.74) as $\tilde{\zeta}$-indefinite and refer to a state which does not satisfy (3.74) as $\tilde{\zeta}$-definite. The following statements hold:

Statement $\tilde{\zeta}$-1 Any biseparable state is also a $\tilde{\zeta}$-indefinite state.

Statement $\tilde{\zeta}$-2 Any $\tilde{\zeta}$-indefinite state satisfies $Q_e = 0$.

Statement $\tilde{\zeta}$-3 A deterministic LOCC transformation from an EP-indefinite state to an EP-definite state is executable if and only if the initial state is $\tilde{\zeta}$-indefinite.

Statement $\tilde{\zeta}$-4 Among truly multipartite states, a deterministic LOCC transformation from a $\tilde{\zeta}$-indefinite state to a $\tilde{\zeta}$-definite state is executable, but the contrary is not executable.

Statement $\tilde{\zeta}$-5 When the initial state is $\tilde{\zeta}$-definite, the deterministic LOCC transformation conserves the entanglement charge Q_e.

Because of the above five statements, the $\tilde{\zeta}$-definite state can be considered as a "charge-definite state." When we transform a $\tilde{\zeta}$-indefinite state into a $\tilde{\zeta}$-definite state, we can choose the value of the entanglement charge Q_e; once the value is determined, we cannot change it anymore with a deterministic LOCC transformation (Fig. 3.6).

Main Theorem 1 and its proof give the rules of the entanglement change by an arbitrary deterministic LOCC transformation. Main Theorem 1 also gives an explicit protocol of determining whether there is an executable deterministic LOCC transformation from an arbitrary three-qubit pure state to another arbitrary three-qubit pure state.

Main Theorem 2 shows that we can reproduce an executable deterministic LOCC transformation from an arbitrary state to another arbitrary state by performing deterministic measurements *three* times at most.
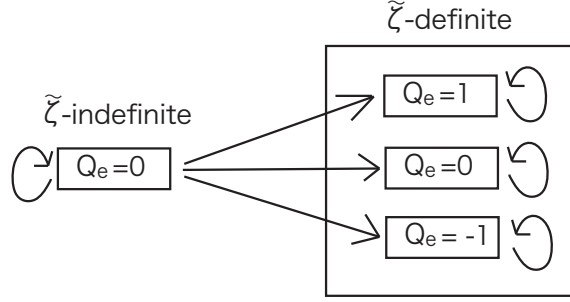
Figure 3.6: The entanglement charge Q_{e} for $\tilde{\zeta}$-definite states and $\tilde{\tilde{\zeta}}$-definite states. The arrows express the executable deterministic LOCC transformations among truly multipartite states. We can *not* execute deterministic LOCC transformations which are not expressed in this figure. For example, we cannot transform a $\tilde{\tilde{\zeta}}$-definite state whose Q_{e} is 1 into another $\tilde{\tilde{\zeta}}$-definite state whose Q_{e} is 0.

Table 3.1: The minimum number of times of measurements to reproduce an arbitrary deterministic LOCC transformation. For the terminology, see the last paragraph of section 3.1.

Initial state	Final state	Times
Truly tripartite state	Truly tripartite state	3
Truly tripartite state	Biseparable state or full-separable state	2
Biseparable state	Biseparable state or full-separable state	1
Full-separable state	Full-separable state	0

Main Theorem 2 *If a deterministic LOCC transformation is executable, we can reproduce it by performing local unitary operations, a deterministic measurement on the qubit A, one on the qubit B and one on the qubit C.*

Main Theorem 2 and the proof of Main Theorem 1 give the minimum number of necessary times of measurements to reproduce an arbitrary deterministic LOCC transformation, as listed in Table 3.1.

3.3   The Summary of the Proofs

In this section, we overview the structure of the proofs of the Main Theorems. We prove Main Theorems 1 and 2 simultaneously in the section 6 in the following three steps:

Step 1 We give a necessary and sufficient condition of the possibility of a two-choice DMT which transforms an arbitrary state $|\psi\rangle$ to another arbitrary state $|\psi'\rangle$.

Step 2 We give a necessary and sufficient condition of the possibility of a C-LOCC transformation from an arbitrary state $|\psi\rangle$ to another arbitrary state $|\psi'\rangle$. We also prove that we can reproduce an arbitrary C-LOCC transformation by performing an A-DMT, a B-DMT and a C-DMT, successively.

Table 3.2: The correspondence between the proofs and the sections.

Case	Step	Section
Case $\mathfrak{A}$	Step 1	6.1.1
	Step 2	6.1.2
	Step 3	6.1.3
Case $\mathfrak{B}$	–	6.2
Case $\mathfrak{C}$	Substitution for Step 1	6.1.1 and 6.3.1
	Step 2	6.3.2
	Step 3	6.3.3
Case $\mathfrak{D}$	Steps 1–3	6.4

Step 3 We show that we can reproduce an executable deterministic LOCC transformation from an arbitrary state $|\psi\rangle$ to an arbitrary state $|\psi'\rangle$ by a C-LOCC transformation. Conversely, we can reproduce a C-LOCC transformation by a deterministic LOCC transformation, because a C-LOCC transformation is also a deterministic LOCC transformation. Then, we find that the condition given in Step 2 is also a necessary and sufficient condition of the possibility of a deterministic LOCC transformation and that we can reproduce an arbitrary deterministic LOCC transformation by performing an A-DMT, a B-DMT and a C-DMT, successively.

All the deterministic LOCC transformations are categorized into any of the following cases determined by the initial and final states:

Case $\mathfrak{A}$: Both of the initial and final states are EP definite and the parameter j_{ABC} of the initial state is not zero.

Case $\mathfrak{B}$: The initial state is EP definite and the final state is EP indefinite.

Case $\mathfrak{C}$: The initial state is EP indefinite and the parameter j_{ABC} of the initial state is not zero.

Case $\mathfrak{D}$: The parameter j_{ABC} of the initial state is zero.

We carry out the proof by performing the above three Steps in Cases $\mathfrak{A}$ and $\mathfrak{D}$. In Case $\mathfrak{B}$, we can prove Main Theorems 1 and 2 directly, not following the three Steps. In Case $\mathfrak{C}$, we go to Step 2 directly before Step 3.

The correspondence between the proofs and the sections is shown in Table 3.2.

3.4 Five useful lemmas

In this section, we prove five lemmas which we use to show Main Theorems.

Lemma 1 *An arbitrary n-choice measurement $\{M_{(i)}|i = 1, ..., n\}$ which is operated on the qubit A of an arbitrary three-qubit pure state $|\psi_{ABC}\rangle$ can be reproduced by local unitary operations and two-choice measurements such as $\{M'_{(i)}|i = 1, 2\}$.*

Proof: We show the present Lemma by mathematical induction with respect to n. If $n = 2$, this Lemma clearly holds. We prove the present Lemma for $n = k + 1$, assuming that the present Lemma holds whenever $2 \leq n \leq k$. It suffices to prove that we can reproduce a $(k+1)$-choice measurement $\{M_{(i)} | i = 1, ..., k+1\}$ by performing a k-choice measurement $\{M_{(1)}, ... M_{(k-1)}, \tilde{M}_{(k)}\}$, a two-choice measurement $\{N_{(l)} | l = k, k+1\}$ and local unitary operations successively. In other words, it suffices to define the matrices $\tilde{M}_{(k)}$, $N_{(k)}$ and $N_{(k+1)}$ which satisfy the following equations:

$$M_{(1)}^\dagger M_{(1)} + ... + M_{(k-1)}^\dagger M_{(k-1)} + \tilde{M}_{(k)}^\dagger \tilde{M}_{(k)} = I, \tag{3.79}$$

$$N_{(k)}^\dagger N_{(k)} + N_{(k+1)}^\dagger N_{(k+1)} = I, \tag{3.80}$$

$$\tilde{M}_{(k)}^\dagger N_{(k)}^\dagger N_{(k)} \tilde{M}_{(k)} = M_{(k)}^\dagger M_{(k)}, \tag{3.81}$$

$$\tilde{M}_{(k)}^\dagger N_{(k+1)}^\dagger N_{(k+1)} \tilde{M}_{(k)} = M_{(k+1)}^\dagger M_{(k+1)}. \tag{3.82}$$

Incidentally, it is correct that (3.81) and (3.82) are not in the forms of $N_{(k)} \tilde{M}_{(k)} = M_{(k)}$, but in the above forms. The reason is that the change of the coefficients of the general Schmidt decomposition $\{\lambda_i, \varphi | i = 0, ..., 4\}$ depends only on the components of $M_{(i)}^\dagger M_{(i)}$ and that it is possible to transform to each other two states with the same entanglement parameters by local unitary transformations.

Henceforth, we explicitly give the matrices $\tilde{M}_{(k)}$, $N_{(k)}$ and $N_{(k+1)}$ which satisfy (3.79)–(3.82). First, we define a matrix $\tilde{M}_{(k)}$ as follows:

$$\tilde{M}_{(k)} = \sqrt{M_{(k)}^\dagger M_{(k)} + M_{(k+1)}^\dagger M_{(k+1)}}. \tag{3.83}$$

It is clear that this matrix $\tilde{M}_{(k)}$ satisfies (3.79) because $\{M_{(i)} | i = 1, ..., k+1\}$ is a measurement. We can take the square root of the matrix $M_{(k)}^\dagger M_{(k)} + M_{(k+1)}^\dagger M_{(k+1)}$ in (3.83), because $M_{(k)}^\dagger M_{(k)} + M_{(k+1)}^\dagger M_{(k+1)}$ is a positive operator.

Next, in order to give the expressions of the measurements $N_{(k)}$ and $N_{(k+1)}$, we define $\tilde{V}$ as a subspace spanned by the basis of the eigenspaces of the matrix $\tilde{M}_{(k)}$ with positive eigenvalues. Because the matrix $\tilde{M}_{(k)}$ is a positive operator, the equation $V = \tilde{V} \oplus K$ holds, where V is the total space and K is the kernel of $\tilde{M}_{(k)}$. If a vector $\vec{x} \in V$ satisfies $M_{(k)} \vec{x} \neq \vec{0}$, then $\vec{x} \in \tilde{V}$. (Proof: Let us assume $\vec{x} \notin \tilde{V}$. Then $\vec{x} \in K$, and hence $\vec{x}^\dagger \left(M_{(k)}^\dagger M_{(k)} + M_{(k+1)}^\dagger M_{(k+1)} \right) \vec{x} = 0$. Therefore, $\vec{x}^\dagger M_{(k)}^\dagger M_{(k)} \vec{x} = 0$. This means that $M_{(k)} \vec{x} = \vec{0}$, and thus we have shown the contraposition.) In the same manner, if the vector $\vec{x} \in V$ satisfies $M_{(k+1)} \vec{x} \neq \vec{0}$, then $\vec{x} \in \tilde{V}$.

Now, we define a matrix $\tilde{M}_{(k)}^{\mathrm{r}}$ by restricting the matrix $\tilde{M}_{(k)}$ to the subspace $\tilde{V}$. The matrix $\tilde{M}_{(k)}^{\mathrm{r}}$ is a regular matrix, and hence it has the inverse matrix $(\tilde{M}_{(k)}^{\mathrm{r}})^{-1}$. Similarly, we define the matrices $(M_{(k)}^\dagger M_{(k)})^{\mathrm{r}}$ and $(M_{(k+1)}^\dagger M_{(k+1)})^{\mathrm{r}}$ by restricting the matrices $M_{(k)}^\dagger M_{(k)}$ and $M_{(k+1)}^\dagger M_{(k+1)}$ to the subspace $\tilde{V}$, respectively.

Let the notation $(\vec{e}_1, ..., \vec{e}_l)$ stand for the eigenvectors of $\tilde{M}_{(k)}$ which constitute an orthonormal basis set of the subspace $\tilde{V}$ and let the notation $(\vec{e}_{l+1}, ..., \vec{e}_n)$ stand for an

orthonormal basis set of the kernel K. Then, the set of $(\vec{e}_1, ..., \vec{e}_l, \vec{e}_{l+1}, ..., \vec{e}_n)$ is an orthonormal basis set of the total space V. Hereafter, we carry out the present Lemma's proof in the representation of this basis set.

Let the notation $(\eta_1, ..., \eta_l)$ stand for the set of the nonzero eigenvalues of the matrix $M_{(k)}^\dagger M_{(k)} + M_{(k+1)}^\dagger M_{(k+1)}$. Then we have

$$\tilde{M}_{(k)} = \begin{pmatrix} \begin{matrix} \sqrt{\eta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\eta_l} \end{matrix} & O \\ O & O \end{pmatrix}, \tag{3.84}$$

$$M_k^\dagger M_{(k)} = \begin{pmatrix} (M_{(k)}^\dagger M_{(k)})^{\mathrm{r}} & O \\ O & O \end{pmatrix}, \tag{3.85}$$

$$M_{(k+1)}^\dagger M_{(k+1)} = \begin{pmatrix} (M_{(k+1)}^\dagger M_{(k+1)})^{\mathrm{r}} & O \\ O & O \end{pmatrix}. \tag{3.86}$$

We define the matrices $N_{(k)}$ and $N_{(k+1)}$ as

$$N_{(k)} = \begin{pmatrix} \sqrt{(M_{(k)}^\dagger M_{(k)})^{\mathrm{r}}} (\tilde{M}_{(k)}^{\mathrm{r}})^{-1} & O \\ O & I \end{pmatrix}, \tag{3.87}$$

$$N_{(k+1)} = \begin{pmatrix} \sqrt{(M_{(k+1)}^\dagger M_{(k+1)})^{\mathrm{r}}} (\tilde{M}_{(k)}^{\mathrm{r}})^{-1} & O \\ O & I \end{pmatrix}. \tag{3.88}$$

It follows that

$$N_{(k+1)}^\dagger N_{(k+1)} + N_{(k)}^\dagger N_{(k)} = \begin{pmatrix} \left[(\tilde{M}_{(k)}^{\mathrm{r}})^{-1} \right]^\dagger \left[(M_{(k+1)}^\dagger M_{(k+1)})^{\mathrm{r}} + (M_{(k)}^\dagger M_{(k)})^{\mathrm{r}} \right] \left[\tilde{M}_{(k)}^{\mathrm{r}} \right]^{-1} & O \\ O & I \end{pmatrix}. \tag{3.89}$$

In the present representation, we have from (3.85)

$$((\tilde{M}_{(k)}^{\mathrm{r}})^{-1})^\dagger = \begin{pmatrix} \sqrt{\eta_1^{-1}} & & \\ & \ddots & \\ & & \sqrt{\eta_l^{-1}} \end{pmatrix}, \tag{3.90}$$

$$(M_{(k+1)}^\dagger M_{(k+1)})^{\mathrm{r}} + (M_{(k)}^\dagger M_{(k)})^{\mathrm{r}} = \begin{pmatrix} \eta_1 & & \\ & \ddots & \\ & & \eta_l \end{pmatrix}, \tag{3.91}$$

$$((\tilde{M}_{(k)}^{\mathrm{r}})^{-1}) = \begin{pmatrix} \sqrt{\eta_1^{-1}} & & \\ & \ddots & \\ & & \sqrt{\eta_l^{-1}} \end{pmatrix}. \tag{3.92}$$

Then (3.89) reduces to (3.80). We can also prove the equations (3.81) and (3.82) by straightforward algebra.□
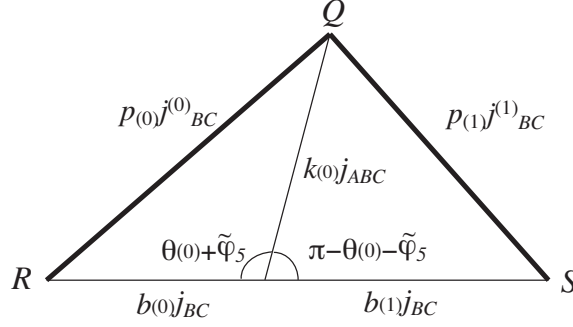
Figure 3.7: A geometric interpretation of the change of j_{BC}.

Thanks to this Lemma, it is possible to substitute two-choice measurements for any measurements of an LOCC transformation on a three-qubit pure state. Hereafter, unless specified otherwise, measurements of LOCC transformations will be two-choice measurements. We will also refer to a two-choice measurement, a two-choice DM and a two-choice DMT simply as a measurement, a DM and a DMT, respectively. Similarly, unless specified otherwise, the notation $\{M_{(i)}\}$ expresses a two-choice measurement. We then express $M_{(0)}^{\dagger}M_{(0)}$ and $M_{(1)}^{\dagger}M_{(1)}$ as

$$M_{(0)}^{\dagger}M_{(0)} = \begin{pmatrix} a_{(0)} & k_{(0)}e^{-i\theta_{(0)}} \\ k_{(0)}e^{i\theta_{(0)}} & b_{(0)} \end{pmatrix} = \begin{pmatrix} a & ke^{-i\theta} \\ ke^{i\theta} & b \end{pmatrix}, \tag{3.93}$$

$$M_{(1)}^{\dagger}M_{(1)} = \begin{pmatrix} a_{(1)} & k_{(1)}e^{-i\theta_{(1)}} \\ k_{(1)}e^{i\theta_{(1)}} & b_{(1)} \end{pmatrix} = \begin{pmatrix} 1 - a & -ke^{-i\theta} \\ -ke^{i\theta} & 1 - b \end{pmatrix} \tag{3.94}$$

in the same manner as in (A.31).

Lemma 2 *Let the notation $\{M_{(i)}|i = 1, 2\}$ stand for an arbitrary two-choice measurement which is operated on the qubit A of a three-qubit pure state $|\psi_{ABC}\rangle$. We refer to each result of the measurements $\{M_{(i)}|i = 1, 2\}$ as $\left|\psi_{ABC}^{(i)}\right\rangle$. Let the notations $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Qe)$ and $(j_{AB}^{(i)}, j_{AC}^{(i)}, j_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)}, Q_e^{(i)})$ stand for the sets of the J-parameters of the states $|\psi_{ABC}\rangle$ and $\left|\psi_{ABC}^{(i)}\right\rangle$, respectively. Then, the following inequalities hold:*

$$j_{BC} \le \sum_{i=0}^{1} p_{(i)}j_{BC}^{(i)} \le \sqrt{j_{BC}^2 + \left[1 - \left(\sum_{k=0}^{1} p_{(i)}\alpha^{(i)}\right)^2\right] j_{ABC}^2}, \tag{3.95}$$

where the probability $p_{(i)}$ and the multiplication factor $\alpha^{(i)}$ are defined in (A.34) and (A.45), respectively.

Proof: The average $\sum_{i=0}^{1} p_{(i)}j_{BC}^{(i)}$ is equal to the length of the heavy line in Fig. 3.7, because we can interpret (A.44) as the cosine theorem and because $b_{(0)} + b_{(1)} = 1$ and $\sum_i \vec{k}_{(i)} = 0$. The end points of the heavy line have to coincide with the end points of
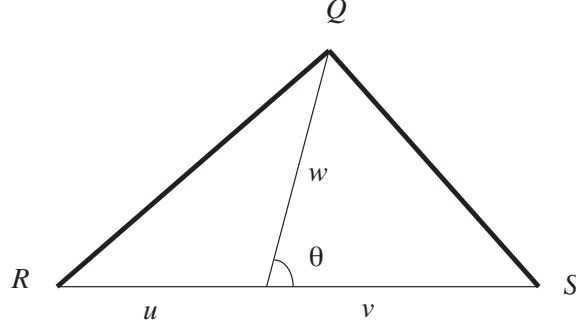
Figure 3.8: Simplification of Fig. 3.7.

the segment RS because $\sum_i \vec{k}_{(i)} = 0$. Then, the left inequality $j_{BC} \leq \sum_{i=1}^{2} p_i j_{BC}^{(i)}$ clearly holds, since a polygonal line is longer than a straight line.

To prove the right inequality of this Lemma, it suffices to show the inequality

$$\sqrt{(bj_{BC} + k\cos\theta j_{ABC})^2 + (k\sin\theta j_{ABC})^2} + \sqrt{[(1-b)j_{BC} - k\cos\theta j_{ABC}]^2 + (k\sin\theta j_{ABC})^2}$$
$$\leq \sqrt{j_{BC}^2 + \{1 - [\sqrt{ab - k^2} + \sqrt{(1-a)(1-b) - k^2}]^2\}j_{ABC}^2} \qquad (3.96)$$

under the conditions $ab - k^2 \geq 0$, $(1-a)(1-b) - k^2 \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq a \leq 1$ and $0 \leq b \leq 1$, where we used the substitutions of the measurement parameters in (A.32) and (A.33):

$$a_{(0)} = a, \quad a_{(1)} = 1 - a, \quad b_{(0)} = b, \quad b_{(i)} = 1 - b, \quad \theta_{(0)} + \tilde{\varphi}_5 = \pi - \theta. \qquad (3.97)$$

The fact that θ can take any value guarantees the last substitution.

Let us find the value of θ which maximizes the left-hand side of (3.96) with the values of the measurement parameters a, b and k fixed. For this purpose, we find the value of the measurement parameter θ which maximizes the length of the heavy line in Fig. 3.8,

$$f(\theta) = \sqrt{u^2 + w^2 + 2uw\cos\theta} + \sqrt{v^2 + w^2 - 2vw\cos\theta}, \qquad (3.98)$$

with the values of u, v and w fixed. Differentiating (3.98) with respect to θ gives the following equation:

$$\frac{\partial f}{\partial \theta} = \frac{-uw\sin\theta}{\sqrt{u^2 + w^2 + 2uw\cos\theta}} + \frac{vw\sin\theta}{\sqrt{v^2 + w^2 - 2vw\cos\theta}}. \qquad (3.99)$$

The equation (3.99) is equal to zero only if $\theta = 0$ or π, or $\cos\theta = w(u-v)/2uv$. The length of the heavy line for $\theta = \pi/2$ is longer than that for $\theta = 0$ and π. Thus, when $\cos\theta = w(u-v)/2uv$, the function f is maximized to be $(u+v)\sqrt{1 + w^2/(uv)}$. Substituting $u = bj_{BC}$, $v = (1-b)j_{BC}$ and $w = kj_{ABC}$ give that

$$\text{the maximum of the left-hand side of (3.96)} = \sqrt{j_{BC}^2 + \frac{k^2 j_{ABC}^2}{b(1-b)}}. \qquad (3.100)$$

Hence, in order to prove Lemma 2, it suffices to show

(the right-hand side of (3.96))^2 $-$ (the maximum of the left-hand side of (3.96))^2

$$= \left\{ 1 - \left[ab - k^2 + (1-a)(1-b) - k^2 + 2\sqrt{ab-k^2}\sqrt{(1-a)(1-b)-k^2} \right] \right\} j_{ABC}^2 - \frac{k^2 j_{ABC}^2}{b(1-b)}$$

$$= \left[a + b - 2ab + 2k^2 - \frac{k^2}{b(1-b)} - 2\sqrt{ab-k^2}\sqrt{(1-a)(1-b)-k^2} \right] j_{ABC}^2 \geq 0, \qquad (3.101)$$

or to show

$$a + b - 2ab + 2k^2 - \frac{k^2}{b(1-b)} \geq 2\sqrt{ab-k^2}\sqrt{(1-a)(1-b)-k^2}. \qquad (3.102)$$

We can rewrite the left-hand side of (3.102) as

$$
\begin{aligned}
a + b - 2ab + 2k^2 - \frac{k^2}{b(1-b)} &= a(1-b) + b(1-a) - \frac{2b^2 - 2b + 1}{b(1-b)} k^2 \\
&= a(1-b) + b(1-a) - \left(\frac{b}{1-b} + \frac{1-b}{b} \right) k^2 \\
&= b \frac{(1-a)(1-b) - k^2}{1-b} + (1-b) \frac{ab - k^2}{b}. \qquad (3.103)
\end{aligned}
$$

Because $ab - k^2 \geq 0$ and $(1-a)(1-b) - k^2 \geq 0$, we find that the left-hand side of (3.102) is nonnegative. The right-hand side of (3.102) is clearly nonnegative. Therefore, (3.102) is equivalent to $g \geq 0$, where

$$g = \text{(the left-hand side of } (3.102))^2 - \text{(the right-hand side of } (3.102))^2. \qquad (3.104)$$

We can simplify the expression of g as follows:

$$
\begin{aligned}
g &= [a(1-b) + b(1-a)]^2 - 2[a(1-b) + b(1-a)] \frac{b^2 + (1-b)^2}{b(1-b)} k^2 + \frac{[b^2 + (1-b)^2]^2}{b^2(1-b)^2} k^4 \\
&= [a(1-b) - b(1-a)]^2 + 2[a(1-b) - b(1-a)] \frac{b^2 - (1-b)^2}{b(1-b)} k^2 + \frac{[b^2 - (1-b)^2]^2}{b^2(1-b)^2} k^4 \\
&= \left[(a-b) + k^2 \frac{(b^2 - (1-b)^2)}{b(1-b)} \right]^2. \qquad (3.105)
\end{aligned}
$$

Thus, the quantity g is nonnegative. Hence we have proved the right inequality of (3.95) and thereby completed the proof of Lemma 2.$\square$

Lemma 2 has the following corollary:

Corollary 1 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the J-parameters of $|\psi\rangle$ and $|\psi'\rangle$ as $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j'_{AB}, j'_{AC}, j'_{BC}, j'_{ABC}, J'_5, Q'_e)$, respectively. If a two-choice A-DMT transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is possible, the following three inequalities hold:*

$$j_{BC}'^2 \sin^2 \varphi'_5 \geq j_{BC}^2 \sin^2 \varphi_5, \qquad (3.106)$$

$$j_{BC}^2 \leq K_{BC}' \equiv j_{BC}'^2 + j_{ABC}'^2 \leq K_{BC} \equiv j_{BC}^2 + j_{ABC}^2, \tag{3.107}$$

$$\Delta'_{norm} \equiv \frac{\Delta'_J}{\alpha^4} = K_5^2 - 4K_{AB}K_{AC}K_{BC}' \geq \Delta_J, \tag{3.108}$$

where (3.106) holds only when both of the states $|\psi\rangle$ and $|\psi'\rangle$ are EP definite, and where $\alpha = j'_{ABC}/j_{ABC}$. The symbol Δ'_{norm} refers to a normalized value of Δ'_J.

Proof: Because of Lemma 2, (A.40)–(A.42) and because the entanglement parameters of the final states of a DMT are the same, the expressions

$$j_{BC} \leq j'_{BC} \leq \sqrt{j_{BC}^2 + (1 - \alpha^2)j_{ABC}^2}, \tag{3.109}$$

$$j'^2_{AB} = \alpha^2 j^2_{AB}, \tag{3.110}$$

$$j'^2_{AC} = \alpha^2 j^2_{AC} \tag{3.111}$$

hold, where $\alpha \equiv j'_{ABC}/j_{ABC}$. Because of (A.48) and because the entanglement parameters of the final states of a DMT are the same, we have

$$j_{BC} \cos \varphi_5 = j'_{BC} \cos \varphi'_5. \tag{3.112}$$

From (3.109) and (3.112), we can derive (3.106) and (3.107) as follows:

$$j'^2_{BC} \sin^2 \varphi'_5 = j'^2_{BC} - j'^2_{BC} \cos^2 \varphi'_5 \geq j^2_{BC} - j^2_{BC} \cos^2 \varphi_5 = j^2_{BC} \sin^2 \varphi_5, \tag{3.113}$$

$$j_{BC}^2 \leq j'^2_{BC} \leq j'^2_{BC} + j'^2_{ABC} = K'_{BC} \leq j_{BC}^2 + (1-\alpha^2)j^2_{ABC} + \alpha^2 j^2_{ABC}$$
$$= j^2_{BC} + j^2_{ABC} = K_{BC}. \tag{3.114}$$

Because of $J_5 = 2j_{AB}j_{AC}j_{BC}\cos\varphi_5$ and (3.109)–(3.112), we obtain

$$J_5' = \alpha^2 J_5. \tag{3.115}$$

From (3.110), (3.111), (3.115) and definition of the α, we obtain

$$K'_{AB} = \alpha^2 K_{AB}, \; K'_{AC} = \alpha^2 K_{AC}, \; K'_5 = \alpha^2 K_5. \tag{3.116}$$

From (3.114) and (3.116), we obtain (3.108):

$$\Delta'_{\text{norm}} \equiv \frac{\Delta'_J}{\alpha^4} = \frac{K'^2_5 - 4K'_{AB}K'_{AC}K'_{BC}}{\alpha^2}$$
$$= K_5^2 - 4K_{AB}K_{AC}K'_{BC} \geq K_5^2 - 4K_{\text{ap}} = \Delta_J. \tag{3.117}$$

This completes the proof of Corollary 1. □

Lemma 3 *For an arbitrary two-choice measurement $\{M_{(i)}|i = 0, 1\}$, the following inequality holds:*

$$\sum_{k=0}^{1} p_{(i)}\alpha^{(i)} \leq 1, \tag{3.118}$$

where $p_{(i)}$ and $\alpha^{(i)}$ are defined by (A.34) and (A.45), respectively.

Proof: Since the measurement $\{M_{(i)}|i = 0, 1\}$ is a two-choice measurement, we use the measurement parameters a, b, k and θ defined in (A.32) and (A.33). Using these parameters, we can express (3.118) to be proved as follows:

$$\sqrt{ab - k^2} + \sqrt{(1-a)(1-b) - k^2} \leq 1. \tag{3.119}$$

In order to prove (3.119), it suffices to show the following inequality because k^2 is non-negative:

$$\sqrt{ab} + \sqrt{(1-a)(1-b)} \leq 1, \tag{3.120}$$

which is followed by

$$ab + (1-a)(1-b) + 2\sqrt{ab(1-a)(1-b)} \leq 1,$$
$$2\sqrt{ab(1-a)(1-b)} \leq 1 - ab - (1-a)(1-b) = a(1-b) + b(1-a). \tag{3.121}$$

The both sides of (3.121) are nonnegative, and thus it suffices to show

$$(a + b - 2ab)^2 \geq 4ab(1-a)(1-b) \tag{3.122}$$

in order to prove the present Lemma, which can be achieved as follows:

$$\begin{aligned} &(a + b - 2ab)^2 - 4ab(1-a)(1-b) \\ = \ &a^2 + b^2 + 2ab + 4a^2b^2 - 4a^2b - 4ab^2 - 4ab + 4a^2b + 4ab^2 - 4a^2b^2 \\ = \ &(a - b)^2 \geq 0. \end{aligned} \tag{3.123}$$

Thus, (3.122) holds, so does (3.118). $\square$

Lemma 4 *Let the notations $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j'_{AB}, j'_{AC}, j'_{BC}, j'_{ABC}, J'_5, Q'_e)$ stand for the sets of the J-parameters of arbitrary three-qubit pure states $|\phi\rangle$ and $|\phi'\rangle$, respectively. If $\Delta_J = 0 \wedge Q'_e = 0$ or $|\phi'\rangle$ is EP-indefinite and if there is a parameter α which satisfies $0 \leq \alpha \leq 1$ and*

$$\begin{pmatrix} j'^2_{AB} \\ j'^2_{AC} \\ j'^2_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \begin{pmatrix} \alpha^2 & & & & \\ & \alpha^2 & & & \\ & & 1 & 1 - \alpha^2 & \\ & & & \alpha^2 & \\ & & & & \alpha^2 \end{pmatrix} \begin{pmatrix} j^2_{AB} \\ j^2_{AC} \\ j^2_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}, \tag{3.124}$$

then we can carry out an A-DMT from the pure state $|\phi\rangle$ to the pure state $|\phi'\rangle$ by the following measurement:

$$M^\dagger_{(0)}M_{(0)} = \begin{pmatrix} a_{(0)} & k_{(0)}e^{-i\theta_{(0)}} \\ k_{(0)}e^{i\theta_{(0)}} & b_{(0)} \end{pmatrix} = \begin{pmatrix} a & ke^{-i\theta} \\ ke^{i\theta} & b \end{pmatrix}, \tag{3.125}$$

$$M^\dagger_{(1)}M_{(1)} = \begin{pmatrix} a_{(1)} & k_{(1)}e^{-i\theta_{(1)}} \\ k_{(1)}e^{i\theta_{(1)}} & b_{(1)} \end{pmatrix} = \begin{pmatrix} 1 - a & -ke^{-i\theta} \\ -ke^{i\theta} & 1 - b \end{pmatrix}, \tag{3.126}$$

where the measurement parameters a, b, k and θ are defined as follows:

$$a = \frac{1}{2} - \frac{(1 - 2\alpha^2)\lambda_1 \sin\varphi}{2\sqrt{\lambda_1^2 \sin^2\varphi + \lambda_0^2(1 - \alpha^2)}}, \tag{3.127}$$
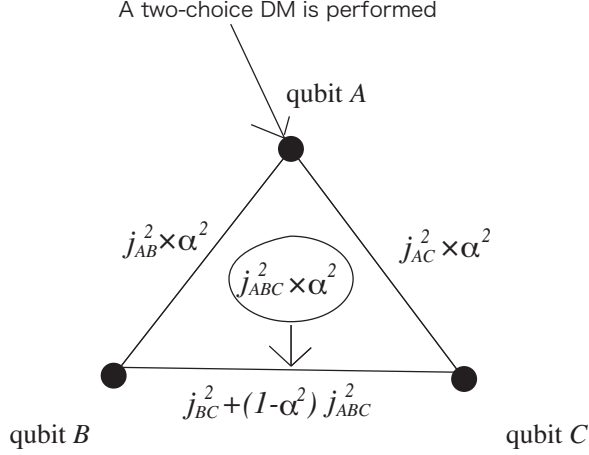
Figure 3.9: Dissipationless entanglement transfer.

$$b = \frac{1}{2} + \frac{\lambda_1 \sin \varphi}{2\sqrt{\lambda_1^2 \sin^2 \varphi + \lambda_0^2(1 - \alpha^2)}}, \tag{3.128}$$

$$k = \frac{\lambda_0(1 - \alpha^2)}{2\sqrt{\lambda_1^2 \sin^2 \varphi + \lambda_0^2(1 - \alpha^2)}}, \tag{3.129}$$

$$\theta = \frac{\pi}{2}. \tag{3.130}$$

If $\lambda_1^2 \sin^2 \varphi + \lambda_0^2(1 - \alpha^2) = 0$, the parameters a, b and k are defined by $a = b = 1/2$ and $k = \sqrt{1 - \alpha^2}/2$.

Comment: We can interpret the above as the rule that describes how a DMT changes the entanglement. When we perform an A-DMT which satisfies the condition of Lemma 4, the change of the entanglement is given by (3.124). We can express this change as in Fig. 3.9. After an A-DMT, the four entanglement parameters, j_{AB}^2, j_{AC}^2, j_{ABC}^2 and J_5, the last of which does not appear in Fig. 3.9, are multiplied by α^2. Note that these four entanglement parameters are related to the qubit A, which is the measured qubit in the A-DMT. The quantity $(1-\alpha^2)j_{ABC}^2$, which is the entanglement lost from j_{ABC}^2, is added to j_{BC}^2, which is the only entanglement parameter that is not related to the measured qubit A. We call this phenomenon the dissipationless entanglement transfer and call the DMT which gives rise to the dissipationless entanglement transfer as a disspationless DMT. (We will present the dissipative entanglement transfer in section 6.1.1 below.) Note that Lemma 4 holds even if $j_{ABC} = 0$.

Proof: We prove the present Lemma by calculating the change of the J-parameters due to the measurement which satisfies (3.125)–(3.130), and showing that the change is expressed as (3.124). We define the normalized states $\{|\phi^{(i)}\rangle\}$ as $|\phi^{(i)}\rangle = M_{(i)} |\phi\rangle / \sqrt{p_{(i)}}$ and let the notation $(j_{AB}^{(i)}, j_{AC}^{(i)}, j_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)}, Q_e^{(i)})$ stand for the set of the J-parameters of $|\phi^{(i)}\rangle$. From (A.34)–(A.36) and (A.40)–(A.46), we derive expressions of the J-parameters $(j_{AB}^{(i)}, j_{AC}^{(i)}, j_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)}, Q_e^{(i)})$ in terms of the J-parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e^{(i)})$ and $\alpha^{(i)}$. We then show that $(j_{AB}^{(0)}, j_{AC}^{(0)}, j_{BC}^{(0)}, j_{ABC}^{(0)}, J_5^{(0)}, Q_e^{(0)})$ and $(j_{AB}^{(1)}, j_{AC}^{(1)}, j_{BC}^{(1)}, j_{ABC}^{(1)}, J_5^{(1)}, Q_e^{(1)})$

are equal to $(j'_{AB}, j'_{AC}, j'_{BC}, j'_{ABC}, J'_5, Q'_e)$, which is related to $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ as in (3.124).

First, let us derive the equation $p_{(i)} = b_{(i)}$. For $i = 0$, we obtain the equation $p_{(0)} = b_{(0)} =: b$ by substituting (3.127)–(3.130) in (A.34) and transforming it as follows:

$$
\begin{aligned}
p_{(0)} &= \lambda_0^2 \left[\frac{1}{2} - \frac{(1 - 2\alpha^2)\lambda_1 \sin\varphi}{2\sqrt{\lambda_1^2 \sin^2\varphi + \lambda_0^2(1 - \alpha^2)}} \right] + (1 - \lambda_0^2) \left[\frac{1}{2} + \frac{\lambda_1 \sin\varphi}{2\sqrt{\lambda_1^2 \sin^2\varphi + \lambda_0^2(1 - \alpha^2)}} \right] \\
&\quad + 2\lambda_0\lambda_1 \frac{\lambda_0(1 - \alpha^2)\sin\varphi}{2\sqrt{\lambda_1^2 \sin^2\varphi + \lambda_0^2(1 - \alpha^2)}} \\
&= \frac{1}{2} + \frac{\lambda_1 \sin\varphi}{2\sqrt{\lambda_1^2 \sin^2\varphi + \lambda_0^2(1 - \alpha^2)}} [-(1 - 2\alpha^2)\lambda_0^2 + (1 - \lambda_0^2) + 2\lambda_0^2(1 - \alpha^2)] \\
&= \frac{1}{2} + \frac{\lambda_1 \sin\varphi}{2\sqrt{\lambda_1^2 \sin^2\varphi + \lambda_0^2(1 - \alpha^2)}} = b.
\end{aligned}
\tag{3.131}
$$

From the equations $p_{(0)} + p_{(1)} = 1$ and $b_{(0)} + b_{(1)} = 1$, we also find that $p_{(1)} = b_{(1)} = 1 - b$. Hence we obtain the following equation:

$$
p_{(i)} = b_{(i)}.
\tag{3.132}
$$

Next, we derive the first, second and fourth rows of (3.124). Because of (A.40)–(A.42), it suffices to prove that $\alpha^{(i)} = \alpha$. The equation (A.45) gives

$$
p_{(i)}\alpha_{(i)} = \sqrt{a_{(i)}b_{(i)} - k_{(i)}^2}.
\tag{3.133}
$$

For $i = 0$, we obtain the equation $p_{(0)}^2 \alpha_{(0)}^2 = \alpha^2 b^2$ by substituting (3.127)–(3.130) in (3.133) and transforming it as follows:

$$
\begin{aligned}
p_{(0)}^2\alpha_{(0)}^2 &= a_{(0)}b_{(0)} - k_{(0)}^2 \\
&= \frac{1}{4} + \frac{\alpha^2\lambda_1\sin\varphi}{2\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)}} - \frac{(1 - 2\alpha^2 + \alpha^4)\lambda_0^2 + (1 - 2\alpha^2)\lambda_1^2\sin^2\varphi}{4[\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)]} \\
&= \frac{\alpha^2\lambda_1\sin\varphi}{2\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)}} + \frac{\alpha^2(1-\alpha^2)\lambda_0^2 + 2\alpha^2\lambda_1^2\sin^2\varphi}{4[\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)]} \\
&= \alpha^2 \left[\frac{1}{2} + \frac{\lambda_1\sin\varphi}{2\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)}} \right]^2 = \alpha^2 b^2.
\end{aligned}
\tag{3.134}
$$

Thus, the equation $p_{(0)} = b$ is followed by the equation $\alpha_{(0)} = \alpha$. The equations $a_{(0)} + a_{(1)} = 1$ and $b_{(0)} + b_{(1)} = 1$ then give that $p_{(1)}^2\alpha_{(1)}^2 = \alpha^2 b_{(1)}^2$ as follows:

$$
\begin{aligned}
p_{(1)}^2\alpha_{(1)}^2 &= a_{(1)}b_{(1)} - k_{(1)}^2 \\
&= (1 - a_{(0)})(1 - b_{(0)}) - k_{(0)}^2 \\
&= 1 - a_{(0)} - b_{(0)} + a_{(0)}b_{(0)} - k_{(0)}^2 \\
&= -\frac{2\alpha^2\lambda_1\sin\varphi}{2\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)}} + \alpha^2 \left[\frac{1}{2} + \frac{\lambda_1\sin\varphi}{2\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)}} \right]^2 \\
&= \alpha^2 \left[\frac{1}{2} - \frac{\lambda_1\sin\varphi}{2\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)}} \right]^2 = \alpha^2(1 - b_{(0)})^2 = \alpha^2 b_{(1)}^2.
\end{aligned}
\tag{3.135}
$$

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Thus, from (3.132), the equation $\alpha_{(1)} = \alpha$ holds. Hence, we obtain the equation $\alpha_{(i)} = \alpha$. Therefore, we have proven the first, second and fourth rows of (3.124).

Next, we derive the third row $(j_{BC}^{(i)})^2 = j'^2_{BC} = j^2_{BC} + (1 - \alpha^2)j^2_{ABC}$. For $i = 0$, we obtain the equation $(j_{BC}^{(i)})^2 = j^2_{BC} + (1 - \alpha^2)j^2_{ABC}$ by substituting (3.127)–(3.132) in (A.44) and transforming it as follows:

$$
\begin{aligned}
(j_{BC}^{(0)})^2 &= \frac{b_{(0)}^2 j^2_{BC} + \lambda_0^2\lambda_4^2 k_{(0)}^2 - 2b_{(0)}k_{(0)}\lambda_0\lambda_4 j_{BC}\cos(\theta_{(0)} + \tilde\varphi_5)}{p_{(0)}^2} \\
&= \frac{b_{(0)}^2 j^2_{BC} + \lambda_0^2\lambda_4^2 k_{(0)}^2 + 2b_{(0)}k_{(0)}\lambda_0\lambda_1\lambda_4^2 \sin\varphi}{b_{(0)}^2} \\
&= j^2_{BC} + j^2_{ABC}\left(\frac{k_{(0)}^2}{b_{(0)}^2} + \frac{2k_{(0)}\lambda_1\sin\varphi}{\lambda_0 b_{(0)}}\right) \\
&= j^2_{BC} + j^2_{ABC}\left\{\left[\frac{\lambda_0(1-\alpha^2)}{\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)} + \lambda_1\sin\varphi}\right]^2 + \frac{2\lambda_1\sin\varphi(1-\alpha^2)}{\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)} + \lambda_1\sin\varphi}\right] \\
&= j^2_{BC} + j^2_{ABC}\frac{\lambda_0^2(1-\alpha^2)^2 + 2(1-\alpha^2)\lambda_1^2\sin^2\varphi + 2\lambda_1\sin\varphi(1-\alpha^2)\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)}}{[\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)} + \lambda_1\sin\varphi]^2} \\
&= j^2_{BC} + j^2_{ABC}(1-\alpha^2)\left[\frac{\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)} + \lambda_1\sin\varphi}{\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)} + \lambda_1\sin\varphi}\right]^2 = j^2_{BC} + j^2_{ABC}(1-\alpha^2). \quad (3.136)
\end{aligned}
$$

We can also apply this procedure to the case of $i = 1$:

$$
\begin{aligned}
(j_{BC}^{(1)})^2 &= \frac{(1-b_{(0)})^2 j^2_{BC} + \lambda_0^2\lambda_4^2 k_{(0)}^2 - 2(1-b_{(0)})k_{(0)}\lambda_0\lambda_4 j_{BC}\cos(\theta_{(1)} + \tilde\varphi_5)}{(1-p_{(0)})^2} \\
&= \frac{(1-b_{(0)})^2 j^2_{BC} + \lambda_0^2\lambda_4^2 k_{(0)}^2 - 2(1-b_{(0)})k_{(0)}\lambda_0\lambda_1\lambda_4^2 \sin\varphi}{(1-b_{(0)})^2} \\
&= j^2_{BC} + j^2_{ABC}\left[\frac{k_{(0)}^2}{(1-b_{(0)})^2} - \frac{2k_{(0)}\lambda_1\sin\varphi}{\lambda_0(1-b_{(0)})}\right] \\
&= j^2_{BC} + j^2_{ABC}\left\{\left[\frac{\lambda_0(1-\alpha^2)}{\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)} - \lambda_1\sin\varphi}\right]^2 - \frac{2\lambda_1\sin\varphi(1-\alpha^2)}{\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)} - \lambda_1\sin\varphi}\right\} \\
&= j^2_{BC} + j^2_{ABC}\frac{\lambda_0^2(1-\alpha^2)^2 + 2(1-\alpha^2)\lambda_1^2\sin^2\varphi - 2\lambda_1\sin\varphi(1-\alpha^2)\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)}}{[\sqrt{\lambda_1^2\sin^2\varphi - \lambda_0^2(1-\alpha^2)} + \lambda_1\sin\varphi]^2} \\
&= j^2_{BC} + j^2_{ABC}(1-\alpha^2)\left[\frac{\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)} - \lambda_1\sin\varphi}{\sqrt{\lambda_1^2\sin^2\varphi + \lambda_0^2(1-\alpha^2)} - \lambda_1\sin\varphi}\right]^2 = j^2_{BC} + j^2_{ABC}(1-\alpha^2). \quad (3.137)
\end{aligned}
$$

Thus the equation $(j_{BC}^{(1)})^2 = j^2_{BC} + (1 - \alpha^2)j^2_{ABC}$ holds. Hence, we obtain the third row of (3.124), $(j_{BC}^{(i)})^2 = j'^2_{BC} = j^2_{BC} + (1 - \alpha^2)j^2_{ABC}$.

Next, let us show the fifth row $J_5^{(i)} = J'_5 = \alpha^2 J_5$. From the equation $J_5 = 2j_{AB}j_{AC}(\lambda_2\lambda_3 - \lambda_1\lambda_4\cos\varphi)$, we obtain the equation $J_5^{(i)} = \alpha^2 J_5$ for $i = 0$ and 1 by using the equations

(A.35)–(A.41), $\alpha^{(i)} = \alpha$, $p_{(i)} = b_{(i)}$ and $\theta_{(i)} = \pm\pi/2$ as follows:

$$
\begin{aligned}
J_5^{(i)} &= 2j_{AB}^{(i)}j_{AC}^{(i)}(\lambda_2^{(i)}\lambda_3^{(i)} - \lambda_1^{(i)}\lambda_4^{(i)}\cos\varphi^{(i)}) \\
&= \alpha^2 2j_{AB}j_{AC}(\lambda_2^{(i)}\lambda_3^{(i)} - \lambda_1^{(i)}\lambda_4^{(i)}\cos\varphi^{(i)}) \\
&= \alpha^2 2j_{AB}j_{AC}\left[\lambda_2\lambda_3 - \frac{\lambda_4(\lambda_0 k_{(i)}\cos\theta_{(i)} + \lambda_1 b_{(i)}\cos\varphi)}{b_{(i)}}\right] \\
&= \alpha^2 2j_{AB}j_{AC}(\lambda_2\lambda_3 - \lambda_1\lambda_4\cos\varphi) \\
&= \alpha^2 J_5, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (3.138)
\end{aligned}
$$

which is the proof of the fifth row of (3.124).

We have already seen that the measurement $\{M_{(i)}\}$ which satisfies (3.127)–(3.130) causes the change of the J-parameters (3.124). Finally, we prove that if $\Delta_J = 0 \wedge Q'_e = 0$ or $|\phi'\rangle$ is EP-indefinite, the measurement $\{M_{(i)}\}$ transforms the state $|\phi\rangle$ into the state $|\phi'\rangle$. If the state $|\phi'\rangle$ is EP-indefinite, the state $|\phi'\rangle$ is determined uniquely only by determining the J-parameters, because any EP-definite state has a zero entanglement charge. Thus, if the state $|\phi'\rangle$ is EP-indefinite, the measurement $\{M_{(i)}\}$ transforms the state $|\phi\rangle$ into the state $|\phi'\rangle$. If the expression $\Delta_J = 0 \wedge Q'_e = 0$ holds, we only have to prove that $Q_e^{(i)} = 0$, because if the equation $Q_e^{(i)} = 0$ holds J-parameters and entanglement charge of $|\phi^{(i)}\rangle$ and $|\phi'\rangle$ are the same. Let us show the equation $Q_e^{(i)} = 0$. Because of $p_{(i)} = b_{(i)}$, $\alpha^{(i)} = \alpha$, (3.124) and (A.35), the following equation holds:

$$
Q_e^{(i)} = \mathrm{sgn}\left[\sin\varphi^{(i)}\left(\frac{b_{(i)}\alpha_{(i)}^2}{p_{(i)}}\lambda_0^2 - \frac{K_5^{(i)}}{2K_{BC}^{(i)}}\right)\right] = \mathrm{sgn}\left[\sin\varphi^{(i)}\alpha^2\left(\lambda_0^2 - \frac{K_5}{2K_{BC}}\right)\right]. \quad (3.139)
$$

If $\Delta_J = 0$, then $\lambda_0^2 - K_5/2K_{BC} = 0$, and hence $Q_e^{(i)} = 0 = Q'_e$. Thus, if the expression $\Delta_J = 0 \wedge Q'_e = 0$ holds, the measurement $\{M_{(i)}\}$ transforms the state $|\phi\rangle$ into the state $|\phi'\rangle$.

We have thus proven that an A-DMT which satisfies (3.124) can be performed by the measurement $\{M_{(i)}|i = 0, 1\}$. This completes the proof of Lemma 4. $\square$

Lemma 4 guarantees that an arbitrary dissipationless DMT is executable. A dissipationless DMT has only a single parameter α. Hereafter, we refer to this parameter as the transfer parameter.

Lemma 5 *Let the notation $\{M_{(i)}|i = 0, 1\}$ stand for an arbitrary two-choice measurement which is operated on a qubit of a three-qubit pure state $|\psi_{ABC}\rangle$. Note that we can operate $\{M_{(i)}|i = 0, 1\}$ on any one of the qubits A, B and C of the state $|\psi_{ABC}\rangle$. We refer to each result of $\{M_{(i)}|i = 0, 1\}$ as $\left|\psi_{ABC}^{(i)}\right\rangle$. Let the notations $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5, Q_e)$ and $(K_{AB}^{(i)}, K_{AC}^{(i)}, K_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)}, Q_e^{(i)})$ stand for the sets of the K-parameters of the states $|\psi_{ABC}\rangle$ and $\left|\psi_{ABC}^{(i)}\right\rangle$, respectively. Then, the following inequality holds:*

$$
\sum_{i=0}^{1} p_{(i)}\sqrt{K_{BC}^{(i)}} \leq \sqrt{K_{BC}}. \quad\quad\quad\quad\quad\quad (3.140)
$$

Proof: First, we prove (3.140) in the case of $j_{ABC} \neq 0$. If the measurement $\{M_{(i)}|i = 0, 1\}$ is performed on the qubit B or C, then $j_{BC}^{(i)} = \alpha^{(i)} j_{BC}$ and $j_{ABC}^{(i)} = \alpha^{(i)} j_{ABC}$, where $\alpha^{(i)}$ is the transfer paremter of the measurement $\{M_{(i)}|i = 0, 1\}$. Thus, because of (A.50), we can obtain (3.140) as follows:

$$\sum_{i=0}^{1} p_{(i)} \sqrt{K_{BC}^{(i)}} = \sum_{i=0}^{1} p_{(i)} \sqrt{(j_{BC}^{(i)})^2 + (j_{ABC}^{(i)})^2} = \sum_{i=0}^{1} p_{(i)} \alpha^{(i)} \sqrt{j_{BC}^2 + j_{ABC}^2} \leq \sqrt{K_{BC}}$$

(3.141)

Now, it suffices to prove (3.140) in the case that the first measurement is performed on the qubit A. Let the notation f stand for the left-hand side of (3.140). In the same manner as in Lemma 2, (A.42) and (A.44) give that

$$
\begin{aligned}
f \;=\; & \sqrt{b^2 j_{BC}^2 + 2bk \cos(\pi - \theta - \tilde{\varphi}_5) j_{BC} j_{ABC} + ab j_{ABC}^2} \\
& + \sqrt{(1-b)^2 j_{BC}^2 - 2(1-b)k \cos(\pi - \theta - \tilde{\varphi}_5) j_{BC} j_{ABC} + (1-a)(1-b) j_{ABC}^2}.
\end{aligned}
$$

(3.142)

We can substitute θ for the phase $\pi - \theta - \tilde{\varphi}_5$, because the range of the phase θ is from 0 to 2π:

$$
\begin{aligned}
f \;=\; & \sqrt{b^2 j_{BC}^2 + 2bk \cos\theta\, j_{BC} j_{ABC} + ab j_{ABC}^2} \\
& + \sqrt{(1-b)^2 j_{BC}^2 - 2(1-b)k \cos\theta\, j_{BC} j_{ABC} + (1-a)(1-b) j_{ABC}^2}. \quad (3.143)
\end{aligned}
$$

In order to find the maximization condition of the quantity f, we differentiate f with respect to $k \cos\theta$:

$$
\begin{aligned}
\frac{\partial f}{\partial(k \cos\theta)} \;=\; & \frac{b j_{BC} j_{ABC}}{\sqrt{b^2 j_{BC}^2 + 2bk \cos\theta\, j_{BC} j_{ABC} + ab j_{ABC}^2}} \\
& - \frac{(1-b) j_{BC} j_{ABC}}{\sqrt{(1-b)^2 j_{BC}^2 - 2(1-b)k \cos\theta\, j_{BC} j_{ABC} + (1-a)(1-b) j_{ABC}^2}}.
\end{aligned}
$$

(3.144)

We can transform the equation $\partial f/\partial(k \cos\theta) = 0$ as follows:

$$
\begin{aligned}
\frac{(1-b)^2}{(1-b)^2 j_{BC}^2 - 2(1-b)k \cos\theta\, j_{BC} j_{ABC} + (1-a)(1-b) j_{ABC}^2} &= \frac{b^2}{b^2 j_{BC}^2 + 2bk \cos\theta\, j_{BC} j_{ABC} + ab j_{ABC}^2}, \\
2b(1-b)^2 k \cos\theta\, j_{BC} j_{ABC} + ab(1-b)^2 j_{ABC}^2 &= -2b^2(1-b)k \cos\theta\, j_{BC} j_{ABC} + (1-a)(1 \\
2k \cos\theta\, j_{BC} j_{ABC} &= (b-a) j_{ABC}^2.
\end{aligned}
$$

Thus, the quantity f becomes the extremum

$$f = \sqrt{K_{BC}} = \sqrt{j_{BC}^2 + j_{ABC}^2}$$

(3.146)

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if and only if (3.145) holds. This extremum is also the maximum because

$$
\frac{\partial^2 f}{\partial (k\cos\theta)^2} = -\frac{b^2 j_{BC}^2 j_{ABC}^2}{\sqrt{b^2 j_{BC}^2 + 2bk\cos\theta j_{BC}j_{ABC} + abj_{ABC}^2}^3}
$$
$$
- \frac{(1-b)^2 j_{BC}^2 j_{ABC}^2}{\sqrt{(1-b)^2 j_{BC}^2 - 2(1-b)k\cos\theta j_{BC}j_{ABC} + (1-a)(1-b)j_{ABC}^2}^3} \le 0.
$$
(3.147)

Hence, the quantity f becomes the maximum $\sqrt{j_{BC}^2 + j_{ABC}^2}$ if $2k\cos\theta j_{BC}j_{ABC} = (b-a)j_{ABC}^2$. The inequalities $ab - k^2 \ge 0$ and $(1-a)(1-b) - k^2 \ge 0$ limit the range of $k\cos\theta$, but this condition can only decrease the maximum of f because there is only one value of $k\cos\theta$ which satisfies the equation $\partial f / \partial (k\cos\theta) = 0$ in $-\infty \le k\cos\theta \le \infty$. Therefore, we have proven (3.140) for $j_{ABC} \ne 0$.

Second, we prove (3.140) in the case of $j_{ABC} = 0$. In the same manner as in the case of $j_{ABC} \ne 0$, it suffices to prove (3.140) in the case that the first measurement is performed on the qubit A. Because of the equation (A.42), the equation $j_{ABC}^{(i)} = 0$ holds. Thus, we only have to prove the following inequality:

$$
\sum_{i=0}^{1} p_{(i)} j_{BC}^{(i)} \le j_{BC},
$$
(3.148)

because $K_{BC}^{(i)} = (j_{BC}^{(i)})^2 + (j_{ABC}^{(i)})^2$ Substituting $j_{ABC} = 0$ in (A.51), we find that (3.148) clearly holds. Thus, the inequality $f \le \sqrt{j_{BC}^2 + j_{ABC}^2} = \sqrt{K_{BC}}$ holds. □

3.5 The Proof of Main Theorems

In this section, we prove Main Theorems, which we reproduce here:

Main Theorem 1 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the K-parameters of $|\psi\rangle$ and $|\psi'\rangle$ as $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5, Q_e)$ and $(K'_{AB}, K'_{AC}, K'_{BC}, j'_{ABC}, J'_5, Q'_e)$, respectively. Then, a necessary and sufficient condition of the possibility of a deterministic LOCC transformation from $|\psi\rangle$ to $|\psi'\rangle$ is that the following two conditions are satisfied:*
Condition 1: There are real numbers $0 \le \zeta_A \le 1$, $0 \le \zeta_B \le 1$, $0 \le \zeta_C \le 1$ and $\zeta_{lower} \le \zeta \le 1$ which satisfy the following equation:

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta \begin{pmatrix} \zeta_A\zeta_B & & & & \\ & \zeta_A\zeta_C & & & \\ & & \zeta_B\zeta_C & & \\ & & & \zeta_A\zeta_B\zeta_C & \\ & & & & \zeta_A\zeta_B\zeta_C \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix},
$$
(3.149)

where

$$
\zeta_{lower} = \frac{Jap}{(K_{AB} - \zeta_C j_{ABC}^2)(K_{AC} - \zeta_B j_{ABC}^2)(K_{BC} - \zeta_A j_{ABC}^2)},
$$
(3.150)

and we refer to ζ, ζ_A, ζ_B and ζ_C as the sub parameter and the main parameters of A, B and C, respectively.

Condition 2: If the state $|\psi'\rangle$ is EP definite, let us check whether the state $|\psi\rangle$ is $\tilde{\zeta}$-definite or not. When the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the condition is

$$Q_e = Q'_e \ \ and \ \ \zeta = \tilde{\zeta}. \tag{3.151}$$

When the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, the condition is

$$|Q'_e| = sgn[(1-\zeta)(\zeta - \zeta_{lower})], \tag{3.152}$$

or in the other words,

$$Q'_e \begin{cases} = 0 & (\zeta = 1 \ or \ \zeta = \zeta_{lower}), \\ \neq 0 & (otherwise). \end{cases} \tag{3.153}$$

Main Theorem 2 *If a deterministic LOCC transformation is executable, then the LOCC transformation can be reproduced by performing local unitary operation, a deterministic measurement on the qubit A, one on the qubit B and one on the qubit C.*

3.5.1 Case $\mathfrak{A}$

First, we prove Main Theorems in Case $\mathfrak{A}$, where both of the initial and final states are EP definite and the parameter j_{ABC} of the initial state is not zero. In the present subsection, we assume that $j_{ABC} \neq 0$, unless specified otherwise.

Step 1 of Case $\mathfrak{A}$

We here give a necessary and sufficient condition of the possibility of a two-choice deterministic measurement transformation (DMT) on the qubit A in the case that the final state is EP definite.

Theorem 1 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. The entanglement parameters of the state $|\psi\rangle$ are referred to as j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5 and Q_e, while the entanglement parameters of the state $|\psi'\rangle$ are referred to as j'_{AB}, j'_{AC}, j'_{BC}, j'_{ABC}, J'_5 and Q'_e. We assume the state $|\psi'\rangle$ to be EP definite. We also assume that $j_{ABC} \neq 0$. Then, a necessary and sufficient condition of the possibility of an A-DMT from the state $|\psi\rangle$ to the EP-definite state $|\psi'\rangle$ whose DM is a two-choice DM is that the following two conditions hold;*

Condition 1 : there are real numbers α_A and β_A which satisfy $0 \leq \alpha_A \leq 1$, $0 \leq \beta_A \leq 1$ and the following equation:

$$\begin{pmatrix} j'^2_{AB} \\ j'^2_{AC} \\ j'^2_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \begin{pmatrix} \alpha_A^2 & & & & \\ & \alpha_A^2 & & & \\ & & 1 & \beta_A(1-\alpha_A^2) & \\ & & & \alpha_A^2 & \\ & & & & \alpha_A^2 \end{pmatrix} \begin{pmatrix} j^2_{AB} \\ j^2_{AC} \\ j^2_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}, \tag{3.154}$$
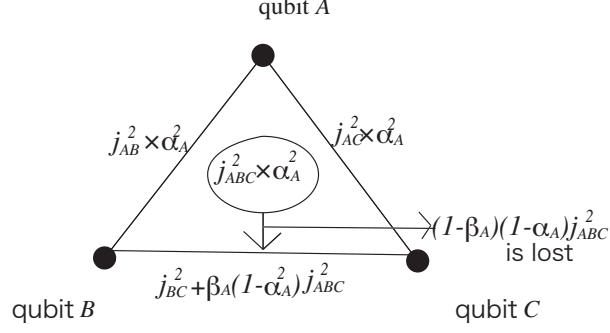
Figure 3.10: Dissipative entanglement transfer

Condition 2: Let us check whether the state $|\psi\rangle$ is $\tilde{\zeta}$-definite or not. When the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the condition is

$$Q_e = Q'_e \ \ and \ \ \beta_A = \tilde{\beta}_A, \tag{3.155}$$

where

$$\tilde{\beta}_A \equiv \frac{4K_{AB}K_{AC}j_{BC}^2 \sin\varphi_5}{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2 \sin\varphi_5}, \tag{3.156}$$

where if $|\psi\rangle$ is EP indefinite, then we define $j_{BC}^2 \sin\varphi_5$ as zero. This parameter corresponds to the ζ-specifying parameter $\tilde{\zeta}$, and thus we refer to $\tilde{\beta}_A$ as β-specifying parameter.
When the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, then the condition is

$$|Q'_e| = sgn[(1 - \beta_A)\beta_A]. \tag{3.157}$$

Comment 1: We can interpret the above as the rule how a DMT changes the entanglement. When an A-DMT transforms a state $|\psi\rangle$ into an EP-definite state $|\psi'\rangle$, the change of the entanglement is expressed by (3.154). We can express this change as in Fig. 3.10. After an A-DMT, the four entanglement parameters, j_{AB}^2, j_{AC}^2, j_{ABC}^2 and J_5, the last of which does not appear in Fig. 3.10, are multiplied by α_A^2. Note that these four entanglement parameters are related to the qubit A, which is the measured qubit in the A-DMT. The quantity $\beta_A(1 - \alpha_A^2)j_{ABC}^2$, which is a part of the entanglement lost from j_{ABC}^2, is added to j_{BC}^2, which is the only entanglement parameter that is not related to the measured qubit A. The quantity $(1 - \beta_A)(1 - \alpha_A^2)j_{ABC}^2$, which is the rest of the entanglement lost from j_{ABC}^2 disappear. We call this phenomenon the dissipative entanglement transfer, and call the DMT which gives rise to the dissipative entanglement transfer as a dissipative DMT. A dissipative DMT has only two single parameters; the transfer parameter α and the other parameter β_A. Hereafter, we refer to this new parameter β_A as the dissipative parameter. Note that Theorem 1 guarantees only a necessary and sufficient condition of an arbitrary DMT whose DM is a two-choice DM. However, this condition holds not only for a two-choice DM but also for an n-choice DM. We prove this statement in section 6.1.4.

Comment 2: Note that Theorem 1 includes not only Step 1 of Case $\mathfrak{A}$, but also a part of Step 1 of Case $\mathfrak{C}$. Indeed, we do not assume that the state $|\psi\rangle$ is EP definite in Theorem 1; we only assume that the state $|\psi'\rangle$ is EP definite. Note that if the state $|\psi\rangle$ is

EP indefinite, the DMT from the state $|\psi\rangle$ to the EP definite state $|\psi'\rangle$ is executable only if $|\psi\rangle$ is $\tilde{\zeta}$-indefinite. Let us prove this statement by reduction to absurdity. Let us assume that the DMT from the state $|\psi\rangle$ to the EP definite state $|\psi'\rangle$ was possible when the state $|\psi\rangle$ was EP indefinite and $\tilde{\zeta}$-definite. Because the state $|\psi\rangle$ is EP indefinite, at least one of the entanglement parameters j_{AB}, j_{AC} and j_{BC} would be zero. Because of (3.154), if j_{AB} or j_{AC} was zero, then the state $|\psi'\rangle$ could not be EP definite, Thus, j_{BC} would have to be the only zero bipartite entanglement parameter in order for $|\psi'\rangle$ to be EP definite. Because of the assumption that $|\psi\rangle$ was $\tilde{\zeta}$-definite, the equation $j_{BC}'^2 = j_{BC}^2 + \tilde{\beta}_A(1-\alpha_A^2)j_{ABC}^2$ would hold. Note that $\tilde{\beta}_A$ was zero, because of (3.156) and j_{BC}. Thus, j_{BC}' would be zero. This would contradict the assumption of Theorem 1 that the state $|\psi'\rangle$ is EP definite. Hence, if the state $|\psi\rangle$ is EP indefinite, the DMT from the state $|\psi\rangle$ to the EP definite state $|\psi'\rangle$ is executable only if the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite.

Proof: Before we describe the proof, we review definitions that are necessary for the proof. The notations λ_0, λ_1, λ_2, λ_3, λ_4 and φ stand for the positive-decomposition coefficients of the generalized Schmidt decomposition of $|\psi\rangle$. We define the measurement parameters a, b, k and θ for a measurement $\{M_{(i)}|i=0,1\}$ as follows:

$$M_{(0)}^\dagger M_{(0)} = \begin{pmatrix} a_{(0)} & k_{(0)}e^{-i\theta_{(0)}} \\ k_{(0)}e^{i\theta_{(0)}} & b_{(0)} \end{pmatrix} = \begin{pmatrix} a & ke^{-i\theta} \\ ke^{i\theta} & b \end{pmatrix}, \tag{3.158}$$

$$M_{(1)}^\dagger M_{(1)} = \begin{pmatrix} a_{(1)} & k_{(1)}e^{-i\theta_{(1)}} \\ k_{(1)}e^{i\theta_{(1)}} & b_{(1)} \end{pmatrix} = \begin{pmatrix} 1-a & -ke^{-i\theta} \\ -ke^{i\theta} & 1-b \end{pmatrix}, \tag{3.159}$$

where we assume that $\sin\theta \geq 0$. We refer to the probability that the result i comes out from the measurement $\{M_{(i)}|i=0,1\}$ as $p_{(i)}$. We define the states $\{|\psi^{(i)}\rangle\}$ as $|\psi^{(i)}\rangle = M_{(i)}|\psi\rangle/\sqrt{p_{(i)}}$. We refer to the probability $p_{(0)}$ as p. We define the entanglement J-parameters of the states $\{|\psi^{(i)}\rangle\}$ as $(j_{AB}^{(i)}, j_{AC}^{(i)}, j_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)}, Q_e^{(i)})$. We can express the probability $p_{(i)}$ as (A.34) with these parameters. We can express the generalized Schmidt coefficients $\lambda_0^{(i)}$, $\lambda_1^{(i)}$, $\lambda_2^{(i)}$, $\lambda_3^{(i)}$, $\lambda_4^{(i)}$ and $\varphi^{(i)}$ of the state $|\psi^{(i)}\rangle$ as in (A.35)–(A.39). Note that we do not specify whether the coefficients $\{\lambda_k^{(0)}, \varphi^{(0)}|k=0,...,4\}$ and $\{\lambda_k^{(1)}, \varphi^{(1)}|k=0,...,4\}$ are positive-decomposition coefficients or negative-decomposition coefficients. We can also express the entanglement parameters $j_{AB}^{(i)}, j_{AC}^{(i)}, j_{BC}^{(i)}, j_{ABC}^{(i)}$ and $\varphi_5^{(i)}$ as in (A.40)–(A.44) and (A.46). Note that we have proven that $j_{AB} \neq 0$ and $j_{AC} \neq 0$ in Comment 2. It follows that the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, if and only if $(\lambda_1\lambda_4\sin\varphi > 0) \vee (\Delta_J > 0)$. This can be easily seen if we note that $4J_{\mathrm{ap}} - J_5^2 = 4j_{AB}^2 j_{AC}^2 \lambda_1^2\lambda_4^2\sin^2\varphi$. Hereafter, we will often use this condition $(\lambda_1\lambda_4\sin\varphi > 0) \vee (\Delta_J > 0)$ in the present proof.

Next, we describe the structure of the proof. We divide the proof into two parts. In the first part, we consider the case where the state $|\psi\rangle$ is $\tilde{\zeta}$-definite. In the second part, we consider the case where tthe state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite. In the first case, we prove the present theorem in the following four steps:

1-1 We note that the two-choice measurement $\{M_{(i)}|i=0,1\}$ is a DM if and only if the

following equations are satisfied:

$$0 \le \frac{ab - k^2}{p^2}, \tag{3.160}$$

$$0 \le \frac{(1-a)(1-b) - k^2}{(1-p)^2}, \tag{3.161}$$

$$0 \le b \le 1, \tag{3.162}$$

$$j_{AB}^{(0)} = j_{AB}^{(1)}, \tag{3.163}$$

$$j_{AC}^{(0)} = j_{AC}^{(1)}, \tag{3.164}$$

$$j_{BC}^{(0)} = j_{BC}^{(1)}, \tag{3.165}$$

$$j_{ABC}^{(0)} = j_{ABC}^{(1)}, \tag{3.166}$$

$$J_5^{(0)} = J_5^{(1)}, \tag{3.167}$$

$$Q_{\mathrm{e}}^{(0)} = Q_{\mathrm{e}}^{(1)}. \tag{3.168}$$

In the above, (3.160)–(3.162) constitute a necessary and sufficient condition that $M_{(0)}^{\dagger}M_{(0)}$ and $M^{\dagger(1)}M_{(1)}$ are positive operators, whereas (3.163)–(3.168) constitute a necessary and sufficient condition that the states $\left|\psi^{(0)}\right\rangle$ and $\left|\psi^{(1)}\right\rangle$ are LU-equivalent. Note that the condition $\sum M_{(i)}^{\dagger}M_{(i)} = I$ is included in the definition of the measurement parameters a, b, k and θ as in (3.158)–(3.159).

1-2 We derive $Q_{\mathrm{e}}^{(i)} = Q_{\mathrm{e}}$ from (3.160)–(3.168). The equation $Q_{\mathrm{e}}^{(i)} = Q_{\mathrm{e}}$ includes (3.168); thus (3.160)–(3.167) with $Q_{\mathrm{e}}^{(i)} = Q_{\mathrm{e}}$ are equivalent to (3.160)–(3.168).

1-3-A In the steps 1-3-A and 1-4-A, we treat the case of $Q_{\mathrm{e}} \ne 0$. In the step 1-3-A, we solve (3.160)–(3.167) and $Q_{\mathrm{e}}^{(i)} = Q_{\mathrm{e}}$, and thereby derive the following five expressions, which are equivalent to (3.160)–(3.167) and $Q_{\mathrm{e}}^{(i)} = Q_{\mathrm{e}}$;

$$b = p\frac{K_5 - Q_{\mathrm{e}}\sqrt{\Delta_J}/(2p-1)}{K_5 - Q_{\mathrm{e}}\sqrt{\Delta_J}}. \tag{3.169}$$

$$k\cos\theta = Q_{\mathrm{e}}\frac{2p(p-1)}{2p-1}\frac{\sqrt{\Delta_J}}{K_5 - Q_{\mathrm{e}}\sqrt{\Delta_J}}\frac{j_{BC}\cos\varphi_5}{j_{ABC}}. \tag{3.170}$$

$$k\sin\theta = \frac{2p(1-p)}{2p-1}\frac{K_5}{K_5 - Q_{\mathrm{e}}\sqrt{\Delta_J}}\frac{j_{BC}\sin\varphi_5}{j_{ABC}}. \tag{3.171}$$

$$a = \frac{-4p(1-p)j_{BC}^2(K_5^2\sin^2\varphi_5 + \Delta_J\cos^2\varphi_5)}{(2p-1)(K_5^2 - \Delta_J)j_{ABC}^2} + \frac{p\{(2p-1)K_5 + Q_{\mathrm{e}}\sqrt{\Delta_J}\}}{(2p-1)(K_5 + Q_{\mathrm{e}}\sqrt{\Delta_J})}. \tag{3.172}$$

$$\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2\sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}(j_{BC}^2\sin^2\varphi_5 + j_{ABC}^2)}} \le p \le 1. \tag{3.173}$$

The equations (3.169)–(3.172) are expressions of a, b, k and θ in terms of p, whereas the inequality (3.173) gives the range of p. Because of the steps 1-1, 1-2 and 1-3-A, we can show that $\{M_{(i)}|i = 0, 1\}$ is a DM if and only if a, b, k, θ and p satisfy (3.169)–(3.173).

1-4-A We prove that if $\{M_{(i)}|i = 0, 1\}$ satisfies (3.169)–(3.173), the initial state $|\psi\rangle$ and the final state $|\psi'\rangle$ satisfy Conditions 1 and 2, and that the transfer parameter α_A which satisfies $0 \leq \alpha_A \leq 1$ and p which satisfies (3.173) have a one-to-one correspondence. Thus, the initial state $|\psi\rangle$ and the final state $|\psi'\rangle$ of any executable DMT satisfy Conditions 1 and 2. Inversely, we can take a set of the measurement parameters (a, b, k, θ) of an executable DM for any states $|\psi\rangle$ and $|\psi'\rangle$ which satisfy Conditions 1 and 2, because we can obtain the probability p from the transfer parameter α_A by using the one-to-one correspondence and because we can obtain the measurement parameters (a, b, k, θ) from the probability p by using (3.169)–(3.172). Thus, we will complete the proof of the present theorem in the case of $Q_e \neq 0$ in the step 1-4-A.

1-3-B In the steps 1-3-B and 1-4-B, we treat the case of $Q_e = 0$. In the step 1-3-B, we prove that if the measurement $\{M_{(i)}|i = 0, 1\}$ is a DM, the initial state $|\psi\rangle$ and the final state $|\psi'\rangle$ of the DMT of the measurement $\{M_{(i)}|i = 0, 1\}$ satisfy Conditions 1 and 2.

1-4-B In the step 1-4-B, we prove that if the state $|\psi\rangle$ and the EP-definite state $|\psi'\rangle$ satisfy Conditions 1 and 2, we can obtain a measurement which transforms the state $|\psi\rangle$ into the EP definite state $|\psi'\rangle$.

Note that we have completed the step 1-1. Thus, in the first case, where the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, we have only to perform the steps 1-2, 1-3-A, 1-3-B, 1-4-A and 1-4-B.

In the second case, where the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, we prove the present theorem in the following two steps:

2-1 We prove that the state $|\psi\rangle$ and the EP-definite state $|\psi'\rangle$ satisfy Conditions 1 and 2, if it is possible to perform a DMT from $|\psi\rangle$ to $|\psi'\rangle$.

2-2 We prove that if $|\psi\rangle$ and $|\psi'\rangle$ satisfy Conditions 1 and 2, we can find the measurement parameters a, b, k and θ whose $\{M_{(i)}|i = 0, 1\}$ is the DM from the state $|\psi\rangle$ to the EP-definite state $|\psi'\rangle$.

In this paragraph and the next one, we prepare for performing the above steps. First, we reduce (3.163)–(3.167) into the forms which are expressed in the measurement parameters a, b, k and θ and the probability p. From (A.40)–(A.42) and (A.45), we see that the equations (3.163), (3.164) and (3.166) are equivalent to

$$\alpha^{(0)} = \alpha^{(1)}. \tag{3.174}$$

Because of (A.28) and (A.48), the equation (3.167) is equivalent to

$$j_{BC}^{(0)} \cos \varphi_5^{(0)} = j_{BC}^{(1)} \cos \varphi_5^{(1)} = j_{BC} \cos \varphi_5. \tag{3.175}$$

Owing to (3.175), the equation (3.165) is reduced to

$$|\lambda_1^{(0)} \lambda_4^{(0)} \sin \varphi^{(0)}| = j_{BC}^{(0)} \sin \varphi_5^{(0)} = j_{BC}^{(1)} \sin \varphi_5^{(1)} = |\lambda_1^{(1)} \lambda_4^{(1)} \sin \varphi^{(1)}| = j_{BC}' \sin \varphi_5'. \tag{3.176}$$

Substituting (A.36), (A.39), (A.45), and (A.46) into (3.174), (3.175) and (3.176), we obtain the equations to be satisfied:

$$\frac{ab - k^2}{p^2} = \frac{(1-a)(1-b) - k^2}{(1-p)^2}, \tag{3.177}$$

$$\frac{bj_{BC}\cos\varphi_5 - kj_{ABC}\cos\theta}{p} = \frac{(1-b)j_{BC}\cos\varphi_5 + kj_{ABC}\cos\theta}{1-p} = j_{BC}\cos\varphi_5, \tag{3.178}$$

$$\left|\frac{bj_{BC}\sin\varphi_5 + kj_{ABC}\sin\theta}{p}\right| = \left|\frac{(1-b)j_{BC}\sin\varphi_5 - kj_{ABC}\sin\theta}{1-p}\right|. \tag{3.179}$$

These three equations are expressed in the measurement parameters a, b, k, θ and the probability p, and are equivalent to (3.163)–(3.167).

Second, we show that if the measurement $M_{(i)}$ is not equivalent to the identity transformation, we can derive the following equations from (3.163)–(3.168):

$$\sin\varphi^{(0)} \geq 0, \tag{3.180}$$

$$\sin\varphi^{(1)} \leq 0. \tag{3.181}$$

The inequality (3.180) is clearly satisfied, because $\sin\theta \geq 0$ and because $\{\lambda_k, \varphi | k = 0, ..., 4\}$ are positive-decomposition coefficients. In order to show (3.181), we show that if $\sin\varphi^{(1)} > 0$, the measurement $M_{(i)}$ is equivalent to the identity transformation. If $\sin\varphi^{(1)} > 0$, then both $\{\lambda_k^{(0)}, \varphi^{(0)} | k = 0, ..., 4\}$ and $\{\lambda_k^{(1)}, \varphi^{(1)} | k = 0, ..., 4\}$ are positive-decomposition coefficients. Thus, if $\sin\varphi^{(1)} > 0$, (3.163)–(3.168) are equivalent to $\{\lambda_k^{(0)}, \varphi^{(0)} | k = 0, ..., 4\} = \{\lambda_k^{(1)}, \varphi^{(1)} | k = 0, ..., 4\}$. From $\lambda_2^{(0)} = \lambda_2^{(1)}$ and (A.37), we obtain $b = p$. From $\lambda_1^{(0)}e^{i\varphi^{(1)}} = \lambda_1^{(1)}e^{i\varphi^{(1)}}$, $b = p$ and (A.36), we obtain $k = 0$. From $\lambda_0^{(0)} = \lambda_0^{(1)}$, $b = p$, $k = 0$ and (A.35), we obtain $a = b$. From $a = b$, $b = p$, $k = 0$ and (A.35)–(A.39), we obtain $\{\lambda_k^{(0)}, \varphi^{(0)} | k = 0, ..., 4\} = \{\lambda_k^{(1)}, \varphi^{(1)} | k = 0, ..., 4\} = \{\lambda_k, \varphi | k = 0, ..., 4\}$. This means that the measurement $M_{(i)}$ is equivalent to the identity transformation. Hence, as a contraposition, if the measurement $M_{(i)}$ is not equivalent to the identity transformation, (3.180) and (3.181) hold.

Let us perform the step 1-2. In other words, we derive $Q_{\mathrm{e}}^{(0)} = Q_{\mathrm{e}}^{(1)} = Q_{\mathrm{e}}$ from (3.163)–(3.168) in the case where the state $|\psi\rangle$ is $\tilde{\zeta}$-definite. In this step, we suppose (3.163)–(3.168) to hold, and hence we refer to $(j_{AB}^{(i)}, j_{AC}^{(i)}, j_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)} Q_{\mathrm{e}}^{(i)})$ as $(j'_{AB}, j'_{AC}, j'_{BC}, j'_{ABC}, J'_5 Q'_{\mathrm{e}})$, $\alpha^{(i)}$ as α, and $K_{AB}^{(i)}, K_{AC}^{(i)}, K_{BC}^{(i)}$ and $K_5^{(i)}$ as $K'_{AB}, K'_{AC}, K'_{BC}$ and K'_5.

First, we show $Q'_{\mathrm{e}} = Q_{\mathrm{e}}$ in the case of $Q_{\mathrm{e}} \neq 0$. There are three possible cases of Q'_{e}; $Q'_{\mathrm{e}} = Q_{\mathrm{e}}$, $Q'_{\mathrm{e}} = -Q_{\mathrm{e}}$ and $Q'_{\mathrm{e}} = 0$. We show that the equations $Q'_{\mathrm{e}} = -Q_{\mathrm{e}}$ and $Q'_{\mathrm{e}} = 0$ are false by reduction to absurdity.

Let us assume that the entanglement charge Q'_{e} were zero. Because of $Q'_{\mathrm{e}} = 0$, (A.19) and (A.11), at least one of the quantityies Δ'_J and $\sin\varphi'$ would be zero. Because of (A.29), if $\sin\varphi'$ was zero, $j'_{BC}\sin\varphi'_5$ would be zero, where the entanglement phase φ_5 is definite because the state $|\psi'\rangle$ is EP-definite. Thus, at least one of the quantities Δ'_J and $j'_{BC}\sin\varphi'_5$ would be zero. As we show below, this contradicts $Q_{\mathrm{e}} \neq 0$. Because of $Q_{\mathrm{e}} \neq 0$, (A.19) and (A.11), none of the quantities Δ_J and $\sin\varphi$ is zero. Because of $\Delta_J \geq 0$ [22], $\Delta_J \neq 0$ and (A.54), the inequality $\Delta'_J > 0$ holds. The other inequality $\sin\varphi > 0$ also

holds, because $\sin \varphi$ is not zero and because we have assumed that $\{\lambda_i, \varphi | i = 0, ..., 4\}$ are positive-decomposition coefficients. Thus, $\lambda_1 \lambda_4 \sin \varphi > 0$ holds, because if one of the coefficients $\{\lambda_i | i = 0, ..., 4\}$ was zero, $\sin \varphi = 0$ would have to hold. Because of $\lambda_1 \lambda_4 \sin \varphi > 0$ and (A.29), the inequality $j_{BC} \sin \varphi_5 > 0$ holds. Because of $j_{BC} \sin \varphi_5 > 0$ and (A.52), the inequality $j'_{BC} \sin \varphi'_5 > 0$ holds. Thus, none of Δ'_J and $j'_{BC} \sin \varphi'_5$ is zero. This is a contradiction, and hence $Q'_e \neq 0$ holds.

Next, we prove $Q'_e \neq -Q_e$ in the case of $Q_e \neq 0$ by reduction to absurdity. Let us assume that $Q'_e = -Q_e$ was valid. Because the entanglement charge Q_e is not zero and because $\{\lambda_i, \varphi | i = 0, ..., 4\}$ are positive-decomposition coefficients, we can express λ_0^2 as (A.20):

$$\lambda_0^2 = \frac{K_5 + Q_e \sqrt{\Delta_J}}{2K_{BC}}. \tag{3.182}$$

Because the entanglement charge Q'_e was not zero, we could also express $(\lambda_0^{(0)})^2$ and $(\lambda_0^{(1)})^2$ as (A.20):

$$(\lambda_0^{(0)})^2 = \frac{K'_5 + Q'_e \sqrt{\Delta'_J}}{2K'_{BC}}, \quad (\lambda_0^{(1)})^2 = \frac{K'_5 - Q'_e \sqrt{\Delta'_J}}{2K'_{BC}}, \tag{3.183}$$

where we used the inequalities $\sin \varphi^{(0)} > 0$ and $\sin \varphi^{(1)} < 0$, which we show below. We have already proven (3.180) and (3.181). Thus, we only have to show that none of $\sin \varphi^{(0)}$ and $\sin \varphi^{(1)}$ is zero. If $\sin \varphi^{(0)}$ was zero, the entanglement charge Q'_e would have to be zero, but this contradicts the inequality $Q'_e \neq 0$. Thus, $\sin \varphi^{(0)}$ would not be zero. In the same manner, we could prove that $\sin \varphi^{(1)}$ would not be zero. Thus, $\sin \varphi^{(0)} > 0$ and $\sin \varphi^{(1)} < 0$ would be correct.

We could derive a contradiction from the assumption $Q'_e = -Q_e$ by using (3.182) and (3.183). Because of (A.35), (3.182) and (3.183), we would obtain

$$(\lambda_0^{(0)})^2 = \frac{K'_5 + Q'_e \sqrt{\Delta'_J}}{2K'_{BC}} = \frac{p}{b} \alpha^2 \lambda_0^2 = \frac{p}{b} \alpha^2 \frac{K_5 + Q_e \sqrt{\Delta_J}}{2K_{BC}}, \tag{3.184}$$

$$(\lambda_0^{(1)})^2 = \frac{K'_5 - Q'_e \sqrt{\Delta'_J}}{2K'_{BC}} = \frac{1-p}{1-b} \alpha^2 \lambda_0^2 = \frac{1-p}{1-b} \alpha^2 \frac{K_5 + Q_e \sqrt{\Delta_J}}{2K_{BC}}. \tag{3.185}$$

Because of the expressions $K'_5 = \alpha^2 K_5$, $\Delta'_J = \alpha^4 \Delta'_{\text{norm}}$ and $K'_{BC}/K_{BC} = (K_5^2 - \Delta'_{\text{norm}})/(K_5^2 - \Delta_J)$ and the assumption $Q'_e = -Q_e$, we would reduce (3.184) and (3.185) into

$$b = p \frac{K_5 + Q_e \sqrt{\Delta_J}}{K_5 - Q_e \sqrt{\Delta'_{\text{norm}}}} \frac{K'_{BC}}{K_{BC}} = p \frac{K_5 + Q_e \sqrt{\Delta'_{\text{norm}}}}{K_5 - Q_e \sqrt{\Delta_J}}, \tag{3.186}$$

$$1 - b = (1-p) \frac{K_5 + Q_e \sqrt{\Delta_J}}{K_5 + Q_e \sqrt{\Delta'_{\text{norm}}}} \frac{K'_{BC}}{K_{BC}} = (1-p) \frac{K_5 - Q_e \sqrt{\Delta'_{\text{norm}}}}{K_5 - Q_e \sqrt{\Delta_J}}. \tag{3.187}$$

Substituting (3.186) into (3.187), we would obtain the probability

$$p = \frac{1 - \sqrt{\Delta_J / \Delta'_{\text{norm}}}}{2}. \tag{3.188}$$

On the other hand, from (A.49), $\sin \varphi^{(0)} > 0$, $\sin \varphi^{(1)} < 0$ and (3.176), we would obtain

$$p j'_{BC} \sin \varphi'_5 - (1-p) j'_{BC} \sin \varphi'_5 = j_{BC} \sin \varphi_5. \tag{3.189}$$

Because of (3.188) and (3.189), we would obtain

$$\frac{1 + j_{BC} \sin \varphi_5 / j'_{BC} \sin \varphi'_5}{2} = \frac{1 - \sqrt{\Delta_J / \Delta'_{\text{norm}}}}{2}. \tag{3.190}$$

The equation holds only if $j_{BC} \sin \varphi_5 = \Delta_J = 0$. However, this contradicts our assumption that the state $|\psi\rangle$ is $\tilde{\zeta}$-definite; note that $\lambda_1^2 \lambda_4^2 \sin^2 \varphi = j_{BC}^2 \sin^2 \varphi_5$. Hence, $Q'_e \neq -Q_e$. Now, we complete the proof that $Q'_e = Q_e$ holds in the case $Q_e = \pm 1$.

Next, we show that $Q'_e = Q_e$ holds in the case of $Q_e = 0$. We prove this by reduction to absurdity. Let us assume that $Q'_e \neq 0$. We can use (3.183) again, because we can derive $\sin \varphi^{(0)} > 0$ and $\sin \varphi^{(1)} < 0$ from the assumption $Q'_e \neq 0$ in the same manner as above. We cannot use (3.182) again, because now the entanglement charge Q_e is zero, but we can use the following equation which is similar to (3.182):

$$\lambda_0^2 = \frac{K_5 + \sqrt{\Delta_J}}{2K_{BC}}. \tag{3.191}$$

Let us show that we can use (3.191). In the section 2, we proved that when the entanglement charge Q_e is zero, the four sets of the coefficients of the general Schmidt decomposition $\{\lambda_i^+, \varphi^+ | i = 0, ..., 4\}$, $\{\lambda_i^-, \varphi^- | i = 0, ..., 4\}$, $\{\lambda_i^+, \tilde{\varphi}^- | i = 0, ..., 4\}$ and $\{\lambda_i^-, \tilde{\varphi}^- | i = 0, ..., 4\}$ are LU-equivalent (Fig. 3.1). Thus, we can choose any one of the four under local unitary transformations. Moreover, because of (A.11), at least one of the sets of positive-decomposition coeffcients $\{\lambda_i^+, \varphi^+ | i = 0, ..., 4\}$ and $\{\lambda_i^-, \varphi^- | i = 0, ..., 4\}$ satisfies (3.191). Thus, we can assume that the positive-decomposition coefficients $\{\lambda_i, \varphi | i = 0, ..., 4\}$ satisfies (3.191) without losing generality. Let us derive a contradiction from the assumption $Q'_e \neq 0$ by using (3.183) and (3.191). Because of (A.35), (3.183) and (3.191), we obtain

$$(\lambda_0^{(0)})^2 = \frac{K'_5 + Q'_e \sqrt{\Delta'_J}}{2K'_{BC}} = \frac{p}{b} \alpha^2 \frac{K_5 + \sqrt{\Delta_J}}{2K_{BC}}, \tag{3.192}$$

$$(\lambda_0^{(1)})^2 = \frac{K'_5 - Q'_e \sqrt{\Delta'_J}}{2K'_{BC}} = \frac{1 - p}{1 - b} \alpha^2 \frac{K_5 + \sqrt{\Delta_J}}{2K_{BC}}. \tag{3.193}$$

In the same manner as we derived (3.188) from (3.184) and (3.185), we can obtain the probability

$$p = \frac{1 + Q'_e \sqrt{\Delta_J / \Delta'_{\text{norm}}}}{2}. \tag{3.194}$$

Let us prove that (3.194) and the assumption $Q'_e \neq 0$ contradict $Q_e = 0$. Because of $Q_e = 0$, at least one of the quantities Δ_J and $\sin \varphi$ is zero. Because the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, at least one of the quantities Δ_J and $\sin \varphi$ is not zero. Thus, one of the quantities Δ_J and $\sin \varphi$ is zero, and the other is not zero. First, we consider the case of $(\sin \varphi = 0) \wedge (\Delta_J \neq 0)$. Because of $\sin \varphi = 0$, the equation (3.179) is equivalent to

$$\frac{k j_{ABC} \sin \theta}{p} = \frac{k j_{ABC} \sin \theta}{1 - p}. \tag{3.195}$$

This equation means $p = 1/2$, but this contradicts (3.194), $\Delta_J \neq 0$ and $Q'_e \neq 0$. Thus, the assumption $Q'_e \neq 0$ leads us to a contradiction in the case of $(\sin \varphi = 0) \wedge (\Delta_J \neq 0)$.

Next, let us consider the case of $(\sin\varphi \neq 0) \wedge (\Delta_J = 0)$. In this case, (3.194) is reduced to $p = 1/2$, because of $\Delta_J = 0$. We can derive $\sin\varphi = 0$, which contradicts $\sin\varphi \neq 0$, from $p = 1/2$. Let us perform this derivation. Because of $\sin\varphi^{(0)} > 0$, $\sin\varphi^{(1)} < 0$, (A.29) and the fact that the state $|\psi'\rangle$ is EP definite, the following equations hold:

$$\lambda_1^{(0)}\lambda_4^{(0)}\sin\varphi^{(0)} = j'_{BC}\sin\varphi'_5, \quad \lambda_1^{(1)}\lambda_4^{(1)}\sin\varphi^{(1)} = -j'_{BC}\sin\varphi'_5. \tag{3.196}$$

From (3.196), (A.49) and $p = 1/2$, we obtain $\lambda_1\lambda_4\sin\varphi = 0$. Because of $\lambda_1\lambda_4\sin\varphi = 0$, at least one of λ_1, λ_4 and $\sin\varphi$ is zero. If λ_1 or λ_4 is zero, $\sin\varphi$ is also zero, because if one of the coefficients $\{\lambda_i | i = 0, ..., 4\}$ is zero, $\sin\varphi$ is also zero. Thus, $\sin\varphi$ is zero, this contradicts $\sin\varphi \neq 0$. Thus, the assumption $Q'_e \neq 0$ also leads us to a contradiction in the case of $(\sin\varphi \neq 0) \wedge (\Delta_J = 0)$. Now, we complete the derivation of a contradiction from the assumption $Q'_e \neq 0$. Thus, the entanglement charge Q'_e has to be zero, and thus $Q_e = Q'_e$ holds in the case where the state $|\psi\rangle$ is $\tilde{\zeta}$-definite.

Next, let us perform the step 1-3-A, where we assume $Q_e \neq 0$. This is equivalent to $(\sin\varphi > 0) \wedge (\Delta_J > 0)$. We have derived the expressions of the measurement parameters a, b, k and θ in the probability p from (3.160)–(3.167) and $Q_e^{(i)} = Q_e$ in the case where the state $|\psi\rangle$ is $\tilde{\zeta}$-definite and where $Q_e \neq 0$. In order to perform the step 1-3-A, it is useful to derive some equations which are equivalent to (3.163)–(3.167) and $Q_e^{(i)} = Q_e$. We have already derived the equations (3.177)–(3.179), which are equivalent to (3.163)–(3.167). Let us derive two equations which are equivalent to $Q_e^{(0)} = Q_e$ and $Q_e^{(1)} = Q_e$, respectively. We can derive these equations from (A.20), if we prove $\sin\varphi^{(0)} > 0$ and $\sin\varphi^{(1)} < 0$. Thus, let us prove $\sin\varphi^{(0)} > 0$ and $\sin\varphi^{(1)} < 0$ at first. In the present case, $(\sin\varphi > 0) \wedge (\Delta_J > 0)$ holds, because of $Q_e = \pm 1$ and (A.19). The inequality $\sin\varphi^{(0)} > 0$ holds, because of (A.36), $\sin\theta \geq 0$ and $\sin\varphi > 0$. Because of (3.176), (3.180) and (3.181), the inequality $\sin\varphi^{(1)} < 0$ also holds. Thus, from $Q_e = Q_e^{(i)}$ and (A.20), we obtain

$$(\lambda_0^{(0)})^2 = \frac{K_5^{(0)} + Q_e\sqrt{\Delta_J^{(0)}}}{2K_{BC}^{(0)}}, \tag{3.197}$$

$$(\lambda_0^{(1)})^2 = \frac{K_5^{(1)} - Q_e\sqrt{\Delta_J^{(1)}}}{2K_{BC}^{(1)}}. \tag{3.198}$$

Because of (A.35), we can turn (3.197) and (3.198) into

$$\frac{K_5 + Q_e\sqrt{\Delta'_{\text{norm}}}}{2K'_{BC}}\alpha^2 = \frac{p}{b}\alpha^2\lambda_0^2 = \frac{p}{b}\frac{K_5 + Q_e\sqrt{\Delta_J}}{2K_{BC}}\alpha^2, \tag{3.199}$$

$$\frac{K_5 - Q_e\sqrt{\Delta'_{\text{norm}}}}{2K'_{BC}}\alpha^2 = \frac{1-p}{1-b}\alpha^2\lambda_0^2 = \frac{1-p}{1-b}\frac{K_5 + Q_e\sqrt{\Delta_J}}{2K_{BC}}\alpha^2. \tag{3.200}$$

The equations (3.199) and (3.200) are what we want.

From (3.177)–(3.179), (3.199) and (3.200), we can express the measurement parameters a, b, k, θ and the quantity Δ'_{norm} in the probability p. First, we obtain the forms of b and Δ'_{norm} which are expressed in the probability p. Because of $K'_{BC}/K_{BC} = (K_5^2 - \Delta'_{\text{norm}})/(K_5^2 - \Delta_J)$, we can transform (3.199) and (3.200) into

$$b = p\frac{K'_{BC}}{K_5 + Q_e\sqrt{\Delta'_{\text{norm}}}}\frac{K_5 + Q_e\sqrt{\Delta_J}}{K_{BC}} = p\frac{K_5 - Q_e\sqrt{\Delta'_{\text{norm}}}}{K_5 - Q_e\sqrt{\Delta_J}}, \tag{3.201}$$

$$1 - b = (1 - p)\frac{K'_{BC}}{K_5 - Q_e\sqrt{\Delta'_{\text{norm}}}}\frac{K_5 + Q_e\sqrt{\Delta_J}}{K_{BC}} = (1 - p)\frac{K_5 + Q_e\sqrt{\Delta'_{\text{norm}}}}{K_5 - Q_e\sqrt{\Delta_J}}. \quad (3.202)$$

We obtain the expression of Δ'_{norm} by substituting (3.201) into (3.202) and transforming it as follows:

$$1 - p\frac{K_5 - Q_e\sqrt{\Delta'_{\text{norm}}}}{K_5 - Q_e\sqrt{\Delta_J}} = (1 - p)\frac{K_5 + Q_e\sqrt{\Delta'_{\text{norm}}}}{K_5 - Q_e\sqrt{\Delta_J}},$$

$$K_5 - Q_e\sqrt{\Delta_J} = K_5 + Q_e\sqrt{\Delta'_{\text{norm}}} - Q_e 2p\sqrt{\Delta'_{\text{norm}}},$$

$$\frac{\sqrt{\Delta_J}}{2p - 1} = \sqrt{\Delta'_{\text{norm}}}. \quad (3.203)$$

We obtain the expression of b by substituting (3.203) into (3.201):

$$b = p\frac{K_5 - Q_e\sqrt{\Delta_J}/(2p - 1)}{K_5 - Q_e\sqrt{\Delta_J}}. \quad (3.204)$$

We obtain the expression of $k\cos\theta$ by substituting (3.204) into (3.178) and transforming it as follows:

$$\frac{(b - p)j_{BC}\cos\varphi_5}{j_{ABC}} = k\cos\theta,$$

$$p\left(\frac{K_5 - Q_e\sqrt{\Delta_J}/(2p - 1)}{K_5 - Q_e\sqrt{\Delta_J}} - 1\right)\frac{j_{BC}\cos\varphi_5}{j_{ABC}} = k\cos\theta,$$

$$Q_e\frac{2p(p - 1)}{2p - 1}\frac{\sqrt{\Delta_J}}{K_5 - Q_e\sqrt{\Delta_J}}\frac{j_{BC}\cos\varphi_5}{j_{ABC}} = k\cos\theta. \quad (3.205)$$

We obtain the expression of $k\sin\theta$ by substituting (3.204) into (3.179) and transforming it as follows:

$$\frac{b}{p}j_{BC}\sin\varphi_5 + \frac{k}{p}j_{ABC}\sin\theta = -\frac{1 - b}{1 - p}j_{BC}\sin\varphi_5 + \frac{k}{1 - p}j_{ABC}\sin\theta,$$

$$\frac{(1 - p)b + p(1 - b)}{p(1 - p)}j_{BC}\sin\varphi_5 = \frac{2p - 1}{p(1 - p)}kj_{ABC}\sin\theta,$$

$$\frac{(1 - 2p)b + p}{2p - 1}j_{BC}\sin\varphi_5 = kj_{ABC}\sin\theta,$$

$$\frac{p}{2p - 1}\left(-\frac{(2p - 1)K_5 - Q_e\sqrt{\Delta_J}}{K_5 - Q_e\sqrt{\Delta_J}} + 1\right)j_{BC}\sin\varphi_5 = kj_{ABC}\sin\theta,$$

$$\frac{2p(1 - p)}{2p - 1}\frac{K_5}{K_5 - Q_e\sqrt{\Delta_J}}\frac{j_{BC}\sin\varphi_5}{j_{ABC}} = k\sin\theta. \quad (3.206)$$

We obtain the expression of a by substituting (3.204), (3.205) and (3.206) into (3.177). First, we express a in b, k, θ and the probability p by transforming (3.177) as follows:

$$\frac{ab - k^2}{p^2} = \frac{(1 - a)(1 - b) - k^2}{(1 - p)^2},$$

$$(1 - 2p)(ab - k^2) = 6p^2(1 - a - b),$$

$$a = \frac{(1 - 2p)k^2 + p^2(1 - b)}{(1 - 2p)b + p^2}. \quad (3.207)$$

Second, we substitute (3.204), (3.205) and (3.206) into (3.207).

$$a = \frac{-4p(1-p)j_{BC}^2(K_5^2\sin^2\varphi_5 + \Delta_J\cos^2\varphi_5)}{(2p-1)(K_5^2 - \Delta_J)j_{ABC}^2} + \frac{p\{(2p-1)K_5 + Q_{\mathrm{e}}\sqrt{\Delta_J}\}}{(2p-1)(K_5 + Q_{\mathrm{e}}\sqrt{\Delta_J})}. \quad (3.208)$$

Now, we have obtained the expressions of a, b, k and θ. We can obtain a measurement $M_{(i)}$ which satisfies (3.163)–(3.168) by defining a, b, k and θ as (3.208), (3.204), (3.205) and (3.206).

Next, we restrict the range of p from (3.160)–(3.162). Because (3.177) is satisfied, the inequality (3.160) is equivalent to (3.161). We derive the range of the probability p from (3.160) and (3.162). First, we substitute (3.207) into (3.160) and transform it as follows:

$$\alpha^2 = \frac{(1-2p)k^2b + p^2b(1-b) - k^2(1-2p)b - k^2p^2}{(1-2p)b + p^2}$$
$$= \frac{b(1-b) - k^2}{(1-2p)b + p^2}. \quad (3.209)$$

Next, we substitute (3.204), (3.205) and (3.206) into (3.209) and transform (3.209) as follows:

$$\alpha^2 = \left\{p^2 - \frac{p(2p-1)K_5 - Q_{\mathrm{e}}p\sqrt{\Delta_J}}{K_5 - Q_{\mathrm{e}}\sqrt{\Delta_J}}\right\}^{-1}$$
$$\left\{\frac{p((2p-1)K_5 - Q_{\mathrm{e}}\sqrt{\Delta_J})(1-p)((2p-1)K_5 + Q_{\mathrm{e}}\sqrt{\Delta_J})}{(2p-1)^2(K_5 - Q_{\mathrm{e}}\sqrt{\Delta_J})^2} - \frac{4p^2(1-p)^2j_{BC}^2(K_5^2\sin^2\varphi_5 + \Delta_J \cdots}{(2p-1)^2(K_5 - Q_{\mathrm{e}}\sqrt{\Delta_J})^2 j_A^2 \cdots}\right.$$
$$= \left\{\frac{p(1-p)(K_5 + Q_{\mathrm{e}}\sqrt{\Delta_J})}{K_5 - Q_{\mathrm{e}}\sqrt{\Delta_J}}\right\}^{-1}\frac{p(1-p)}{(2p-1)^2(K_5 - Q_{\mathrm{e}}\sqrt{\Delta_J})^2}$$
$$\left\{(2p-1)^2K_5^2 - \Delta_J - \frac{4p(1-p)j_{BC}^2(K_5^2\sin^2\varphi_5 + \Delta_J\cos^2\varphi_5)}{j_{ABC}^2}\right\}$$
$$= \frac{\{(2p-1)^2K_5^2 - \Delta_J\}j_{ABC}^2 + \{(2p-1)^2 - 1\}j_{BC}^2(K_5^2\sin^2\varphi_5 + \Delta_J\cos^2\varphi_5)}{(2p-1)^2(K_5^2 - \Delta_J)j_{ABC}^2},$$
$$(2p-1)^2\{(K_5^2 - \Delta_J)\alpha^2 j_{ABC}^2 - K_5^2 j_{ABC}^2 - j_{BC}^2(K_5^2\sin^2\varphi_5 + \Delta_J\cos^2\varphi_5)\}$$
$$= -\Delta_J j_{ABC}^2 - j_{BC}^2(K_5^2\sin^2\varphi_5 + \Delta_J\cos^2\varphi_5),$$
$$(2p-1)^2 = \frac{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2\sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}(j_{BC}^2\sin^2\varphi_5 + (1-\alpha^2)j_{ABC}^2)}.$$

The equation (3.210) expresses $(2p-1)^2$ as a monotonically decreasing function of α^2. Thus, from (3.160), we obtain

$$(2p-1)^2 \geq \frac{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2\sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}(j_{BC}^2\sin^2\varphi_5 + j_{ABC}^2)}. \quad (3.211)$$

Because of the expressions (3.203), (3.211) and $\Delta'_{\mathrm{norm}} \geq \Delta_J$, the expression $1 \geq p \geq 1/2$ holds. Thus, (3.211) is equivalent to

$$\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2\sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}(j_{BC}^2\sin^2\varphi_5 + j_{ABC}^2)}} \leq p \leq 1. \quad (3.212)$$

Now, we have restricted the range of the probability p from (3.160). Next, let us show that (3.212) satisfies (3.162). Because of (3.204), we obtain

$$\frac{db}{dp} = \frac{K_5}{K_5 - Q_e\sqrt{\Delta_J}} + Q_e \frac{1}{(2p-1)^2} \frac{\sqrt{\Delta_J}}{K_5 - Q_e\sqrt{\Delta_J}}. \tag{3.213}$$

Because of (3.213), if $Q_e = 1$, b is a monotonically increasing function of the probability p. If $Q_e = -1$, we can derive the following expression from (3.213):

$$\frac{db}{dp} = 0 \Rightarrow p = \frac{1}{2}\left(1 - Q_e\frac{\Delta_J}{K_5}\right). \tag{3.214}$$

Because of (3.204), we have

$$b|_{p=1} = 1, \tag{3.215}$$

$$b|_{p=\frac{1}{2}\left(1-Q_e\frac{\Delta_J}{K_5}\right)} = 0 \quad (\text{if } Q_e = -1) \tag{3.216}$$

$$b\Big|_{p=\frac{1}{2}+\frac{1}{2}\sqrt{\frac{\Delta_J+4K_{AB}K_{AC}j_{BC}^2\sin^2\varphi_5}{\Delta_J+4K_{AB}K_{AC}(j_{BC}^2\sin^2\varphi_5+j_{ABC}^2)}}}$$

$$= \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2\sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}(j_{BC}^2\sin^2\varphi_5 + j_{ABC}^2)}}\right) \frac{K_5 - Q_e\sqrt{\Delta_J}\sqrt{\frac{\Delta_J+4K_{AB}K_{AC}(j_{BC}^2\sin^2\varphi_5+j_{ABC}^2)}{\Delta_J+4K_{AB}K_{AC}j_{BC}^2\sin^2\varphi_5}}}{K_5 - Q_e\sqrt{\Delta_J}}$$

$$\geq \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2\sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}(j_{BC}^2\sin^2\varphi_5 + j_{ABC}^2)}}\right) \frac{K_5 - Q_e\sqrt{\Delta_J}\sqrt{\frac{K_5^2}{\Delta_J}}}{K_5 - Q_e\sqrt{\Delta_J}} \geq 0. \tag{3.217}$$

Because of (3.213)–(3.217), the inequalities $0 \leq b \leq 1$ hold. Now, we have obtained the equations which are equivalent to (3.160)–(3.168): (3.204)–(3.206), (3.208) and (3.212). In other words, we have completed the step 1-3-A.

Next, let us perform the step 1-4-A. First, let us prove that if the measurement $\{M_{(i)}|i = 0, 1\}$ satisfies (3.204)–(3.206), (3.208) and (3.212), then Condition 1 is satisfied. The expressions (3.204)–(3.206), (3.208) and (3.212) are equivalent to (3.160)–(3.168), and thus the entanglement parameters of the states $|\psi\rangle$ and $|\psi'\rangle$ are the same. Thus, if the measurement $\{M_{(i)}|i = 0, 1\}$ satisfies (3.204)–(3.206), (3.208) and (3.212), there are the transfer parameter α_A and the dissipation parameter β_A which satisfy Condition 1 because of Lemma 2, (3.163)–(3.167), (A.40)–(A.42) and (A.46). Second, we prove that if the measurement $\{M_{(i)}|i = 0, 1\}$ satisfies (3.204)–(3.206), (3.208) and (3.212), then Condition 2 is satisfied. We have already proven that $Q_e^{(i)} = Q_e$, and thus we only have to show that if the measurement parameters a, b, k and θ satisfy (3.204)–(3.206), (3.208) and (3.212), then $\beta_A = \tilde{\beta}_A$ holds. Because of Lemma 2, (3.165) and $\alpha^{(0)} = \alpha^{(1)}$, we can take $0 \leq \beta_A \leq 1$ which satisfies that $j_{BC}'^2 = j_{BC}^2 + \beta_A(1 - \alpha_A^2)j_{ABC}^2$. We obtain $\tilde{\beta}_A = \beta_A$ by substituting the dissipation parameter β_A and (3.203) into (3.210) and transforming

it as follows:

$$\frac{\Delta_J}{\Delta'_{\text{norm}}} = (2p-1)^2 = \frac{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2 \sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}(j_{BC}^2 \sin^2\varphi_5 + (1-\alpha^2)j_{ABC}^2)},$$

$$\frac{\Delta_J}{\Delta_J + 4K_{AB}K_{AC}j_{ABC}^2(1-\beta_A)(1-\alpha_A^2)} = \frac{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2 \sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}(j_{BC}^2 \sin^2\varphi_5 + (1-\alpha^2)j_{ABC}^2)},$$

$$(1-\beta_A)4K_{AB}K_{AC}j_{ABC}^2(1-\alpha_A^2) = \frac{\Delta_J 4K_{AB}K_{AC}j_{ABC}^2(1-\alpha_A^2)}{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2 \sin^2\varphi_5},$$

$$\beta_A = \frac{4K_{AB}K_{AC}j_{BC}^2 \sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2 \sin^2\varphi_5} = \tilde{\beta}_A. \qquad (3.218)$$

Finally, we prove that the transfer parameter α_A and the probability p have a one-to-one correspondence. Because of (3.210), the quantity $(2p-1)^2$ is a monotonic function of the transfer parameter α. Moreover, if $p=1$, the equation $\alpha = 1$ holds, and if the probability p takes the lower limit of (3.212), the equation $\alpha = 0$ holds. Thus, the transfer parameter α and the probability p have a one-to-one correspondence. Now, we have completed the proof in the case of $Q_{\text{e}} = \pm 1$; when the measurement parameters a, b, k and θ are defined as (3.204)–(3.206) and (3.208), the equations (3.154), $Q_{\text{e}} = Q'_{\text{e}}$ and $\beta_A = \tilde{\beta}_A$ are satisfied, and the transfer parameter α_A can take any value from 0 to 1. Inversely, if Conditions 1 and 2 of Theorem 1 are satisfied, we can take the probability p which corresponds to the transfer parameter α_A and take a measurement which executes the deterministic transformation from the state $|\psi\rangle$ to the EP-definite state $|\psi'\rangle$. Hence, we have proven the present theorem in the case of $Q_{\text{e}} = \pm 1$.

Next, we perform the step 1-3-B, where we assume $Q_{\text{e}} = 0$. We prove that if a measurement satisfies (3.160)–(3.168) and $Q_{\text{e}}^{(i)} = Q_{\text{e}} = 0$, Conditions 1 and 2 are satisfied. Hereafter, in the steps 1-3-B and 1-4-B, we assume that the state $|\psi\rangle$ is $\tilde{\zeta}$-definite and that (3.160)–(3.168) and $Q_{\text{e}}^{(i)} = Q_{\text{e}} = 0$ hold. Because of Lemma 2, (3.163)–(3.167), (A.40)–(A.42) and (A.46), there are the transfer parameter α_A and the dissipation parameter β_A which satisfy Condition 1. In the step 1-2, we have already proven $Q_{\text{e}} = Q'_{\text{e}}$. Thus, we only have to prove $\tilde{\beta}_A = \beta_A$. Because of $Q_{\text{e}} = 0$, (A.19) and (A.30), at least one of Δ'_J and $\sin\varphi'_5$ is zero. First, we prove $\tilde{\beta}_A = \beta_A$ in the case where Δ'_J is zero. Because of (A.54), the expression $\Delta'_J = 0 \Rightarrow \Delta_J = 0$ holds. The equation $\Delta_J = \Delta'_{\text{norm}}$ holds if and only if $\beta_A = 1$ holds, because of the equation $\Delta'_{\text{norm}} - \Delta_J = 4(1-\beta_A)K_{\text{ap}}$. Thus, if Δ'_J is zero, the dissipation parameter β_A is one. As we show below, if Δ'_J is zero, the β-specifying parameter $\tilde{\beta}_A$ is one. Let us show this. Because the the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, $\Delta_J = 0 \Rightarrow \lambda_1\lambda_4 \sin\varphi = j_{BC} \sin\varphi_5 > 0$ has to hold. Thus if $\Delta'_J = 0$, then $\tilde{\beta}_A = 1$. Thus, if Δ'_J is zero, $\tilde{\beta}_A = \beta_A$ holds. Next, let us prove $\tilde{\beta}_A = \beta_A$ in the case where Δ'_J is not zero. If Δ'_J is not zero, then $\sin\varphi'_5$ must be zero, because at least one of Δ'_J and $\sin\varphi'_5$ is zero. Because of $\sin\varphi'_5 = 0$ and (A.52), $j_{BC} \sin\varphi_5 = 0$ holds. Thus, $j_{BC} \sin\varphi_5 = j'_{BC} \sin\varphi'_5 = 0$. This means $j_{BC} = j'_{BC}$, because $j'_{BC} \cos\varphi'_5 = j_{BC} \cos\varphi_5$ also holds. Thus, $\beta_A = 0$ has to hold, because of (3.154). Incidentally, the β-specifying parameter $\tilde{\beta}_A$ is equal to 0, because of $\sin\varphi_5 = 0$ and the definition of the β-specifying parameter $\tilde{\beta}_A$. Thus, $\tilde{\beta}_A = \beta_A$.

Next, let us perform the step 1-4-B. In other words, we prove that if the state $|\psi\rangle$ and the EP-definite state $|\psi'\rangle$ satisfy Conditions 1 and 2, we can take a measurement which transforms the state $|\psi\rangle$ into the EP-definite state $|\psi'\rangle$. Because of the assumption that $Q_{\text{e}} = 0$, the expression $\Delta_J = 0 \lor \sin\varphi = 0$ holds. If $\Delta_J = 0$, Lemma 4 guarantees that we

can take the measurement which transforms the state $|\psi\rangle$ to the EP-definite state $|\psi'\rangle$, because of $0 = Q_e = Q'_e$. If $\sin\varphi = 0$, as we show below, the measurement transforms the state $|\psi\rangle$ into the EP-definite state $|\psi'\rangle$ is a measurement M whose measurement parameters a, b, k and θ satisfy (3.204)–(3.206) and (3.208), whose entanglement charge Q_e is substituted by unity and where the probability p satisfies (3.210) and $p \geq 1/2$. We have already shown that (3.204)–(3.206), (3.208), (3.210) and $p \geq 1/2$ satisfy (3.177)–(3.179), (3.199), (3.200), (3.218), (3.160) and (3.162). Because of (3.177)–(3.179), the measurement M satisfies (3.163)–(3.167). Because of (3.179), (3.206) and $\sin\varphi = 0$, the equations $j^{(0)}\sin\varphi_5^{(0)} = j_{BC}^{(1)}\sin\varphi_5^{(1)} = 0$ hold. The equation $j^{(0)}\sin\varphi_5^{(0)} = 0$ holds only if $\sin\varphi^{(0)} = 0$ or the state $|\psi^{(0)}\rangle$ is EP indefinite. Thus, $Q_e^{(0)} = 0$. In the same manner, $Q_e^{(1)} = 0$ has to hold, and thus, the measurement M satisfies (3.168). Because the measurement M satisfies (3.177) and (3.160), the measurement M satisfies (3.161). Thus, the measurement M satisfies (3.163)–(3.162). Hence, the measurement M transforms the state $|\psi\rangle$ to the EP-definite state $|\psi'\rangle$ because the measurement M satisfies (3.160)–(3.167), $Q_e = Q_e^{(i)}$, (3.218) and (3.210). Now we have shown Theorem 1 in the case of $Q_e = 0 \wedge$ (the state $|\psi\rangle$ is $\tilde{\zeta}$-definite).

Finally, we perform the steps 2-1 and 2-2. Because the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite if and only if Δ_J and $\lambda_1\lambda_4\sin\varphi$ are zero, we assume that $\Delta_J = 0$ and $\lambda_1\lambda_4\sin\varphi = 0$ in the steps 2-1 and 2-2. First, we perform the step 2-1. Namely, we show that Conditions 1 and 2 are necessary conditions. Because of Lemma 2, (A.40)–(A.42) and (A.46), Condition 1 is necessary. Let us show that Condition 2 is necessary. Because of Lemma 2, we can take $0 \leq \beta_A \leq 1$ which satisfies that $j_{BC}'^2 = j_{BC}^2 + \beta_A(1-\alpha_A^2)j_{ABC}^2$. If $0 < \beta_A < 1$ holds, then we have

$$j_{BC}'^2 \;>\; j_{BC}^2, \tag{3.219}$$

$$K'_{BC} \;=\; K_{BC} - (1-\beta_A)(1-\alpha_A^2)j_{ABC}^2 < K_{BC}, \tag{3.220}$$

$$\begin{aligned}
\lambda_1'^2\lambda_4'^2\sin^2\varphi' &= j_{BC}'^2\sin^2\varphi_5' = j_{BC}'^2 - j_{BC}'^2\cos^2\varphi_5' \\
&= j_{BC}'^2 - j_{BC}^2\cos^2\varphi_5 > j_{BC}^2 - j_{BC}^2\cos^2\varphi_5 \\
&= j_{BC}^2\sin^2\varphi_5 = \lambda_1^2\lambda_4^2\sin^2\varphi,
\end{aligned} \tag{3.221}$$

$$\Delta'_{\mathrm{norm}} \;\equiv\; \frac{\Delta'_J}{\alpha_A^4} = K_5^2 - 4K_{AB}K_{AC}K'_{BC} > K_5^2 - 4K_{\mathrm{ap}} = \Delta_J. \tag{3.222}$$

Thus, if $0 < \beta_A < 1$, then $\lambda_1'\lambda_4'\sin\varphi' > 0$ and $\Delta'_J > 0$ hold. Hence, if $0 < \beta_A < 1$, then $Q'_e = \pm 1$. If $\beta_A = 1$, we have

$$K'_{BC} = j_{BC}^2 + (1-\alpha_A^2)j_{ABC}^2 + \alpha_A^2 j_{ABC}^2 = j_{BC}^2 + j_{ABC}^2 = K_{BC}, \tag{3.223}$$

$$\Delta'_{\mathrm{norm}} = K_5^2 - 4K_{AB}K_{AC}K'_{BC} = K_5^2 - 4K_{\mathrm{ap}} = \Delta_J = 0. \tag{3.224}$$

Thus, if $\beta_A = 1$, $\Delta'_J = 0$ holds. Hence, if $\beta_A = 1$, then $Q'_e = 0$. If $\beta_A = 0$, we have

$$j_{BC}'^2 \;=\; j_{BC}^2, \tag{3.225}$$

$$\begin{aligned}
\lambda_1'^2\lambda_4'^2\sin^2\varphi' &= j_{BC}'^2\sin^2\varphi_5' = j_{BC}'^2 - j_{BC}'^2\cos^2\varphi_5' \\
&= j_{BC}'^2 - j_{BC}^2\cos^2\varphi_5 = j_{BC}^2 - j_{BC}^2\cos^2\varphi_5 = j_{BC}^2\sin^2\varphi_5 \\
&= \lambda_1^2\lambda_4^2\sin^2\varphi = 0.
\end{aligned} \tag{3.226}$$

Thus, if $\beta_A = 0$, $\lambda_1' \lambda_4' \sin \varphi' = 0$ holds. Note that $\sin \varphi' = 0$ follows from $\lambda_1' \lambda_4' \sin \varphi' = 0$: if λ_1' or λ_4' is zero, then $\sin \varphi'$ is also zero, because then there is a zero in $\{\lambda_i' | i = 0, ..., 4\}$. Hence, if $\beta_A = 0$, then $Q_e' = 0$. Hence, $|Q_e'| = \mathrm{sgn}[\beta_A(1 - \beta_A)]$ has to hold.

Next, let us perform the step 2-2. In other words, we prove that Conditions 1 and 2 are sufficient conditions. To show this, we only have to show that if Conditions 1 and 2 are satisfied, there is a measurement which transforms the state $|\psi\rangle$ into the EP-definite state $|\psi'\rangle$. We define the measurement parameters a, b, k and θ as follows:

$$a = \frac{1}{2} + Q_e \frac{\sqrt{4 K_{AB} K_{AC} j_{ABC}^2 (1 - \alpha_A^2)(1 - \beta_A)}}{2 K_5}, \tag{3.227}$$

$$b = \frac{1}{2} - Q_e \frac{\sqrt{4 K_{AB} K_{AC} j_{ABC}^2 (1 - \alpha_A^2)(1 - \beta_A)}}{2 K_5}, \tag{3.228}$$

$$k \cos \theta = -Q_e \frac{\sqrt{4 K_{AB} K_{AC} j_{ABC}^2 (1 - \alpha_A^2)(1 - \beta_A)}}{2 K_5} \frac{j_{BC} \cos \varphi_5}{j_{ABC}}, \tag{3.229}$$

$$k \sin \theta = \frac{\sqrt{\beta_A (1 - \alpha_A^2)}}{2}. \tag{3.230}$$

These four parameters satisfy the following equations:

$$
\begin{aligned}
a &= 1 - b, & (3.231) \\
p &= \lambda_0^2 a + (1 - \lambda_0^2) b + 2 \lambda_0 \lambda_1 k \cos(\theta - \varphi) = \lambda_0^2 + (1 - 2\lambda_0^2) b + 2 \lambda_0 \lambda_1 k \cos \theta \cos \varphi \\
&= \frac{1}{2} - Q_e \frac{\sqrt{4 K_{AB} K_{AC} j_{ABC}^2 (1 - \beta_A)(1 - \alpha_A^2)}}{2 K_5} \left(1 - \frac{K_5}{K_{BC}} + \frac{2\lambda_0}{\lambda_4} \lambda_1 \lambda_4 \cos \varphi \frac{j_{BC} \cos \varphi_5}{j_{ABC}} \right) \\
&= \frac{1}{2} - Q_e \frac{\sqrt{4 K_{AB} K_{AC} j_{ABC}^2 (1 - \beta_A)(1 - \alpha_A^2)}}{2 K_5} \left(1 - \frac{K_5}{K_{BC}} + \frac{2\lambda_0^2}{j_{ABC}^2} (\lambda_2 \lambda_3 - j_{BC} \cos \varphi_5) j_{BC} \cos \varphi_5 \right) \\
&= \frac{1}{2} - Q_e \frac{\sqrt{4 K_{AB} K_{AC} j_{ABC}^2 (1 - \beta_A)(1 - \alpha_A^2)}}{2 K_5} \left(1 - \frac{K_5}{K_{BC}} + \frac{1}{j_{ABC}^2} (J_5 - j_{BC}^2 \frac{K_5}{K_{BC}}) \right) = \frac{1}{2}, \quad (3.232)
\end{aligned}
$$

where we use $\lambda_0^2 = K_5 / K_{BC}$ and (A.29). From (3.232) and $a = 1 - b$, we obtain (3.174). From (3.232) and (3.227)–(3.230), we obtain

$$\frac{ab - k^2}{p^2} = 4 \left(\frac{1}{4} - \frac{4 K_{AB} K_{AC} (1 - \alpha_A^2)(1 - \beta_A)}{4 K_5^2} (j_{ABC}^2 + j_{BC}^2) - \frac{\beta_A (1 - \alpha_A^2)}{4} \right) = \alpha_A^2, \tag{3.233}$$

where we used $\Delta_J = K_5^2 - 4 K_{ap} = 0$. Now, we have shown (3.154) without its fourth column. Next, we obtain $j_{BC}'^2 \sin^2 \varphi_5' = j_{BC}^2 \sin^2 \varphi_5 + \beta_A (1 - \alpha_A^2) j_{ABC}^2$ by using $j_{BC}^2 \sin^2 \varphi_5 = 0$:

$$\lambda_1^{(i)} \lambda_4^{(i)} \sin \varphi^{(i)} = \pm \frac{k \sin \theta j_{ABC}}{p} = \pm \sqrt{\beta_A (1 - \alpha_A^2) j_{ABC}^2}, \tag{3.234}$$

$$(j_{BC}^{(i)})^2 \sin^2 \varphi_5^{(i)} = (\lambda_1^{(i)} \lambda_4^{(i)} \sin \varphi^{(i)})^2 = \beta_A (1 - \alpha_A^2) j_{ABC}^2, \tag{3.235}$$

where the double sign $\pm$ takes $+$ if $i = 0$ and takes $-$ if $i = 1$. Now, we have shown that the measurement whose the measurement parameters a, b, k and θ satisfy (3.227)–(3.230)

satisfies Condition 1. Let us show that the measurement realizes Condition 2:

$$
\begin{aligned}
(\lambda_0^{(i)})^2 = \frac{p_{(i)}}{b_{(i)}} \alpha_A^2 \frac{K_5}{2K_{BC}} &= \frac{\alpha_A^2 K_5^2}{2K_{BC}(K_5 \mp Q_e\sqrt{4K_{AB}K_{AC}j_{ABC}^2(1-\beta_A)(1-\alpha_A^2)})} \\
&= \frac{K_5^2\alpha_A^2(K_5 \pm Q_e\sqrt{4K_{AB}K_{AC}j_{ABC}^2(1-\beta_A)(1-\alpha_A^2)})}{2K_{BC}(K_5^2 - 4K_{AB}K_{AC}j_{ABC}^2(1-\beta_A)(1-\alpha_A^2))} \\
&= \frac{\alpha_A^2(K_5 \pm Q_e\sqrt{4K_{AB}K_{AC}j_{ABC}^2(1-\beta_A)(1-\alpha_A^2)})}{2(K_{BC} - (1-\beta_A)(1-\alpha_A^2)j_{ABC}^2)} \\
&= \frac{K_5' \pm Q_e\sqrt{4K_{AB}K_{AC}j_{ABC}^2(1-\beta_A)(1-\alpha_A^2)}}{2K_{BC}'}, \quad (3.236)
\end{aligned}
$$

where if $i = 0$, the double sign $\pm$ is $+$, and if $i = 1$, the double sign $\pm$ is $-$. From the definition of the entanglement charge $Q_e^{(i)}$, (3.234) and (3.236), we obtain $Q_e^{(0)} = Q_e^{(1)}$ and $|Q_e'| = \text{sgn}[\beta_A(1-\beta_A)]$. Thus, the measurement satisfies Condition 2. Thus, we have completed the seventh step. Hence, we have completed the proof of Theorem 1. $\square$

Theorem 1 can be expressed in K-parameters as follows.

Theorem 1′ *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the K-entanglement parameters of the state $|\psi\rangle$ and the state $|\psi'\rangle$ as $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5, Q_e)$ and $(K_{AB}', K_{AC}', K_{BC}', j_{ABC}', J_5', Q_e')$, respectively. We assume the state $|\psi'\rangle$ to be EP definite. We also assume that $j_{ABC} \neq 0$. Then, a necessary and sufficient condition of the possibility of an A-DMT from the state $|\psi\rangle$ to the EP-definite state $|\psi'\rangle$ is that the following two conditions are satisfied:*
Condition 1: There are real numbers $0 \leq \zeta_A \leq 1$ and $\zeta_{lower}^{(A)} \leq \zeta^{(A)} \leq 1$ which satisfy the following equation:

$$
\begin{pmatrix} K_{AB}' \\ K_{AC}' \\ K_{BC}' \\ j_{ABC}'^2 \\ J_5' \end{pmatrix} = \zeta^{(A)} \begin{pmatrix} \zeta_A & & & & \\ & \zeta_A & & & \\ & & 1 & & \\ & & & \zeta_A & \\ & & & & \zeta_A \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j_{ABC}^2 \\ J_5 \end{pmatrix}, \quad (3.237)
$$

where

$$
\zeta_{lower}^{(A)} = \frac{j_{BC}^2}{(K_{BC} - \zeta_A j_{ABC}^2)}. \quad (3.238)
$$

If $\zeta_A = 1$ and $j_{BC} = 0$ hold, we define the lower bound $\zeta_{lower}^{(A)}$ to be unity.
Condition 2: Let us check whether the state $|\psi\rangle$ is $\tilde{\zeta}$-definite or not. When the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the condition is

$$
Q_e = Q_e' \text{ and } \zeta^{(A)} = \tilde{\zeta}^{(A)}. \quad (3.239)
$$

where

$$
\tilde{\zeta}^{(A)} = \frac{j_{BC}^2(\Delta_J + 4K_{ap}\sin^2\varphi_5)}{4K_{ap}j_{BC}^2\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j_{ABC}^2)}, \quad (3.240)
$$

where if the state $|\psi\rangle$ is EP indefinite, then we define $j_{BC}^2\sin^2\varphi_5$ as zero. When the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, the condition is

$$
|Q_e'| - \text{sgn}[(1-\zeta^{(A)})(\zeta^{(A)} - \zeta_{lower}^{(A)})], \quad (3.241)
$$

or in other words,

$$Q'_e = 0 \ \ for \ \zeta^{(A)} = 1 \ \ or \ \ \zeta^{(A)} = \zeta^{(A)}_{lower}, \tag{3.242}$$

$$Q'_e \neq 0 \ \ otherwise. \tag{3.243}$$

Comment:Theorem 1 guarantees that we can specify an A-DMT by determining its parameters $\zeta^{(A)}$ and ζ_A. Hereafter we refer to ζ_A and $\zeta^{(A)}$ as the main parameter of A and sub parameter of A, respectively. Note that if the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, we only have to determine the main parameter of A ζ_A in order to specify an A-DMT. This is the reason why ζ_A is refferred to as "main."

Proof: First, we show that Condition 1 of Theorem 1 is equivalent to Condition 1 of Theorem 1 . Using $K_{BC} = j_{BC}^2 + j_{ABC}^2$ and Condition 1 of Theorem 1, we have the following equations:

$$\alpha_A^2 = \zeta^{(A)}\zeta_A, \tag{3.244}$$

$$K'_{BC} = \zeta^{(A)}K_{BC} = K_{BC} - (1 - \beta_A)(1 - \alpha_A^2)j_{ABC}^2. \tag{3.245}$$

From these equations, we obtain the expression of the main parameter of A ζ_A in terms of the transfer parameter α_A and the dissipation parameter β_A:

$$\zeta_A = \frac{K_{BC}\alpha_A^2}{K_{BC} - (1 - \beta_A)(1 - \alpha_A^2)j_{ABC}^2}. \tag{3.246}$$

Thus, if $0 \leq \alpha_A \leq 1$ and $0 \leq \beta_A \leq 1$ hold, then

$$0 \leq \zeta_A = \frac{K_{BC} - (1 - \alpha_A^2)K_{BC}}{K_{BC} - (1 - \beta_A)(1 - \alpha_A^2)j_{ABC}^2} \leq 1, \tag{3.247}$$

where $\zeta_A = 0$ when $\alpha_A = 0$, while $\zeta_A = 1$ when $\alpha_A = 1$. From (3.244) and (3.245), we also obtain the expression of sub parameter of A $\zeta^{(A)}$ in terms of the dissipation parameter β_A and the main parameter ζ_A:

$$\zeta^{(A)}K_{BC} = K_{BC} - (1 - \beta_A)(1 - \zeta^{(A)}\zeta_A)j_{ABC}^2, \tag{3.248}$$

$$\zeta^{(A)} = \frac{K_{BC} - (1 - \beta_A)j_{ABC}^2}{K_{BC} - (1 - \beta_A)\zeta_A j_{ABC}^2}. \tag{3.249}$$

Thus, if $0 \leq \alpha_A \leq 1$ and $0 \leq \beta_A \leq 1$ hold, then

$$\frac{j_{BC}^2}{K_{BC} - \zeta_A j_{ABC}^2} = \zeta^{(A)}_{\text{lower}} \leq \zeta^{(A)} \leq 1 \tag{3.250}$$

holds, where $\zeta^{(A)} = \zeta^{(A)}_{\text{lower}}$ holds, while $\beta_A = 0$ and $\zeta^{(A)} = 1$ holds when $\beta_A = 1$. Hence, Condition 1 of Theorem 1 is equivalent to Condition 1 of Theorem 1 .

Second, we show that Condition 2 of Theorem 1 is equivalent to Condition 2 of Theorem 1 . We prove that if the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, $\zeta^A = \tilde{\zeta}^{(A)}$ holds. Because of (3.249) and because $\beta_A = \tilde{\beta}_A$ as shown in Theorem 1,

$$
\begin{aligned}
\zeta^{(A)} &= \frac{K_{BC} - \dfrac{\Delta_J}{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2 \sin^2\varphi_5}j_{ABC}^2}{K_{BC} - \dfrac{\Delta_J}{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2 \sin^2\varphi_5}\zeta_A j_{ABC}^2} \\
&= \frac{j_{BC}^2(\Delta_J + 4K_{\text{ap}}\sin^2\varphi_5)}{4K_{\text{ap}}j_{BC}^2\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j_{ABC}^2)} = \tilde{\zeta}.
\end{aligned}
\tag{3.251}
$$

Thus, if the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, $\zeta^{(A)} = \tilde{\zeta}^{(A)}$ holds, and thus Condition 2 of Theorem 1 is equivalent to Condition 2 of Theorem 1 . Next, we treat the case where the the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite. As we proved in the previous paragraph, the conditions $\zeta^{(A)} = \zeta^{(A)}_{\text{lower}}$ and $\zeta^{(A)} = 1$ are equivalent to the conditions $\beta_A = 0$ and $\beta_A = 1$, respectively. Thus, $|Q'_{\text{e}}| = \text{sgn}[(1 - \zeta^{(A)})(\zeta^{(A)} - \zeta^{(A)}_{\text{lower}})]$ is equivalent to $|Q'_{\text{e}}| = \text{sgn}[\beta_A(1 - \beta_A)]$. Thus, Condition 2 of Theorem 1 is equivalent to Condition 2 of Theorem 1 . □

Step 2 of Case $\mathfrak{A}$

In the present section 6.1.2, we obtain a necessary and sufficient condition of the possibility of a C-LOCC transformation from $|\psi\rangle$ to $|\psi'\rangle$, where $|\psi\rangle$ and $|\psi'\rangle$ are EP definite. This corresponds to Step 2 of Case $\mathfrak{A}$.

Lemma 6 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the entanglement parameters of the states $|\psi\rangle$ and $|\psi'\rangle$ as $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5, Q_e)$ and $(K'_{AB}, K'_{AC}, K'_{BC}, j'_{ABC}, J'_5, Q'_e)$, respectively. We assume that both states $|\psi\rangle$ and $|\psi'\rangle$ are EP definite. We also assume that $j_{ABC} \neq 0$. Then, a necessary and sufficient condition of the possibility of a C-LOCC transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is that the following two conditions are satisfied:*
Condition 1: There are real numbers $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$, $0 \leq \zeta_C \leq 1$ and $\zeta_{lower} \leq \zeta \leq 1$ which satisfy the following equation:

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta \begin{pmatrix} \zeta_A\zeta_B & & & & \\ & \zeta_A\zeta_C & & & \\ & & \zeta_B\zeta_C & & \\ & & & \zeta_A\zeta_B\zeta_C & \\ & & & & \zeta_A\zeta_B\zeta_C \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}, \qquad (3.252)
$$

where

$$
\zeta_{lower} = \frac{j^2_{AB}j^2_{AC}j^2_{BC}}{(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}, \qquad (3.253)
$$

and we refer to ζ, ζ_A, ζ_B and ζ_C as the sub parameter and the main parameters of A, B and C, respectively.
Condition 2: let us check whether the state $|\psi\rangle$ is $\tilde{\zeta}$-definite or not. When the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the condition is

$$
Q_e = Q'_e \ \text{ and } \ \zeta = \tilde{\zeta}, \qquad (3.254)
$$

where

$$
\tilde{\zeta} = \frac{J_{ap}(\Delta_J + 4K_{ap}\sin^2\varphi_5)}{4K_{ap}J_{ap}\sin^2\varphi_5 + \Delta_J(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}. \qquad (3.255)
$$

When the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, the condition is

$$
|Q'_e| = sgn[(1 - \zeta)(\zeta - \zeta_{lower})], \qquad (3.256)
$$

or in the other words,

$$
Q'_e \begin{cases} = 0 & (\zeta = 1 \ \text{or} \ \zeta = \zeta_{lower}), \\ \neq 0 & (otherwise). \end{cases} \qquad (3.257)
$$

Proof: We first describe the structure of the proof. We divide the proof in two parts. In the first part, we consider the case where the the state $|\psi\rangle$ is $\tilde{\zeta}$-definite. In the second part, we consider the case where the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite. In the first case, we prove the present Lemma in the following three steps:

1-1 We prove that if we can transform the state $|\psi\rangle$ into the state $|\psi'\rangle$ by performing an A-DMT and a B-DMT successively, we can also transform $|\psi\rangle$ into $|\psi'\rangle$ by performing another B-DMT and another A-DMT successively. This holds not only for an A-DMT and a B-DMT, but also a B-DMT and a C-DMT, or an A-DMT and a C-DMT. Thus, we can reproduce an arbitrary C-LOCC by performing the following: we perform first A-DMTs, second B-DMTs, and third C-DMTs.

1-2 We prove that if we can transform the state $|\psi\rangle$ into the state $|\psi'\rangle$ by performing two A-DMTs successively, we can also transform the state $|\psi\rangle$ into the state $|\psi'\rangle$ by performing an A-DMT. This statement means that we can reproduce an arbitrary C-LOCC by performing an A-DMT, a B-DMT and a C-DMT successively. Thus, a C-LOCC transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is executable if and only if we can transform the state $|\psi\rangle$ into the state $|\psi'\rangle$ by performing an A-DMT, a B-DMT and a C-DMT successively.

1-3 We prove that a necessary and sufficient condition of the possibility of the transformation which is constituted by an A-DMT, a B-DMT and a C-DMT is Conditions 1 and 2 of the present Lemma.

In the second case where the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, we prove the present Lemma in the following four steps:

2-1 We prove that if a DMT from an EP-definite state which is $\tilde{\zeta}$-indefinite to another EP-definite state is executable, then the final state is $\tilde{\zeta}$-definite. Because of this statement, the final state of the first DMT of an arbitrary C-LOCC transformation between EP-definite states is $\tilde{\zeta}$-definite. Thus, an arbitrary C-LOCC transformation can be reproduced by performing four DMTs successively: the first DMT of the C-LOCC T_{f}, an A-DMT T_A, a B-DMT T_B and a C-DMT T_C. The first DMT may be either an A-, a B- or a C-DMT. We can assume that the first DMT T_{f} is an A-DMT without losing generality.

2-2 We prove that a necessary and sufficient condition of the possibility of the transformation which is constituted by the DMTs T_{f}, T_A, T_B and T_C is Conditions 1 and 2 of the present Lemma.

Now, let us perform the step 1-1. We perform this by showing a necessary and sufficient condition of the possibility of performing an A-DMT and a B-DMT successively and that of the possibility of performing a B-DMT and an A-DMT successively are the same. Because of Theorem 1 , we can realize the transformation $|\psi\rangle \rightarrow |\psi''\rangle \rightarrow |\psi'\rangle$ by operating an A-DMT and a B-DMT successively if and only if $Q'_{\mathrm{e}} = Q_{\mathrm{e}}$ and there are $0 \leq \zeta_A \leq 1$

and $0 \leq \zeta_B \leq 1$ which satisfy the following equation:

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta^{(A)}\zeta'^{(B)} \begin{pmatrix} \zeta_B & & & & \\ & 1 & & & \\ & & \zeta_B & & \\ & & & \zeta_B & \\ & & & & \zeta_B \end{pmatrix} \begin{pmatrix} \zeta_A & & & & \\ & \zeta_A & & & \\ & & 1 & & \\ & & & \zeta_A & \\ & & & & \zeta_A \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix},
$$
(3.258)

where the sub parameter of A $\zeta^{(A)}$ and the sub parameter of B are given by

$$
\zeta^{(A)} = \frac{j^2_{BC}(\Delta_J + 4K_{\mathrm{ap}}\sin^2\varphi_5)}{4j^2_{BC}K_{\mathrm{ap}}\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j^2_{ABC})},
$$
(3.259)

$$
\zeta'^{(B)} = \frac{(j''_{AC})^2(\Delta''_J + 4K''_{AB}K''_{AC}K''_{BC}\sin^2\varphi''_5)}{4(j''_{AC})^2 K''_{AB}K''_{AC}K''_{BC}\sin^2\varphi''_5 + \Delta''_J(K''_{AC} - \zeta_B(j''_{ABC})^2)},
$$
(3.260)

and where j''_{AC}, K''_{AB}, etc. are the entanglement parameters of the state $|\psi''\rangle$. On the other hand, we can realize the transformation the $|\psi\rangle \to |\psi'''\rangle \to |\psi'\rangle$ by operating a B-DMT and an A-DMT successively if and only if $Q'_\mathrm{e} = Q_\mathrm{e}$ and there are $0 \leq \zeta_A \leq 1$ and $0 \leq \zeta_B \leq 1$ which satisfy the following equation:

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta^{(B)}\zeta'^{(A)} \begin{pmatrix} \zeta_A & & & & \\ & \zeta_A & & & \\ & & 1 & & \\ & & & \zeta_A & \\ & & & & \zeta_A \end{pmatrix} \begin{pmatrix} \zeta_B & & & & \\ & 1 & & & \\ & & \zeta_B & & \\ & & & \zeta_B & \\ & & & & \zeta_B \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix},
$$
(3.261)

where

$$
\zeta^{(B)} = \frac{j^2_{AC}(\Delta_J + 4K_{\mathrm{ap}}\sin^2\varphi_5)}{4j^2_{AC}K_{\mathrm{ap}}\sin^2\varphi_5 + \Delta_J(K_{AC} - \zeta_B j^2_{ABC})},
$$
(3.262)

$$
\zeta'^{(A)} = \frac{(j'''_{BC})^2(\Delta'''_J + 4K'''_{AB}K'''_{AC}K'''_{BC}\sin^2\varphi'''_5)}{4(j'''_{BC})^2 K'''_{AB}K'''_{AC}K'''_{BC}\sin^2\varphi'''_5 + \Delta'''_J(K'''_{BC} - \zeta_A(j'''_{ABC})^2)},
$$
(3.263)

and where j'''_{AC}, K'''_{AB}, etc. are the entanglement parameters of $|\psi''\rangle$.

Note that (3.261)–(3.263) are equivalent to (3.258)–(3.260) with the labels A and B are exchanged. Hence, in order to perform the step 1-1, we only have to show that $\zeta^{(A)}\zeta'^{(B)}$ is symmetric with respect to the labels A and B. Because of (A.40)–(A.42) and (A.43), the equation (3.260) can be transformed as follows:

$$
\begin{aligned}
\zeta'^{(B)} &= \frac{j^2_{AC}(\Delta''_{\mathrm{norm}} + 4K_{AB}K_{AC}K''_{BC}\sin^2\varphi''_5)}{4j^2_{AC}K_{AB}K_{AC}K''_{BC}\sin^2\varphi''_5 + \Delta''_{\mathrm{norm}}(K_{AC} - \zeta_B j^2_{ABC})} \\
&= \frac{j^2_{AC}(\Delta''_{\mathrm{norm}} + 4K_{AB}K_{AC}K''_{BC}\sin^2\varphi''_5)}{j^2_{AC}(\Delta''_{\mathrm{norm}} + 4K_{AB}K_{AC}K''_{BC}\sin^2\varphi''_5) + \Delta''_{\mathrm{norm}}(1 - \zeta_B)j^2_{ABC}} \\
&= \frac{j^2_{AC}}{j^2_{AC} + \Delta''_{\mathrm{norm}}(1 - \zeta_B)j^2_{ABC}/(\Delta''_{\mathrm{norm}} + 4K_{AB}K_{AC}K''_{BC}\sin^2\varphi''_5)},
\end{aligned}
$$
(3.264)

where $\Delta''_{\mathrm{norm}} \equiv \Delta''_J/(\zeta_A\zeta^{(A)})^2$. Let us express Δ''_{norm} and $\Delta''_{\mathrm{norm}}+4K_{AB}K_{AC}K''_{BC}\sin^2\varphi''_5$ in terms of K_{AB}, K_{AC}, K_{BC}, j_{ABC} and J_5. From $\Delta''_{\mathrm{norm}} = K_5^2 - 4K_{AB}K_{AC}K''_{BC}$ and

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(3.259), we obtain

$$
\begin{aligned}
\Delta''_{\text{norm}} &= \Delta_J + 4K_{\text{ap}}(1 - \zeta^{(A)}) \\
&= \Delta_J + 4K_{\text{ap}}\frac{\Delta_J(1 - \zeta_A)j^2_{ABC}}{4j^2_{BC}K_{\text{ap}}\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j^2_{ABC})} \\
&= \Delta_J\frac{4j^2_{BC}K_{\text{ap}}\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j^2_{ABC}) + 4K_{\text{ap}}(1 - \zeta_A)j^2_{ABC}}{4j^2_{BC}K_{\text{ap}}\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j^2_{ABC})}. \quad (3.265)
\end{aligned}
$$

On the other hand, from $\sin^2\varphi_5 = 1 - J_5^2/4j^2_{AB}j^2_{AC}j^2_{BC}$, we obtain

$$
\begin{aligned}
\Delta''_{\text{norm}} &+ 4K_{AB}K_{AC}K''_{BC}\sin^2\varphi''_5 \\
&= \Delta_J + 4K_{\text{ap}}(1 - \zeta^{(A)}) + 4K_{\text{ap}}\zeta^{(A)}\left(1 - \frac{J''^2_5}{4J''_{\text{ap}}}\right) \\
&= \Delta_J + 4K_{\text{ap}}\left(1 - \zeta^{(A)}\frac{j''^2_{AB}j''^2_{AC}j''^2_{BC}\cos^2\varphi''_5}{j''^2_{AB}j''^2_{AC}j''^2_{BC}}\right) = \Delta_J + 4K_{\text{ap}}\left(1 - \frac{j^2_{BC}\cos^2\varphi_5}{K_{BC} - \zeta_A j^2_{ABC}}\right) \\
&= \frac{4j^2_{BC}K_{\text{ap}}\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j^2_{ABC}) + 4K_{\text{ap}}(1 - \zeta_A)j^2_{ABC}}{K_{BC} - \zeta_A j^2_{ABC}}. \quad (3.266)
\end{aligned}
$$

Substituting (3.265) and (3.266) into (3.264) and using (3.259) with it, we obtain

$$
\zeta^{(A)}\zeta'^{(B)} = \frac{j^2_{AC}j^2_{BC}(\Delta_J + 4K_{\text{ap}}\sin^2\varphi_5)}{j^2_{AC}j^2_{BC}4K_{\text{ap}}\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})}. \quad (3.267)
$$

Thus, $\zeta^{(A)}\zeta'^{(B)}$ is symmetric with respect to the exchange of the labels A and B. Hence, if we can transform the state $|\psi\rangle$ into the state $|\psi'\rangle$ by performing an A-DMT and a B-DMT successively, we can also transform the state $|\psi\rangle$ into the state $|\psi'\rangle$ by performing another B-DMT and another A-DMT successively. This holds not only for an A-DMT and a B-DMT, but also a B-DMT and a C-DMT, or an A-DMT and a C-DMT. Thus, we can reproduce an arbitrary C-LOCC by performing as follows: we perform first A-DMTs, second B-DMTs and third C-DMTs.

Next, we perform the step 1-2. We perform this by obtaining specifically the A-DMT which can be substituted for two A-DMTs. When we operate the transformation $|\psi\rangle \to |\psi''\rangle \to |\psi'\rangle$ by operating two A-DMTs T_A and T'_A whose main parameters are ζ_A and ζ'_A, respectively, the entanglement parameters change as follows:

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta^{(A)}\zeta'^{(A)}\begin{pmatrix} \zeta'_A & & & & \\ & \zeta'_A & & & \\ & & 1 & & \\ & & & \zeta'_A & \\ & & & & \zeta'_A \end{pmatrix}\begin{pmatrix} \zeta_A & & & & \\ & \zeta_A & & & \\ & & 1 & & \\ & & & \zeta_A & \\ & & & & \zeta_A \end{pmatrix}\begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}, \quad (3.268)
$$

where $\zeta^{(A)}$ and $\zeta'^{(A)}$, which are the sub parameters of A of DMTs T_A and T_f, are given

$$
\zeta^{(A)} = \frac{j^2_{BC}(\Delta_J + 4K_{\text{ap}}\sin^2\varphi_5)}{4j^2_{BC}K_{\text{ap}}\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j^2_{ABC})}, \quad (3.269)
$$

$$\zeta'^{(A)} = \frac{j_{BC}''^2(\Delta_J'' + 4K_{AB}''K_{AC}''K_{BC}'' \sin^2 \varphi_5'')}{4j_{BC}''^2 K_{AB}''K_{AC}''K_{BC}'' \sin^2 \varphi_5'' + \Delta_J''(K_{BC}'' - \zeta_A' j_{ABC}''^2)}. \tag{3.270}$$

In the same manner as in deriving (3.264), we obtain

$$\zeta'^{(A)} = \frac{j_{BC}''^2(\Delta_{\mathrm{norm}}'' + 4K_{AB}K_{AC}K_{BC}'' \sin^2 \varphi_5'')}{4j_{BC}''^2 K_{AB}K_{AC}K_{BC}'' \sin^2 \varphi_5'' + \Delta_{\mathrm{norm}}''(K_{BC}'' - \zeta_A' j_{ABC}''^2)}$$

$$= \frac{1}{1 + \Delta_{\mathrm{norm}}''(1 - \zeta_A')j_{ABC}''^2/j_{BC}''^2(\Delta_{\mathrm{norm}}'' + 4K_{AB}K_{AC}K_{BC}'' \sin^2 \varphi_5'')}. \tag{3.271}$$

Substituting (3.265) and (3.266) into (3.271) and using (3.269) with it, we obtain

$$\zeta^{(A)}\zeta'^{(A)} = \frac{j_{BC}^2(\Delta_J + 4K_{\mathrm{ap}} \sin^2 \varphi_5)}{4j_{BC}^2 K_{\mathrm{ap}} \sin^2 \varphi_5 + \Delta_J(K_{BC} - \zeta_A\zeta_A' j_{ABC}^2)}. \tag{3.272}$$

Thus, the successive operation of the DMTs T_A and T_A' can be reproduced by an A-DMT whose transfer parameter is $\zeta_A\zeta_A'$. Hence, an arbitrary C-LOCC transformation from a $\tilde{\zeta}$-definite state can be reproduced by performing an A-DMT, a B-DMT and a C-DMT successively. The converse also holds, because successive operation of an A-DMT, a B-DMT and a C-DMTs is a C-LOCC transformation.

Next, we perform the step 1-3. We can realize the transformation $|\psi\rangle \to |\psi''\rangle \to |\psi'''\rangle \to |\psi'\rangle$ by operating an A-DMT, a B-DMT and a C-DMT successively if and only if $Q_{\mathrm{e}}' = Q_{\mathrm{e}}$ and there are $0 \le \zeta_A \le 1$, $0 \le \zeta_B \le 1$ and $0 \le \zeta_C \le 1$ which satisfy the following equation:

$$\begin{pmatrix} K_{AB}' \\ K_{AC}' \\ K_{BC}' \\ j_{ABC}'^2 \\ J_5' \end{pmatrix} = \zeta^{(A)}\zeta'^{(B)}\zeta''^{(C)} \begin{pmatrix} \zeta_A\zeta_B & & & & \\ & \zeta_A\zeta_C & & & \\ & & \zeta_B\zeta_C & & \\ & & & \zeta_A\zeta_B\zeta_C & \\ & & & & \zeta_A\zeta_B\zeta_C \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j_{ABC}^2 \\ J_5 \end{pmatrix}, \tag{3.273}$$

where

$$\zeta^{(A)} = \frac{j_{BC}^2(\Delta_J + 4K_{\mathrm{ap}} \sin^2 \varphi_5)}{4j_{BC}^2 K_{\mathrm{ap}} \sin^2 \varphi_5 + \Delta_J(K_{BC} - \zeta_A j_{ABC}^2)}, \tag{3.274}$$

$$\zeta'^{(B)} = \frac{(j_{AC}'')^2(\Delta_J'' + 4K_{AB}''K_{AC}''K_{BC}'' \sin^2 \varphi_5'')}{4(j_{AC}'')^2 K_{AB}''K_{AC}''K_{BC}'' \sin^2 \varphi_5'' + \Delta_J''(K_{AC}'' - \zeta_B(j_{ABC}'')^2)}, \tag{3.275}$$

$$\zeta''^{(C)} = \frac{j_{AB}'''^2(\Delta_J''' + 4K_{AB}'''K_{AC}'''K_{BC}''' \sin^2 \varphi_5''')}{4j_{AB}'''^2 K_{AB}'''K_{AC}'''K_{BC}''' \sin^2 \varphi_5''' + \Delta_J'''(K_{AB}''' - \zeta_C(j_{ABC}''')^2)}. \tag{3.276}$$

Thus, in order to perform the step 1-3, we prove that $\zeta^{(A)}\zeta'^{(B)}\zeta''^{(C)}$ is equal to the ζ-specifying parameter $\tilde{\zeta}$ of Condition 2 of the present Lemma. We have already shown that $\zeta^{(A)}\zeta'^{(B)}$ follows (3.267). In the same manner as in deriving (3.264), we obtain

$$\zeta''^{(C)} = \frac{j_{AB}^2}{j_{AB}^2 + \Delta_{\mathrm{norm}}'''(1 - \zeta_C)j_{ABC}^2/(\Delta_{\mathrm{norm}}''' + 4K_{\mathrm{ap}}\zeta^{(A)}\zeta'^{(B)} \sin^2 \varphi_5''')}, \tag{3.277}$$

where

$$\Delta_{\mathrm{norm}}''' \equiv K_5^2 - 4K_{\mathrm{ap}}\zeta^{(A)}\zeta'^{(B)}. \tag{3.278}$$

In the same manner as in deriving (3.265) and (3.266), we obtain

$$
\begin{aligned}
\Delta'''_{\text{norm}} &= \Delta_J + 4K_{\text{ap}}(1 - \zeta^{(A)}\zeta'^{(B)}) \\
&= \Delta_J + 4K_{\text{ap}}\frac{\Delta_J\{(K_{AC} - \zeta_A j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC}) - j^2_{AB}j^2_{AC}\}}{j^2_{AC}j^2_{BC}4K_{\text{ap}}\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})} \\
&= \Delta_J\frac{j^2_{AC}j^2_{BC}4K_{\text{ap}}\sin^2\varphi_5 + K_5^2(K_{BC} - \zeta_A j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC}) - 4K_{\text{ap}}j^2_{AB}j^2_{AC}}{j^2_{AC}j^2_{BC}4K_{\text{ap}}\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})}
\end{aligned}
\tag{3.279}
$$

and

$$
\begin{aligned}
\Delta'''_{\text{norm}} + 4K_{\text{ap}}\sin^2\varphi_5'''\zeta^{(A)}\zeta'^{(B)} &= \Delta_J + 4K_{\text{ap}}\left(1 + \frac{j^2_{AC}j^2_{BC}\cos^2\varphi_5}{(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}\right) \\
&= \frac{j^2_{AC}j^2_{BC}4K_{\text{ap}}\sin^2\varphi_5 + K_5^2(K_{BC} - \zeta_A j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC}) - 4K_{\text{ap}}j^2_{AB}j^2_{AC}}{(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}.
\end{aligned}
\tag{3.280}
$$

From (3.277), (3.279), (3.280) and (3.267), we obtain

$$
\begin{aligned}
\zeta^{(A)}\zeta'^{(B)}\zeta''^{(C)} &= \frac{J_{\text{ap}}(\Delta_J + 4K_{\text{ap}}\sin^2\varphi_5)}{4K_{\text{ap}}J_{\text{ap}}\sin^2\varphi_5 + \Delta_J(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})} \\
&= \tilde{\zeta}.
\end{aligned}
\tag{3.281}
$$

Hence, we have performed the step 1-3, and thus have shown the present Lemma in the case where the state $|\psi\rangle$ is $\tilde{\zeta}$-definite.

Next, we perform the step 2-1, where the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite. Because of Theorem 1, the following relation among the entanglement parameters of the initial state $|\psi\rangle$ and the final state $|\psi'\rangle$ of an A-DMT holds:

$$
\begin{aligned}
j'^2_{BC}\sin^2\varphi_5' &= j'^2_{BC} - j'^2_{BC}\cos^2\varphi_5' = j^2_{BC} + \beta_A(1 - \alpha_A^2)j^2_{ABC} - j^2_{BC}\cos^2\varphi_5 \\
&= j^2_{BC}\sin^2\varphi_5 + \beta_A(1 - \alpha_A^2)j^2_{ABC}, \tag{3.282} \\
\Delta_J' &= \alpha_A^4(K_5^2 - 4K_{AB}K_{AC}K'_{BC}) = \alpha_A^4(K_5^2 - 4K_{AB}K_{AC}(j'^2_{BC} + j'^2_{ABC})) \\
&= \alpha_A^4(K_5^2 - 4K_{AB}K_{AC}(j^2_{BC} + \beta_A(1 - \alpha_A^2)j^2_{ABC} + \alpha_A^2 j^2_{ABC})) \\
&= \alpha_A^4(K_5^2 - 4K_{AB}K_{AC}(K_{BC} - (1 - \beta_A)(1 - \alpha_A^2)j^2_{ABC} + \alpha_A^2 j^2_{ABC})) \\
&= \alpha_A^4(\Delta_J + (1 - \beta_A)(1 - \alpha_A^2)j^2_{ABC}K_{AB}K_{AC}), \tag{3.283}
\end{aligned}
$$

where j_{BC} and K_{BC} etc. and j'_{BC} and K'_{BC} etc. are the entanglement parameters of the EP-definite states $|\psi\rangle$ and $|\psi'\rangle$ respectively. Because the EP-definite state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, both of Δ_J and $j_{BC}\sin\varphi_5$ are zero. Substituting $\Delta_J = 0$ and $j_{BC}\sin\varphi_5 = 0$ into (3.282) and (3.283), we can see that both of Δ_J' and $j'_{BC}\sin\varphi_5'$ are zero only if the transfer parameter α_A is one. The equation $\alpha_A = 1$ means that the A-DMT is the identity transformation. Thus, we have completed the step 2-1. Thus, an arbitrary C-LOCC transformation can be reproduced by performing four DMTs successively: the first DMT of the C-LOCC T^{f}, an A-DMT T_A, a B-DMT T_B and a C-DMT T_C. The first DMT can be either an A-, a B- or a C-DMT. We can assume that the first DMT T^{f} is an A-DMT without losing generality.

Next, we perform the step 2-2. We refer to the final state after the first DMT T^{f} as $|\psi''\rangle$ and refer to the entanglement K-parameters of the state $|\psi''\rangle$ as $(K''_{AB}, K''_{AC}, K''_{BC}, j''_{ABC}, J''_5, Q''_{\mathrm{e}})$. A necessary and sufficient condition of the possibility of the C-LOCC transformation is the existence of the parameters $0 \le \zeta_{A0} \le 1$, $0 \le \zeta_A \le 1$, $0 \le \zeta_B \le 1$, $0 \le \zeta_C \le 1$ and $j^2_{BC}/(K_{BC} - \zeta_{A0}j^2_{ABC}) \le \zeta_0 \le 1$ which satisfy the following equation:

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \tilde{\zeta}'\zeta_0 \begin{pmatrix} \zeta_{A0}\zeta_A\zeta_B & & & & \\ & \zeta_{A0}\zeta_A\zeta_C & & & \\ & & \zeta_B\zeta_C & & \\ & & & \zeta_{A0}\zeta_A\zeta_B\zeta_C & \\ & & & & \zeta_{A0}\zeta_A\zeta_B\zeta_C \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix} \tag{3.284}
$$

$$
|Q'_{\mathrm{e}}| = \mathrm{sgn}\left[(1 - \zeta_0)\left(\zeta_0 - \frac{j^2_{BC}}{K_{BC} - \zeta_{A0}j^2_{ABC}}\right)\right], \tag{3.285}
$$

where

$$
\tilde{\zeta}' = \frac{j''^2_{AB}j''^2_{AC}j''^2_{BC}(\Delta''_J + 4K''_3 \sin^2 \varphi''_5)}{4K''_3 j''^2_{AB}j''^2_{AC}j''^2_{BC} \sin^2 \varphi''_5 + \Delta''_J(K''_{AB} - \zeta_C j''^2_{ABC})(K''_{AC} - \zeta_B j''^2_{ABC})(K''_{BC} - \zeta_A j''^2_{ABC})}. \tag{3.286}
$$

Let us prove that these conditions are equivalent to Conditions 1 and 2 of the present Lemma. In other words, we prove the following statements:

Statement 1 The following inequality holds:

$$
\frac{j^2_{AB}j^2_{AC}j^2_{BC}}{(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A \zeta_{A0} j^2_{ABC})} \le \zeta_0\tilde{\zeta}' \le 1, \tag{3.287}
$$

where the left equality of (3.287) holds if and only if ζ_0 is equal to $j^2_{BC}/(K_{BC} - \zeta_{A0}j^2_{ABC})$, while the right equality of (3.287) holds if and only if ζ_0 is equal to 1.

Statement 2 The equation (3.285) is equivalent to

$$
|Q'_{\mathrm{e}}| = \mathrm{sgn}\left[(1 - \zeta_0\tilde{\zeta}')\left(\zeta_0\tilde{\zeta}' - \frac{j^2_{AB}j^2_{AC}j^2_{BC}}{(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A \zeta_{A0} j^2_{ABC})}\right)\right]. \tag{3.288}
$$

First, we prove Statement 1. In order to perform the proof, we use the following equations which hold because $|\psi\rangle$ is $\tilde{\zeta}$-indefinite if and only if $\Delta_J = 0$ and $\sin\varphi_5 = 0$:

$$K_5^2 = \Delta_J + 4K_{\mathrm{ap}} = 4K_{\mathrm{ap}}, \tag{3.289}$$

$$\Delta_J + 4K_{\mathrm{ap}}(1-\zeta_0) = 4K_{\mathrm{ap}}(1-\zeta_0), \tag{3.290}$$

$$\sin^2\varphi_5'' = 1 - \frac{J_5''^2}{4j_{AB}''^2 j_{AC}''^2 j_{BC}''^2} = 1 - \frac{j_{BC}^2\cos^2\varphi_5}{\zeta_0(K_{BC}-\zeta_{A0}j_{ABC}^2)} = 1 - \frac{j_{BC}^2}{\zeta_0(K_{BC}-\zeta_{A0}j_{ABC}^2)} \tag{3.291}$$

$$\Delta_{\mathrm{norm}}'' = \frac{\Delta_J''}{\zeta_{A0}^2\zeta_0^2} = K_5^2 - 4K_{\mathrm{ap}}\zeta_0 = 4K_{\mathrm{ap}} - 4K_{\mathrm{ap}}\zeta_0, \tag{3.292}$$

$$\begin{aligned}
\Delta_{\mathrm{norm}}'' + 4K_{\mathrm{ap}}\zeta_0\sin^2\varphi_5'' &= \frac{\Delta_J''}{\zeta_{A0}^2\zeta_0^2} + 4K_{\mathrm{ap}}\zeta_0\left(1 - \frac{j_{BC}^2}{\zeta_0(K_{BC}-\zeta_{A0}j_{ABC}^2)}\right) \\
&= K_5^2 - 4K_{\mathrm{ap}}\zeta_0 + 4K_{\mathrm{ap}}\zeta_0\left(1 - \frac{j_{BC}^2}{\zeta_0(K_{BC}-\zeta_{A0}j_{ABC}^2)}\right) \\
&= 4K_{\mathrm{ap}} - 4K_{\mathrm{ap}}\zeta_0 + 4K_{\mathrm{ap}}\zeta_0\left(1 - \frac{j_{BC}^2}{\zeta_0(K_{BC}-\zeta_{A0}j_{ABC}^2)}\right) \\
&= 4K_{\mathrm{ap}}\frac{(1-\zeta_{A0})j_{ABC}^2}{K_{BC}-\zeta_{A0}j_{ABC}^2}. \tag{3.293}
\end{aligned}$$

Because of these equations, (A.40)–(A.43) and $j_{BC}'' = \zeta_0(K_{BC}-\zeta_{A0}j_{ABC}^2)$, (3.286) can be transformed as follows: Substituting (A.40)–(A.43) and $K_{BC}'' = \zeta_0 K_{BC}$ into (3.286), we obtain

$$\tilde{\zeta}' = \frac{j_{AB}^2 j_{AC}^2 j_{BC}''^2(\Delta_{\mathrm{norm}}'' + 4K_{\mathrm{ap}}\zeta_0\sin^2\varphi_5'')}{4\zeta_0 K_{\mathrm{ap}}j_{AB}^2 j_{AC}^2 j_{BC}''^2\sin^2\varphi_5'' + \Delta_{\mathrm{norm}}''(K_{AB}-\zeta_C j_{ABC}^2)(K_{AC}-\zeta_B j_{ABC}''^2)\zeta_0(K_{BC}-\zeta_A\zeta_{A0}j_{ABC}^2)}. \tag{3.294}$$

Substituting (3.291)–(3.293) into (3.294), we obtain

$$\tilde{\zeta}' = \tag{3.295}$$
$$\frac{j_{AB}^2 j_{AC}^2(1-\zeta_{A0})j_{ABC}^2}{j_{AB}^2 j_{AC}^2(\zeta_0(K_{BC}-\zeta_{A0}j_{ABC}^2) - j_{BC}^2) + (1-\zeta_0)(K_{AB}-\zeta_C j_{ABC}^2)(K_{AC}-\zeta_B j_{ABC}^2)(K_{BC}-\zeta_A\zeta_{A0}j_{ABC}^2)}$$

Note that we can express the denominater of (3.295) as

$$\zeta_0(j_{AB}^2 j_{AC}^2(K_{BC}-\zeta_{A0}j_{ABC}^2) - (K_{AB}-\zeta_C j_{ABC}^2)(K_{AC}-\zeta_B j_{ABC}^2)(K_{BC}-\zeta_A\zeta_{A0}j_{ABC}^2)) + (\text{the part irrelevant to } \tag{3.296}$$

Because of

$$j_{AB}^2 j_{AC}^2(K_{BC}-\zeta_{A0}j_{ABC}^2) - (K_{AB}-\zeta_C j_{ABC}^2)(K_{AC}-\zeta_B j_{ABC}^2)(K_{BC}-\zeta_A\zeta_{A0}j_{ABC}^2) \le 0, \tag{3.297}$$

(3.295) and (3.296), the denominator of $\tilde{\zeta}'$ is a monotonically decreasing function of ζ_0. Thus, the quantity $\tilde{\zeta}'$ is a monotonically increasing function of ζ_0. Additionally, from (3.295), we obtain

$$\zeta_0\tilde{\zeta}'|_{\zeta_0=1} = 1, \tag{3.298}$$

$$\zeta_0\tilde{\zeta}'|_{\zeta_0=j_{BC}^2/(K_{BC}-\zeta_{A0}j_{ABC}^2)} = \frac{j_{AB}^2 j_{AC}^2 j_{BC}^2}{(K_{AB}-\zeta_C j_{ABC}^2)(K_{AC}-\zeta_B j_{ABC}^2)(K_{BC}-\zeta_A\zeta_{A0}j_{ABC}^2)} \tag{3.299}$$

Thus,

$$\frac{J_{\mathrm{ap}}}{(K_{AB} - \zeta_C j_{ABC}^2)(K_{AC} - \zeta_B j_{ABC}^2)(K_{BC} - \zeta_A \zeta_{A0} j_{ABC}^2)} \leq \zeta_0 \tilde{\zeta} \leq 1 \qquad (3.300)$$

holds, where the left equality of (3.300) holds if and only if ζ_0 is equal to $j_{BC}^2/(K_{BC} - \zeta_{A0} j_{ABC}^2)$, while the right equality of (3.300) holds if and only if ζ_0 is equal to one. Now we have completed the proof of Statement 1.

Next, we prove Statement 2. Note that the left equality of (3.300) holds if and only if ζ_0 is equal to $j_{BC}^2/(K_{BC} - \zeta_{A0} j_{ABC}^2)$, and that the right equality of (3.300) holds if and only if ζ_0 is equal to one. Thus, (3.285) is equivalent to (3.288).

Hence, Conditions 1 and 2 of the present Lemma is a necessary and sufficient condition even if $|\psi\rangle$ is $\tilde{\zeta}$-indefinite. Because of (3.291), (3.298), (3.299) and (3.288), we can reproduce the sequence of operations $T^{\mathrm{f}} \to T_A \to T_B \to T_C$ by the following three DMTs: an A-DMT whose main and sub parameters are $\zeta_A \zeta_{A0}$ and ζ_0, respectively, a B-DMT whose main parameter is ζ_B, and a C-DMT whose main parameter is ζ_C. $\square$

Note that the proof of Lemma 6 guarantees that an arbitrary C-LOCC can be reproduced by performing an A-DMT, a B-DMT and a C-DMT.

Step 3 of Case $\mathfrak{A}$

Conditions 1 and 2 of Lemma 6 is necessary and sufficient not only for a C-LOCC transformation between EP-definite states but also for a deterministic LOCC transformation between EP-definite states. We show this statement in the following theorem. This corresponds to Step 3 of Case $\mathfrak{A}$.

Theorem 2 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the entanglement K-parameters of the states $|\psi\rangle$ and $|\psi'\rangle$ as $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5, Q_e)$ and $(K'_{AB}, K'_{AC}, K'_{BC}, j'_{ABC}, J'_5, Q'e)$, respectively. We assume both $|\psi\rangle$ and $|\psi'\rangle$ to be EP-definite. We also assume that $j_{ABC} \neq 0$. Then, a necessary and sufficient condition of the possibility of a deterministic LOCC transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is that the following two conditions are satisfied:*
Condition 1: There are real numbers $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$, $0 \leq \zeta_C \leq 1$ and $\zeta_{lower} \leq \zeta \leq 1$ which satisfy the following equation:

$$\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta \begin{pmatrix} \zeta_A \zeta_B & & & & \\ & \zeta_A \zeta_C & & & \\ & & \zeta_B \zeta_C & & \\ & & & \zeta_A \zeta_B \zeta_C & \\ & & & & \zeta_A \zeta_B \zeta_C \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}, \qquad (3.301)$$

where

$$\zeta_{lower} = \frac{J_{ap}}{(K_{AB} - \zeta_C j_{ABC}^2)(K_{AC} - \zeta_B j_{ABC}^2)(K_{BC} - \zeta_A j_{ABC}^2)}, \qquad (3.302)$$

and we refer to ζ, ζ_A, ζ_B and ζ_C as the sub parameter and the main parameters of A, B and C, respectively.

Condition 2: Let us check whether the state $|\psi\rangle$ is $\tilde{\zeta}$-definite or not. When the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the condition is

$$Q_e = Q'_e \ \ and \ \ \zeta = \tilde{\zeta}, \tag{3.303}$$

where

$$\tilde{\zeta} = \frac{J_{ap}(\Delta_J + 4K_{ap}\sin^2\varphi_5)}{4K_{ap}J_{ap}\sin^2\varphi_5 + \Delta_J(K_{AB} - \zeta_C j_{ABC}^2)(K_{AC} - \zeta_B j_{ABC}^2)(K_{BC} - \zeta_A j_{ABC}^2)}. \tag{3.304}$$

When the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, the condition is

$$|Q'_e| = sgn[(1-\zeta)(\zeta - \zeta_{lower})], \tag{3.305}$$

or in the other words,

$$Q'_e \begin{cases} = 0 & (\zeta = 1 \ \ or \ \ \zeta = \zeta_{lower}), \\ \neq 0 & (otherwise). \end{cases} \tag{3.306}$$

Proof: To prove the present Theorem, it suffices to prove the following statement S: "An arbitrary deterministic LOCC transformation can be reproduced by a C-LOCC transformation." Once we prove this statement, we can use Lemma 6 in the section 6.1.2 to prove Theorem 2. We prove the statement S by mathematical induction with respect to N, which is the number of times measurement are performed in the deterministic LOCC transformation.

First, we define how to count the number of times of the measurement. Let the notation $\mathbb{T}$ stands for an arbitrary LOCC transformation. We fix the order of measurements in the LOCC transformation $\mathbb{T}$ cyclically: If the first measurement of the LOCC transformation $\mathbb{T}$ is performed on the qubit A, the second one is on the qubit B, the third one is on the qubit C, the fourth one returns to the qubit A, and so on. If the first measurement is performed on the qubit B, the second one is on the qubit C, and so on. We can attain such a fixed order by inserting the identity transformation as a measurement. The LOCC transformation $\mathbb{T}$ may have branches and the numbers of times the measurements are performed may be different in different branches. We refer to the largest of the numbers as the number N. We can make the number of each branch equal to N by inserting the identity transformations. An example is given in Fig. 3.11. We use this counting procedure in the proofs of other theorems, too.

Next, let us prove Theorem 2 by mathematical induction with respect to the number N. An arbitrary deterministic LOCC transformation with $N = 1$ is also a C-LOCC transformation. Thus the statement S clearly holds for $N = 1$. We prove the statement S for $N = k + 1$, assuming that the statement S is proved whenever $1 \le N \le k$.

Let the notation T stand for a deterministic LOCC transformation with $N = k + 1$. We can assume that the first measurement of the deterministic LOCC transformation T is performed on the qubit A without loss of generality, because the state $|\psi\rangle$ is an arbitrary EP-definite state. We define states $\{|\psi^{(i)}\rangle\}$ as results of the first measurement $\{M_{(i)}|i = 0, 1\}$ (Fig. 3.12). For $M_{(i)}$, we define the measurement parameters $a_{(i)}$, $b_{(i)}$, $k_{(i)}$ and $\theta_{(i)}$ as follows:

$$M_{(i)}^\dagger M_{(i)} = \begin{pmatrix} a_{(i)} & k_{(i)}e^{-i\theta_{(i)}} \\ k_{(i)}e^{i\theta_{(i)}} & b_{(i)} \end{pmatrix}. \tag{3.307}$$
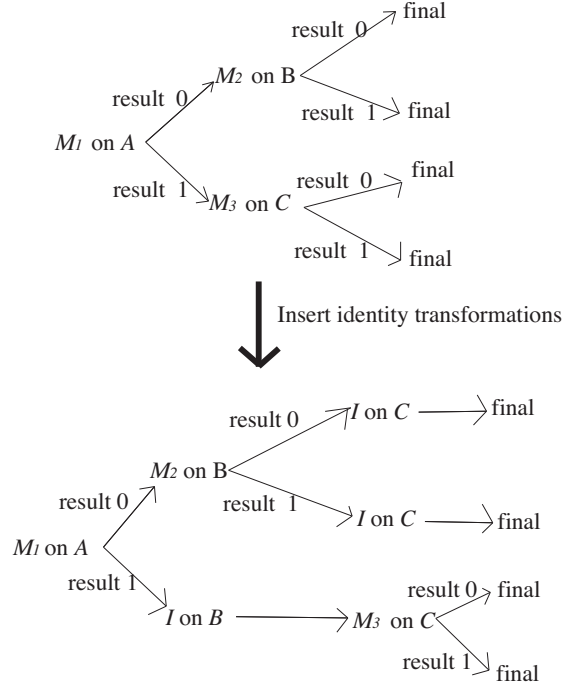
Figure 3.11: The method of counting the number N. In this figure, M_1, M_2 and M_3 denote measurements and I denotes the identity transformation. The number N is 3 in this example.

We can assume that $\sin \theta_{(0)} \geq 0$ without losing generality. Because of the assumption for $N = k$, a deterministic LOCC transformation with $N = k$ from the state $\left| \psi^{(i)} \right\rangle$ to the final state $\left| \psi' \right\rangle$ can be reproduced by a C-LOCC transformation $T_{(i)}$. Lemma 6 tells us that, we can express the C-LOCC transformation $T_{(i)}$ in terms of DMTs $T_A^{(i)}$, $T_B^{(i)}$ and $T_C^{(i)}$, whose DMs are performed on the qubits A, B and C, respectively (Fig. 3.12).

Let the notation $(K_{AB}^{(i)}, K_{AC}^{(i)}, K_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)}, Q_{\mathrm{e}}^{(i)})$ stand for the set of the entanglement K-parameters of the state $\left| \psi^{(i)} \right\rangle$. Lemma 6 and (A.43) give that the C-LOCC transformation $T_{(i)}$ changes the entanglement parameters as follows:

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ J'_{ABC} \\ J'_5 \end{pmatrix} = \zeta^{A(i)} \zeta^{B(i)} \zeta^{C(i)} \begin{pmatrix} \zeta_A^{(i)} \zeta_B^{(i)} & & & & \\ & \zeta_A^{(i)} \zeta_C^{(i)} & & & \\ & & \zeta_B^{(i)} \zeta_C^{(i)} & & \\ & & & \zeta_A^{(i)} \zeta_B^{(i)} \zeta_C^{(i)} & \\ & & & & \zeta_A^{(i)} \zeta_B^{(i)} \zeta_C^{(i)} \end{pmatrix} \begin{pmatrix} (\alpha^{(i)})^2 K_{AB} \\ (\alpha^{(i)})^2 K_{AC} \\ K_{BC}^{(i)} \\ (\alpha^{(i)})^2 j_{ABC}^2 \\ J_5^{(i)} \end{pmatrix},
$$

(3.308)

where $(\zeta_A^{(i)}, \zeta^{A(i)})$, $(\zeta_B^{(i)}, \zeta^{B(i)})$ and $(\zeta_C^{(i)}, \zeta^{C(i)})$ are the sets of the main and sub parameters of the DMTs $T_A^{(i)}$, $T_B^{(i)}$ and $T_C^{(i)}$, respectively.

Here, the product $(\alpha^{(i)})^2 \zeta_A^{(i)} \zeta_B^{(i)} \zeta_C^{(i)} \zeta^{A(i)} \zeta^{B(i)} \zeta^{C(i)}$ must be independent of i because $j'^2_{ABC}/j^2_{ABC} = (\alpha^{(i)})^2 \zeta_A^{(i)} \zeta_B^{(i)} \zeta_C^{(i)} \zeta^{A(i)} \zeta^{B(i)} \zeta^{C(i)}$. Similarly, the product $(\alpha^{(i)})^2 \zeta_A^{(i)} \zeta_C^{(i)} \zeta^{A(i)} \zeta^{B(i)} \zeta^{C(i)}$ must be independent of i because $K'_{AC}/K_{AC} = (\alpha^{(i)})^2 \zeta_A^{(i)} \zeta_C^{(i)} \zeta^{A(i)} \zeta^{B(i)} \zeta^{C(i)}$. Hence, the
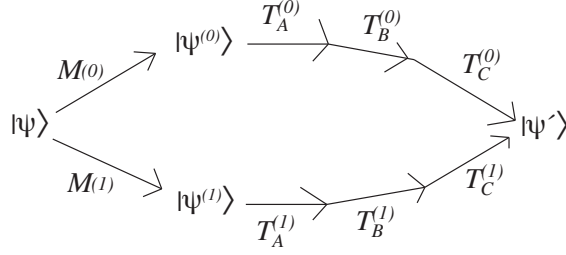
Figure 3.12: The deterministic transformation from the state $|\psi^{(i)}\rangle$ to the state $|\psi'\rangle$ is reproduced by a C-LOCC transformation. We refer to the transformation from the state $|\psi^{(i)}\rangle$ to state $|\psi'\rangle$ as $T_{(i)}$.

main parameter $\zeta_B^{(i)}$ must be independent of i. In the same manner, we can show that the main parameter $\zeta_C^{(i)}$ must be also independent of i. Note that the DMT $T_A^{(i)}$ is a DMT between EP-definite states and that the step 2-1 of Lemma 6, (A.52) and (A.54) guarantees that the final state of a DMT between EP-definite states is $\tilde{\zeta}$-definite. Thus, the states after DMTs $T_A^{(i)}$ and $T_B^{(i)}$ are $\tilde{\zeta}$-definite. Thus, the sub parameter $\zeta^{C(i)}$ is equal to the ζ-specifying parameter of the state after each DMT $T_B^{(i)}$. Hence, the sub parameter $\zeta^{C(i)}$ is a monotonously increasing function of the main parameter $\zeta_C^{(i)}$, because the ζ-specifying parameter of the state after each DMT $T_B^{(i)}$ is a monotonously increasing function of $\zeta_C^{(i)}$. Thus, the quantity $\zeta^{C(i)}\zeta_C^{(i)}$ is a monotonously increasing function of the main parameter $\zeta_C^{(i)}$. This means that if the main parameter $\zeta_C^{(i)}$ is specified, $\zeta^{C(i)}\zeta_C^{(i)}$ is also determined uniequly. Thus, $\zeta^{C(i)}\zeta_C^{(i)}$ is independent of i because the main parameter $\zeta_C^{(i)}$ is independent of i. Hence, the entanglement parameters of the initial state of each DMT $T_C^{(i)}$ is independent of i, because $\zeta_C^{(i)}$, $\zeta^{C(i)}\zeta_C^{(i)}$ and the state $|\psi''\rangle$ are independent of i. In the same manner, the entanglement parameters of the initial state of each DMT $T_B^{(i)}$ are independent of i. Let us refer to the initial EP-definite state as $|\psi''\rangle$ (Fig. 3.13).

Therefore, we only have to prove the following statement S_2: "A deterministic LOCC transformation which consists of the measurement $M_{(i)}$ and the DMT $T_A^{(i)}$ can be reproduced by an A-DMT whose DM is a two-choice measurement." The reason why we have to prove the statement S_2 is that a C-LOCC transformation consists of DMTs whose DMs are two-choice measurements. We prove the statement S_2 by showing that an A-DMT whose DM is a two-choice measurement can realize the change of the entanglement parameters which is caused by performing the measurement $M_{(i)}$ and the DMT $T_A^{(i)}$ successively. We refer to the A-DMT as T_A. Then, we can reproduce the deterministic LOCC T by performing the DMTs T_A, T_B and T_C successively (Fig. 3.13). The number of times of the measurement of the transformation T is $k+1$, and thus if we have proven the statement S_2 we can also prove Theorem 2 by completing mathematical induction.

We denote the set of entanglement J-parameters of the state $|\psi''\rangle$ in Fig. 3.13 by $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5, Q''_e)$. We prove the statement S_2 by showing that the sets of J-parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5, Q''_e)$ satisfy Conditions 1 and 2 of Theorem 1.

Let us prove that the sets of J-parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5, Q$ satisfy Condition 1 of Theorem 1 by examining the change of the J-parameters $j_{AB}, j_{AC},$
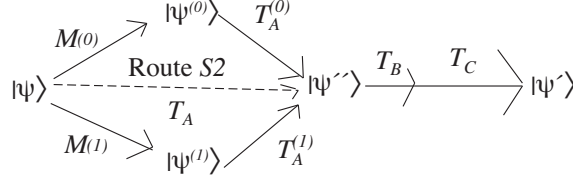
Figure 3.13: The concept of the statement S_2. The Route S_2 is realized by an A-DMT whose DM is a two-choice measurement.

j_{BC}, j_{ABC} and J_5. First, we examine the change of the J-parameters j_{AB}, j_{AC} and j_{ABC}. From (A.40)–(A.42), we have the following three equations:

$$j''^2_{AB} = (\alpha_A^{(i)} \alpha^{(i)})^2 j^2_{AB}, \tag{3.309}$$

$$j''^2_{AC} = (\alpha_A^{(i)} \alpha^{(i)})^2 j^2_{AC}, \tag{3.310}$$

$$j''^2_{ABC} = (\alpha_A^{(i)} \alpha^{(i)})^2 j^2_{ABC}, \tag{3.311}$$

where $\alpha_A^{(i)}$ is the transfer parameter of the transformation $T_A^{(i)}$.

Second, we examine the change of the J-parameter J_5. From $J_5'' = (\alpha_A^{(i)})^2 J_5^{(i)}$, it follows that the product $(\alpha_A^{(i)})^2 J_5^{(i)}$ must be independent of i. Note that the product $(\alpha_A^{(i)} \alpha^{(i)})^2$ must be independent of i because $j''^2_{ABC} = (\alpha_A^{(i)} \alpha^{(i)} j_{ABC})^2$. Hence, we have $J_5^{(i)} \propto (\alpha^{(i)})^2$. Thus, we can take a constant γ which satisfies the equation $J_5^{(i)} = \gamma (\alpha^{(i)})^2 J_5$. The equations (A.28), (A.40) and (A.41) give that $J_5^{(i)} = 2(\alpha^{(i)})^2 j_{AB} j_{AC} j_{BC}^{(i)} \cos \varphi_5^{(i)}$ and $J_5 = 2 j_{AB} j_{AC} j_{BC} \cos \varphi_5$. Hence, we obtain the equation $j_{BC}^{(i)} \cos \varphi_5^{(i)} = \gamma j_{BC} \cos \varphi_5$. This equation and (A.48) give that $\gamma = 1$. Then, the equations

$$j_{BC}^{(i)} \cos \varphi_5^{(i)} = j_{BC} \cos \varphi_5 \ \text{ and } \ J_5^{(i)} = (\alpha^{(i)})^2 J_5 \tag{3.312}$$

hold. This means that we have obtained the change of the J-parameter J_5 in the form

$$J_5'' = (\alpha_A^{(i)} \alpha^{(i)})^2 J_5. \tag{3.313}$$

Third, we examine the change of the J-parameter j_{BC} and prove that the sets of J-parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J_5'')$ satisfy Condition 1 of Theorem 1. Because of Lemma 5, at least one of $K_{BC}^{(0)}$ and $K_{BC}^{(1)}$ is less than or equal to K_{BC}. Because both of $T_A^{(i)}$ are A-DMTs, the inequalities $K_{BC}'' \le K_{BC}^{(i)}$ hold. Thus, we obtain $K_{BC}'' \le K_{BC}$. Because of Lemma 2, at least one of $j_{BC}^{(i)}$ is more than or equal to j_{BC}. Because both of $T_A^{(i)}$ are A-DMTs, the inequalities $j_{BC}^{(i)} \le j_{BC}''$ hold. Thus, we obtain $j_{BC} \le j_{BC}''$. Because of $K_{BC}'' \le K_{BC}$ and $j_{BC} \le j_{BC}''$, we can find a parameter $0 \le \beta_A \le 1$ which satisfies

$$j''^2_{BC} = j^2_{BC} + \beta_A (1 - (\alpha^{(i)} \alpha_A^{(i)})^2) j^2_{ABC}. \tag{3.314}$$

Note that (3.309)–(3.311), (3.312) and (3.314) mean that $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J_5'')$ satisfy Condition 1 of Theorem 1.

Next, let us prove that $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J_5'', Q_e'')$ satisfy Condition 2 of Theorem 1.

First, we prove this in the case where the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite by examining the change of the entanglement charge Q_e''. In other words, we prove that the entanglement charge Q_e'' is zero if and only if the dissipation parameter β_A is zero or one. Because of (A.19) and (A.30), Q_e'' is zero if and only if at least one of Δ_J'' and $\sin\varphi_5''$ is zero. First we prove that $\sin\varphi_5'' > 0$ if and only if $\beta_A > 0$. Because of (3.312) and because both of $T_A^{(i)}$ are A-DMTs, we obtain

$$j_{BC}'' \cos\varphi_5'' = j_{BC} \cos\varphi_5. \tag{3.315}$$

From (3.314) and (3.315), we obtain

$$j_{BC}''^2 \sin^2\varphi_5'' = j_{BC}^2 \sin^2\varphi_5 + \beta_A(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j_{ABC}^2 = \beta_A(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j_{ABC}^2. \tag{3.316}$$

Thus, $j_{BC}'' \sin\varphi_5'' > 0$ if and only if $\beta_A > 0$. Because the state $|\psi''\rangle$ is EP-definite, $j_{BC}'' > 0$ holds, and thus $\sin\varphi_5'' > 0$ if and only if $\beta_A > 0$. Next, we prove that $\Delta_J'' > 0$ if and only if $1 > \beta$. From (3.309)–(3.311), (3.312) and (3.314), we obtain

$$
\begin{aligned}
\Delta_{\text{norm}}'' &\equiv \frac{\Delta_J''}{(\alpha^{(i)}\alpha_A^{(i)})^2} = K_5^2 - 4K_{AB}K_{AC}K_{BC}'' \\
&= K_5^2 - 4K_{AB}K_{AC}(j_{BC}^2 + \beta_A(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j_{ABC}^2 + (\alpha^{(i)}\alpha_A^{(i)})^2 j_{ABC}^2) \\
&= \Delta_J + 4K_{AB}K_{AC}j_{ABC}^2(1 - \beta_A)(1 - (\alpha^{(i)}\alpha_A^{(i)})^2) \\
&= 4K_{AB}K_{AC}j_{ABC}^2(1 - \beta_A)(1 - (\alpha^{(i)}\alpha_A^{(i)})^2).
\end{aligned}
\tag{3.317}
$$

Thus, $\Delta_J'' > 0$ if and only if $1 > \beta_A$. Hence, the entanglement charge Q_e'' is zero if and only if the dissipation parameter β_A is zero or one. Thus, the entanglement charge Q_e'' and the dissipation parameter β_A satisfy Condition 2 of Theorem 1.

Now, we have proven that if the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, the sets of J-parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j_{AB}'', j_{AC}'', j_{BC}'', j_{ABC}'', J_5'', Q_e'')$ satisfy Conditions 1 and 2 of Theorem 1. Thus, we can reproduce the transformation from the state $|\psi\rangle$ to the state $|\psi''\rangle$ by an A-DMT whose transfer parameter and dissipative parameter are $\alpha^{(i)}\alpha_A^{(i)}$ and the dissipation parameter β_A, respectively. Hence, we have completed the proof of the statement S_2 in the case where the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite.

Next, let us prove that $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j_{AB}'', j_{AC}'', j_{BC}'', j_{ABC}'', J_5'', Q_e'')$ satisfy Condition 2 of Theorem 1 in the case where he state $|\psi\rangle$ is ζ-definite. In other words, we prove the equations $Q_e = Q_e''$ and $\beta_A = \tilde{\beta}_A$.

First, we prove $Q_e = Q_e''$. We prove this equation by using the equations (3.322) and (3.323), which we derive below. In order to derive (3.322) and (3.323), we first show that the state $|\psi^{(i)}\rangle$ is $\tilde{\zeta}$-definite by reduction to absurdity. Let us assume that the state $|\psi^{(i)}\rangle$ is $\tilde{\zeta}$-indefinite. Because $|\psi^{(i)}\rangle$ is $\tilde{\zeta}$-indefinite, both of $\sin\varphi_5^{(i)}$ and $\Delta_J^{(i)}$ would be zero. We could take a real number $\beta^{(i)}$ which satisfies $(j_{BC}^{(i)})^2 = j_{BC}^2 + \beta^{(i)}(1 - (\alpha^{(i)})^2)j_{ABC}^2$. Note that the number $\beta^{(i)}$ can be more than one or negative, which can differ from dissipation parameter β_A. Because of (3.312), we would obtain

$$(j_{BC}^{(i)} \sin\varphi_5^{(i)})^2 = j_{BC}^2 \sin^2\varphi_5 + \beta^{(i)}(1 - (\alpha^{(i)})^2)j_{ABC}^2. \tag{3.318}$$

Because of (3.318) and $\sin\varphi_5^{(i)} = 0$, at least one of the following two expressions would hold:

$$((\alpha^{(i)})^2 > 1) \wedge (\beta^{(i)} > 0), \quad ((\alpha^{(i)})^2 \leq 1) \wedge (\beta^{(i)} \leq 0). \tag{3.319}$$

On the other hand, we will later derive the following expression:

$$((\alpha^{(i)})^2 > 1) \wedge (\beta^{(i)} < 1), \quad ((\alpha^{(i)})^2 \leq 1) \wedge (\beta^{(i)} \geq 1). \tag{3.320}$$

The expressions (3.319) and (3.320) contradict each other; Because of Lemma 3, $(\alpha^{(i)})^2 \leq 1$ holds for at least one of i. Thus, for at least one of i, $(\beta^{(i)} \leq 0) \wedge (\beta^{(i)} \geq 1)$ would hold. This is a contradiction. Thus, if we derive (3.320) we complete the proof that the states $|\psi^{(i)}\rangle$ are $\tilde{\zeta}$-definite; if the states $|\psi^{(i)}\rangle$ were $\tilde{\zeta}$-indefinite, there would be a contradiction.

Let us derive (3.320). First, we transform $\Delta_{\text{norm}}^{(i)} \equiv \Delta_J^{(i)}/(\alpha^{(i)})^4$ as follows:

$$
\begin{aligned}
\Delta_{\text{norm}}^{(i)} &\equiv \frac{\Delta_J^{(i)}}{(\alpha^{(i)})^4} = \frac{(K_5^{(i)})^2 - 4K_{\text{ap}}^{(i)}}{(\alpha^{(i)})^4} = (\alpha^{(i)})^4 \frac{K_5^2 - 4K_{AB}K_{AC}K_{BC}^{(i)}}{(\alpha^{(i)})^4} \\
&= K_5^2 - 4K_{AB}K_{AC}K_{BC}^{(i)} = K_5^2 - 4K_{AB}K_{AC}(j_{BC}^2 + \beta^{(i)}(1-(\alpha^{(i)})^2)j_{ABC}^2 + (\alpha^{(i)})^2 j_{ABC}^2) \\
&= K_5^2 - 4K_{\text{ap}} + 4K_{AB}K_{AC}(1-\beta^{(i)})(1-(\alpha^{(i)})^2)j_{ABC}^2 \\
&= \Delta_J + 4K_{AB}K_{AC}(1-\beta^{(i)})(1-(\alpha^{(i)})^2)j_{ABC}^2.
\end{aligned}
\tag{3.321}
$$

Because of (3.321) and $\Delta_J^{(i)} = 0$, (3.320) has to hold. Thus, the states $|\psi^{(i)}\rangle$ are $\tilde{\zeta}$-definite.

Next, let us derive the following equations which we use to prove $Q_{\text{e}} = Q_{\text{e}}''$;

$$
\begin{aligned}
1 - \tilde{\beta}_A'' &= \frac{\Delta_{\text{norm}}''}{\Delta_{\text{norm}}'' + 4K_{AB}K_{AC}j_{BC}''^2 \sin^2 \varphi_5''} = \frac{\Delta_{\text{norm}}^{(i)}}{\Delta_{\text{norm}}^{(i)} + 4K_{AB}K_{AC}(j_{BC}^{(i)})^2 \sin^2 \varphi_5^{(i)}} \\
&= 1 - \tilde{\beta}_A^{(i)}
\end{aligned}
\tag{3.322}
$$

$$
\frac{Q_{\text{e}}'' t \sqrt{\Delta_J}}{\sqrt{\Delta_{\text{norm}}''}} = \frac{j_{BC} \sin \varphi_5}{j_{BC}'' \sin \varphi_5''}, \tag{3.323}
$$

where $\tilde{\beta}_A^{(i)}$, Δ_{norm}'' and $\Delta_{\text{norm}}^{(i)}$ are the β-specifying parameter of the state $|\psi^{(i)}\rangle$, $\Delta_{\text{norm}}'' \equiv \Delta_J''/(\alpha^{(i)}\alpha_A^{(i)})^4 = K_5^2 - 4K_{AB}K_{AC}K_{BC}''$ and $\Delta_{\text{norm}}^{(i)} \equiv \Delta_J^{(i)}/(\alpha^{(i)})^4 = K_5^2 - 4K_{AB}K_{AC}K_{BC}^{(i)}$, respectively, and where t is determined as follows: If $Q_{\text{e}} = 1$, t is $+1$. If $Q_{\text{e}} = -1$, t is -1. If $Q_{\text{e}} = 0$, t can be either $+1$ or -1. Because $T_A^{(i)}$ is an A-DMT, we obtain

$$
\begin{aligned}
1 - \tilde{\beta}''_A &= \frac{\Delta_J''}{\Delta_J'' + 4K_{AB}'' K_{AC}'' j_{BC}''^2 \sin^2 \varphi_5''} \\
&= \frac{\Delta_J^{(i)} + 4K_{AB}^{(i)} K_{AC}^{(i)} (j_{ABC}^{(i)})^2 (1-\beta_A^{(i)})(1-(\alpha^{(i)})^2)}{\Delta_J^{(i)} + 4K_{AB}^{(i)} K_{AC}^{(i)} (j_{ABC}^{(i)})^2 (1-(\alpha^{(i)})^2) + 4K_{AB}^{(i)} K_{AC}^{(i)} (j_{BC}^{(i)})^2 \sin^2 \varphi_5^{(i)}} \\
&= \frac{\Delta_J^{(i)} \frac{\Delta_J^{(i)} + 4K_{AB}^{(i)} K_{AC}^{(i)} (j_{ABC}^{(i)})^2 (1-(\alpha^{(i)})^2) + 4K_{AB}^{(i)} K_{AC}^{(i)} (j_{BC}^{(i)})^2 \sin^2 \varphi_5^{(i)}}{\Delta_J^{(i)} + 4K_{AB}^{(i)} K_{AC}^{(i)} (j_{BC}^{(i)})^2 \sin^2 \varphi_5^{(i)}}}{\Delta_J^{(i)} + 4K_{AB}^{(i)} K_{AC}^{(i)} (j_{ABC}^{(i)})^2 (1-(\alpha^{(i)})^2) + 4K_{AB}^{(i)} K_{AC}^{(i)} (j_{BC}^{(i)})^2 \sin^2 \varphi_5^{(i)}} \\
&= \frac{\Delta_J^{(i)}}{\Delta_J^{(i)} + 4K_{AB}^{(i)} K_{AC}^{(i)} (j_{BC}^{(i)})^2 \sin^2 \varphi_5^{(i)}} = 1 - \tilde{\beta}_A^{(i)}.
\end{aligned}
\tag{3.324}
$$

The equation (3.324) can be transformed to (3.322) with

$$
\frac{(j_{BC}^{(i)})^2 \sin^2 \varphi_5^{(i)}}{j_{BC}''^2 \sin^2 \varphi_5''} = \frac{\Delta_{\text{norm}}^{(i)}}{\Delta_{\text{norm}}''}. \tag{3.325}
$$

Because a DMT between states which are $\tilde{\zeta}$-definite conserves the entanglement charge, the expression $Q_{\mathrm{e}}^{(0)} = Q_{\mathrm{e}}^{(1)} = Q_{\mathrm{e}}'' \neq 0$ holds. Because we have assumed $\sin\theta^{(0)} \geq 0$, the equation $Q_{\mathrm{e}}^{(0)} = Q_{\mathrm{e}}^{(1)} = Q_{\mathrm{e}}'' \neq 0$ is equivalent to the following equations:

$$(\lambda_0^{(0)})^2 = \frac{K_5 + Q_{\mathrm{e}}''\sqrt{\Delta_{\mathrm{norm}}^{(0)}}}{2K_{BC}^{(0)}}(\alpha^{(0)})^2 = \frac{p}{b}(\alpha^{(0)})^2 \frac{K_5 + t\sqrt{\Delta_J}}{2K_{BC}}, \qquad (3.326)$$

$$(\lambda_0^{(1)})^2 = \frac{K_5 + Q_{\mathrm{e}}''s\sqrt{\Delta_{\mathrm{norm}}^{(1)}}}{2K_{BC}^{(1)}}(\alpha^{(1)})^2 = \frac{1-p}{1-b}(\alpha^{(1)})^2 \frac{K_5 + t\sqrt{\Delta_J}}{2K_{BC}}, \qquad (3.327)$$

where s and t are determined as follows: if $\sin\varphi^{(1)} \geq 0$, s is $+1$. If $\sin\varphi^{(1)} < 0$, s is -1. If $Q_{\mathrm{e}} = 1$, t is $+1$. If $Q_{\mathrm{e}} = -1$, t is -1. If $Q_{\mathrm{e}} = 0$, t can be either $+1$ or -1. Because of (3.326), (3.327) and $K_{BC}^{(i)}/K_{BC} = (K_5^2 - \Delta_{\mathrm{norm}}^{(i)})/(K_5^2 - \Delta_J)$, we obtain the expression of the probability p in terms of Δ_J and $\Delta_{\mathrm{norm}}^{(i)}$:

$$p = \frac{Q_{\mathrm{e}}''t\sqrt{\Delta_J} - s\sqrt{\Delta_{\mathrm{norm}}^{(1)}}}{\sqrt{\Delta_{\mathrm{norm}}^{(0)}} - s\sqrt{\Delta_{\mathrm{norm}}^{(1)}}}. \qquad (3.328)$$

Because of (A.49),

$$pj_{BC}^{(0)}\sin\varphi_5^{(0)} + s(1-p)j_{BC}^{(1)}\sin\varphi_5^{(1)} = j_{BC}\sin\varphi_5. \qquad (3.329)$$

From (3.329), we obtain

$$p = \frac{j_{BC}\sin\varphi_5 - sj_{BC}^{(1)}\sin\varphi_5^{(1)}}{j_{BC}^{(0)}\sin\varphi_5^{(0)} - sj_{BC}^{(1)}\sin\varphi_5^{(1)}}. \qquad (3.330)$$

From (3.325), (3.328) and (3.330), we arrive at (3.323).

Let us prove $Q_{\mathrm{e}} = Q_{\mathrm{e}}''$ by using (3.322) and (3.323). We perform this proof in the following three steps. First, we prove that if $Q_{\mathrm{e}} \neq 0$ holds, $Q_{\mathrm{e}}'' \neq 0$ also holds. Second, we prove $Q_{\mathrm{e}} = Q_{\mathrm{e}}'' = 0$ in the case where $Q_{\mathrm{e}} = 0$. Finally, we prove $Q_{\mathrm{e}} = Q_{\mathrm{e}}''$ in the case where $Q_{\mathrm{e}} \neq 0$.

Let us prove that if $Q_{\mathrm{e}} \neq 0$ holds, $Q_{\mathrm{e}}'' \neq 0$ also holds by reduction to absurdity. Let us assume that $Q_{\mathrm{e}} \neq 0$ and $Q_{\mathrm{e}}'' = 0$ held. Because a DMT between $\tilde{\zeta}$-definite states conserves the entanglement charge, the expression $Q_{\mathrm{e}}^{(0)} = Q_{\mathrm{e}}^{(1)} = Q_{\mathrm{e}}'' = 0$ would hold. Because of $Q_{\mathrm{e}}^{(0)} = Q_{\mathrm{e}}^{(1)} = 0$, at least one of $\sin\varphi_5^{(i)}$ and $\Delta_J^{(i)}$ would be zero. Thus, at least one of

$$(\Delta_J^{(0)} \neq 0) \wedge (\sin\varphi_5^{(0)} = 0) \qquad (3.331)$$

and

$$(\Delta_J^{(0)} = 0) \wedge (\sin\varphi_5^{(0)} \neq 0) \qquad (3.332)$$

would hold and at least one of

$$(\Delta_J^{(1)} \neq 0) \wedge (\sin\varphi_5^{(1)} = 0) \qquad (3.333)$$

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and
$$(\Delta_J^{(1)} = 0) \wedge (\sin \varphi_5^{(1)} \neq 0) \tag{3.334}$$

would hold. Note that substitution of (3.331) and (3.334) in (3.325) makes a contradiction and that substitution of (3.332) and (3.333) in (3.325) makes a contradiction. Thus, there can be only two pairs which could be valid; ((3.331) and (3.333)) and ((3.332) and (3.334)). Because of $Q_{\mathrm{e}} \neq 0$, both of $\sin \varphi_5^{(i)}$ and $\Delta_J^{(i)}$ would be more than zero. If (3.331) and (3.333) held, $(\sin \varphi_5^{(0)} = 0) \wedge (\sin \varphi_5^{(1)} = 0) \wedge (\sin \varphi_5 \neq 0)$ would hold. This contradicts (A.49). If (3.332) and (3.334) held, $(\Delta_J^{(0)} = 0) \wedge (\Delta_J^{(1)} = 0) \wedge (\Delta_J = 0)$ would hold. This contradicts Lemma 5, because

$$0 = \Delta_J^{(i)} = (\alpha^{(i)})^4 \Delta_{\mathrm{norm}}^{(i)} = (\alpha^{(i)})^4 (K_5^2 - 4K_{AB}K_{AC}K_{BC}^{(i)}) < \Delta_J \neq 0. \tag{3.335}$$

Hence, if $Q_{\mathrm{e}} \neq 0$ and $Q_{\mathrm{e}}'' = 0$ held, there would be a contradiction, and thus if $Q_{\mathrm{e}} \neq 0$ holds, $Q_{\mathrm{e}}'' \neq 0$ also holds.

Next, we show that $Q_{\mathrm{e}} = Q_{\mathrm{e}}''$ in the case where $Q_{\mathrm{e}} \neq 0$. If $Q_{\mathrm{e}} \neq 0$ holds, $\Delta_J > 0$, $j_{BC} \sin \varphi_5 > 0$ and $Q_{\mathrm{e}}'' \neq 0$ hold. Because of $Q_{\mathrm{e}}'' \neq 0$, the equation (3.323) holds. Thus, because of the definition of t and the equation (3.323), if $Q_{\mathrm{e}} \neq 0$, then $Q_{\mathrm{e}} = Q_{\mathrm{e}}''$ holds. If $Q_{\mathrm{e}} = 0$, the expression $\Delta_J = 0 \vee j_{BC} \sin \varphi_5 = 0$ holds. Because we assumed that the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, $\Delta_J = 0 \wedge j_{BC} \sin \varphi_5 = 0$ does not hold. This contradicts (3.323). Thus, if $Q_{\mathrm{e}} = 0$, $Q_{\mathrm{e}}'' = 0$ has to hold, because if $Q_{\mathrm{e}}'' \neq 0$ then (3.323) holds. Hence, $Q_{\mathrm{e}} = Q_{\mathrm{e}}''$ holds even if $Q_{\mathrm{e}} = 0$.

Next, we prove $\beta_A = \tilde{\beta}_A$. In other words, we show that

$$j_{BC}''^2 = j_{BC}^2 + \tilde{\beta}_A(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j_{ABC}^2. \tag{3.336}$$

First, we prove this in the case of $Q_{\mathrm{e}} \neq 0$. From (3.325), (3.323) and $Q_{\mathrm{e}} = Q_{\mathrm{e}}''$, we obtain

$$\frac{j_{BC}^2 \sin^2 \varphi_5}{j_{BC}''^2 \sin^2 \varphi_5''} = \frac{\Delta_J}{\Delta_{\mathrm{norm}}''},$$

$$j_{BC}^2 \sin^2 \varphi_5(\Delta_J - 4K_{AB}K_{AC}(j_{BC}''^2 - j_{BC}^2 + ((\alpha^{(i)}\alpha_A^{(i)})^2 - 1)j_{ABC}^2)) = j_{BC}''^2 \sin^2 \varphi_5'' \Delta_J,$$

$$-4K_{AB}K_{AC}j_{BC}^2 \sin^2 \varphi_5(j_{BC}''^2 - j_{BC}^2) + 4K_{AB}K_{AC}j_{BC}^2 \sin^2 \varphi_5(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j_{ABC}^2 = (j_{BC}''^2 - j_{BC}^2)\Delta_J,$$

$$j_{BC}''^2 = j_{BC}^2 + \frac{4K_{AB}K_{AC}j_{BC}^2 \sin^2 \varphi_5}{\Delta_J + 4K_{AB}K_{AC}j_{BC}^2 \sin^2 \varphi_5}(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j_{ABC}^2$$

$$= j_{BC}^2 + \tilde{\beta}_A(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j_{ABC}^2, \tag{3.337}$$

where we used

$$j_{BC}''^2 \sin^2 \varphi_5'' - j_{BC}^2 \sin^2 \varphi_5 = j_{BC}''^2 - j_{BC}^2. \tag{3.338}$$

Thus, if $Q_{\mathrm{e}} \neq 0$, we can reproduce the transformation from the state $|\psi\rangle$ to the state $|\psi''\rangle$ with an A-DMT whose transfer parameter is $(\alpha^{(i)}\alpha_A^{(i)})^2$.

Second, we prove $\beta_A = \tilde{\beta}_A$ in the case of $Q_{\mathrm{e}} = 0$. Because of $Q_{\mathrm{e}} = Q_{\mathrm{e}}''$, then $Q_{\mathrm{e}}'' = 0$ has to hold. The equation $Q_{\mathrm{e}} = 0$ is equivalent to $\sin \varphi_5 = 0 \vee \Delta_J = 0$. We have already shown that $\Delta_{\mathrm{norm}}'' \geq \Delta_J$ and $j_{BC}'' \sin \varphi_5'' \geq j_{BC}^2 \sin^2 \varphi_5$ hold. In the present case, $\sin \varphi_5 = 0 \wedge \Delta_J = 0$ does not hold, because the state $|\psi\rangle$ is $\tilde{\zeta}$-definite. Thus, if $\Delta_J = 0$,

$\Delta''_{\text{norm}} = 0$ has to hold. Hence, we can obtain $j''^2_{BC} = j^2_{BC} + \tilde{\beta}_A(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j^2_{ABC}$ as follows:

$$\Delta_J = K_5^2 - 4K_{\text{ap}} = K_5^2 - 4K_{AB}K_{AC}K''_{BC} = \Delta''_{\text{norm}} = 0,$$
$$K_{BC} = K''_{BC},$$
$$j''^2_{BC} = j^2_{BC} + (1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j^2_{ABC} = j^2_{BC} + \frac{4K_{AB}K_{AC}j^2_{BC}\sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}j^2_{BC}\sin^2\varphi_5}(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j^2_{ABC}$$
$$= j^2_{BC} + \tilde{\beta}_A(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j^2_{ABC}. \tag{3.339}$$

Furthermore, if $\sin\varphi_5 = 0$, $\sin\varphi_5' = 0$ has to hold. Hence, we can obtain as follows:

$$j^2_{BC}\sin^2\varphi_5 = j''^2_{BC}\sin^2\varphi_5'' = 0,$$
$$j''^2_{BC} = j^2_{BC} = j^2_{BC} + \frac{4K_{AB}K_{AC}j^2_{BC}\sin^2\varphi_5}{\Delta_J + 4K_{AB}K_{AC}j^2_{BC}\sin^2\varphi_5}(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j^2_{ABC}$$
$$= j^2_{BC} + \tilde{\beta}_A(1 - (\alpha^{(i)}\alpha_A^{(i)})^2)j^2_{ABC}. \tag{3.340}$$

Thus, we have shown (3.340). Hence, if $Q_e = 0$, we can reproduce the transformation from the state $|\psi\rangle$ to the state $|\psi''\rangle$ with an A-DMT whose transfer parameter is $(\alpha^{(i)}\alpha_A^{(i)})^2$. Now, we have completed the proof of the statement S_2. Namely, we have completed the proof of Theorem 2. □

Thus, we have proven Main Theorem 1 in Case $\mathfrak{A}$. Because of the statement S of Theorem 2 and the fact that any C-LOCC can be reproduced by three DMTs, we have also proven Main Theorem 2 in Case $\mathfrak{A}$.

A necessary and sufficient condition of an n-choice DMT between EP-definite states

Theorem 2 guarantees that the condition of Theorem 1 holds not only for an arbitrary two-choice DM, but for an arbitrary n-choice DM.

Corollary 2 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the K-parameters of the states $|\psi\rangle$ and $|\psi'\rangle$ as $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5, Q_e)$ and $(K'_{AB}, K'_{AC}, K'_{BC}, j'_{ABC}, J'_5, Q'_e)$, respectively. We assume $|\psi'\rangle$ to be EP definite. We also assume that $j_{ABC} \neq 0$. Then, a necessary and sufficient condition of the possibility of an n-choice A-DMT from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is that the following two conditions are satisfied:*
Condition 1: There are real numbers $0 \leq \zeta_A \leq 1$ and $\zeta_{lower}^{(A)} \leq \zeta \leq 1$ which satisfy the following equation:

$$\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta \begin{pmatrix} \zeta_A & & & & \\ & \zeta_A & & & \\ & & 1 & & \\ & & & \zeta_A & \\ & & & & \zeta_A \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}. \tag{3.341}$$

where

$$\zeta_{lower}^{(A)} = \frac{j^2_{BC}}{(K_{BC} - \zeta_A j^2_{ABC})}. \tag{3.342}$$

If $\zeta_A = 1$ and $j_{BC} = 0$ hold, we define the lower bound $\zeta_{lower}^{(A)}$ to be unity.

Condition 2: Let us check whether the state $|\psi\rangle$ is $\tilde{\zeta}$-definite or not. When the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the condition is

$$Q_e = Q'_e \ \ and \ \ \zeta^{(A)} = \tilde{\zeta}^{(A)}. \tag{3.343}$$

where

$$\tilde{\zeta}^{(A)} = \frac{j_{BC}^2(\Delta_J + 4K_{ap}\sin^2\varphi_5)}{4K_{ap}j_{BC}^2\sin^2\varphi_5 + \Delta_J(K_{BC} - \zeta_A j_{ABC}^2)}, \tag{3.344}$$

where if the state $|\psi\rangle$ is EP indefinite, then we define $j_{BC}^2 \sin^2\varphi_5$ as zero. When the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, the condition is

$$|Q'_e| - sgn[(1 - \zeta^{(A)})(\zeta^{(A)} - \zeta_{lower}^{(A)})], \tag{3.345}$$

or in other words,

$$Q'_e = 0 \ \ for \ \ \zeta^{(A)} = 1 \ \ or \ \ \zeta^{(A)} = \zeta_{lower}^{(A)}, \tag{3.346}$$

$$Q'_e \neq 0 \ \ otherwise. \tag{3.347}$$

Comment

This corollary guarantees that the condition of Theorem 1 holds for an arbitrary n-choice measurement.

Proof: Lemma 1 guarantees that an n-choice measurement performed on the qubit A is equivalent to a deterministic LOCC transformation whose measurements are performed only on the qubit A. We refer to a deterministic LOCC transformation whose measurements are performed only on the qubit A as T_{LA}. We prove the statement S_3: "The deterministic LOCC transformation T_{LA} from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is executable if and only if Conditions of Theorem 2 whose $\zeta_B = \zeta_C = 1$ are satisfied." If we can prove the statement S_3, we can also prove the present Corollary: Because Conditions of Theorem 2 whose $\zeta_B = \zeta_C = 1$ are equivalent to Conditions of the present Corollary.

Theorem 2 guarantees that the deterministic LOCC transformation T_{LA} is executable if Conditions of Theorem 2 whose ζ_B and ζ_C are equal to one are satisfied. Thus we only have to prove that if T_{LA} is executable then $\alpha_B = \alpha_C = 1$ holds, in order to prove the statement S_3. If $\alpha_B \neq 1$ or $\alpha_C \neq 1$, then we have $j_{AB} : j_{AC} : j_{ABC} \neq j'_{AB} : j'_{AC} : j'_{ABC}$. However, the measurements of the deterministic transformation T_{LA} are performed only on the qubit A, and thus (A.40)–(A.42) give that $K_{AB} : K_{AC} : j_{ABC}^2 = K'_{AB} : K'_{AC} : j'^2_{ABC}$. Then, if the deterministic LOCC transformation T_{LA} is executable, we have $\zeta_B = \zeta_C = 1$. $\square$

3.5.2 Case $\mathfrak{B}$

In this subsection, we consider an arbitrary deterministic LOCC transformation from an arbitrary EP-definite state to an arbitrary EP-indefinite state (Case $\mathfrak{B}$). In this case, we can prove Main Theorems directly, not following Steps 1–3. A deterministic LOCC transformation from an EP-definite state to an EP-indefinite state is executable only if the final state is biseparable or full-separable. We prove Theorem 3, which includes the above statement. In this subsection, we prove theorems including the case of $j_{ABC} = 0$.

Theorem 3 *Let the notation T_{SL} stand for an LOCC transformation from an arbitrary EP-definite state $|\psi\rangle$ to arbitrary EP-indefinite states $\{|\psi^{(i)}\rangle\}$. The subscript SL stands for Stochastic LOCC. Then, if this LOCC transformation T_{SL} is executable, there must be full-separable states or biseparable states in the set $\{|\psi^{(i)}\rangle\}$.*

Proof: In the same manner as Theorem 2, we prove the present theorem by mathematical induction with respect to N, which is the number of times measurements are performed in the LOCC transformation T_{SL}. Let the notations $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j_{AB}^{(i)}, j_{AC}^{(i)}, j_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)}, Q_e^{(i)})$ stand for the sets of the J-parameters of the EP-definite state $|\psi\rangle$ and the EP-indefinite states $|\psi^{(i)}\rangle$, respectively.

First, we prove the present theorem for $N = 1$. Because of the arbitrariness of the state $|\psi\rangle$, we can assume that the first measurement $\{M_{(i)}|i = 0, 1\}$ of the LOCC transformation T_{SL} is performed on the qubit A without loss of generality. Thus, the operator $M_{(i)}$ makes j_{AB}, j_{AC} and j_{ABC} evenly multiplied by a real number $\alpha^{(i)}$. The state $|\psi\rangle$ is EP definite, and hence j_{AB}, j_{AC} and j_{BC} are all positive. Because the state $|\psi^{(i)}\rangle$ is EP indefinite, at least one of $j_{AB}^{(i)}$, $j_{AC}^{(i)}$ and $j_{BC}^{(i)}$ has to be zero for all i. When $j_{AB}^{(i)}$ or $j_{AC}^{(i)}$ is zero, the multiplication factor $\alpha^{(i)}$ must be zero, and therefore all of $j_{AB}^{(i)}$, $j_{AC}^{(i)}$ and $j_{ABC}^{(i)}$ must be zero. Then, the parameter $J_5^{(i)}$ also must be zero because of the expressions (A.26), (A.27) and $j_{AB}^{(i)} j_{AC}^{(i)} j_{BC}^{(i)} = 0$. Thus, in the case of $j_{AB}^{(i)} = 0$ or $j_{AC}^{(i)} = 0$, the EP-indefinite state $|\psi^{(i)}\rangle$ is a full-separable state with $j_{BC}^{(i)} = 0$ or a biseparable state with $j_{BC}^{(i)} \neq 0$. Hence, if there were neither a full-separable state nor a biseparable state in the set of EP-indefinite states $\{|\psi^{(i)}\rangle\}$, the expressions $j_{AC}^{(i)} \neq 0$, $j_{AB}^{(i)} \neq 0$ and $j_{BC}^{(i)} = 0$ would hold for all i. Because of Lemma 2, however, at least one of $j_{BC}^{(0)}$ and $j_{BC}^{(1)}$ would be greater than or equal to j_{BC}, which is positive. This is a contradiction, and thus the expression $j_{BC}^{(i)} \neq 0$ has to hold for at least one of i. We have thereby shown the present theorem for $N = 1$.

Now, we prove Theorem 3 for $N = k + 1$, assuming that Theorem 3 holds whenever $1 \leq N \leq k$. Let us assume that the number of times of measurements in the LOCC transformation T_{SL} from the EP-definite state $|\psi\rangle$ to the EP-indefinite states $\{|\psi^{(i)}\rangle\}$ is $k+1$. Because of the assumption for $1 \leq N \leq k$, the situation before the last measurement has to be either of the following two situations:

(i) All states are already EP indefinite, and there are full-separable states or biseparable states among them.

(ii) Some states are EP definite.

In the case of (i), there are full-separable states or biseparable states in the final EP-indefinite states $\{|\psi^{(i)}\rangle\}$ because an arbitrary full-separable state or an arbitrary biseparable state can be transformed only into full-separable states or biseparable states by a measurement.

In the case of (ii), if there were neither a full-separable state nor a biseparable state in the EP-indefinite states $\{|\psi^{(i)}\rangle\}$, there would have to be a measurement which could transform an EP-definite state to EP-indefinite states which are neither full-separable states nor biseparable states. Because of the theorem for $N = 1$, this is impossible.

Therefore, there must be either full-separable states or biseparable states in the EP-indefinite states $\{|\psi^{(i)}\rangle\}$ in the case (ii) as well as in the case (i). This completes the

proof of Theorem 3. □

The set of full-separable states and biseparable states which have the same kind of bipartite entanglement is a totally ordered set [14]. In other words, if an EP-definite state $|\psi\rangle$ and an EP-indefinite state $|\psi'\rangle$ belong to such a set, there is an executable deterministic LOCC transformation from the EP-definite state $|\psi\rangle$ to the EP-indefinite state $|\psi'\rangle$ if and only if the bipartite entanglement of the state $|\psi\rangle$ is greater than or equal to that of the state $|\psi'\rangle$. Hence, for a deterministic LOCC transformation from an EP-definite state $|\psi\rangle$ to an EP-indefinite state $|\psi'\rangle$, if we can obtain the upper limit of the bipartite entanglement of the EP-indefinite state $|\psi'\rangle$, we can reproduce the transformation from the EP-definite state $|\psi\rangle$ to the EP-indefinite state $|\psi'\rangle$ in the following two steps:

Step$_{TtB}$ 1 We carry out a deterministic LOCC transformation from an EP-definite state $|\psi\rangle$ to a biseparable state $|\psi''\rangle$ whose the bipartite entanglement is equal to the upper limit of that of $|\psi'\rangle$.

Step$_{TtB}$ 2 We carry out a deterministic LOCC transformation from the EP-definite state $|\psi''\rangle$ to the EP-indefinite state $|\psi'\rangle$.

The following Theorem 4 gives the upper limit of Step$_{TtB}$ 1. In fact, this theorem holds not only for LOCC transformations from an arbitrary EP-definite state, but for LOCC transformations from a general state.

Theorem 4 *Let the notation T_L stand for an arbitrary LOCC transformation from an arbitrary state $|\psi\rangle$ to arbitrary EP-indefinite states $\{|\psi'^{(i)}\rangle\}$. We refer to the sets of the J-parameters of the state $|\psi\rangle$ and the EP-indefinite states $|\psi'^{(i)}\rangle$ as $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Qe)$ and $(j'^{(i)}_{AB}, j'^{(i)}_{AC}, j'^{(i)}_{BC}, j'^{(i)}_{ABC}, J'^{(i)}_5, Q'^{(i)}_e)$, respectively. We also assume that $Q'^{(i)}_e = j'^{(i)}_{AB} = j'^{(i)}_{AC} = j'^{(i)}_{ABC} = J'^{(i)}_5 = 0$ for any i. Then, the inequality $j'_{BC} \leq \sqrt{j^2_{BC} + j^2_{ABC}}$ holds, where the notation j'_{BC} stands for the minimum of $j'^{(i)}_{BC}$.*

Proof: We prove the present theorem by mathematical induction with respect to N, which is the number of times measurements are performed in the LOCC transformation T_L. For $N = 1$, the EP-indefinite states $\{|\psi'^{(i)}\rangle\}$ must be achieved by one measurement. Then, as was shown in the proof of Theorem 3, the multiplication factors of this measurement are equal to zero for any i. Thus, (A.51) with $\alpha^{(i)} = 0$ gives that $j'_{BC} \leq \sqrt{j^2_{BC} + j^2_{ABC}}$.

Next, we prove Theorem 4 for $N = k + 1$, assuming that Theorem 4 holds whenever $1 \leq N \leq k$. Let the notion $\{|\psi^{(i)}\rangle\}$ stand for results of the first measurement of the LOCC transformation T_L. We refer to the set of the J-parameters of $|\psi^{(i)}\rangle$ as $(j^{(i)}_{AB}, j^{(i)}_{AC}, j^{(i)}_{BC}, j^{(i)}_{ABC}, j^{(i)}_5)$. The states $\{|\psi^{(i)}\rangle\}$ are transformed to the EP-indefinite states $\{|\psi'^{(i)}\rangle\}$ after k times of measurements. Because of the assumption for $1 \leq N \leq k$, the inequality $j'_{BC} \leq \sqrt{(j^{(i)}_{BC})^2 + (j^{(i)}_{ABC})^2}$ holds for any i. Thus, for the failure of Theorem 4 in the case of $N = k + 1$, the inequality $\sqrt{(j^{(i)}_{BC})^2 + (j^{(i)}_{ABC})^2} > \sqrt{j^2_{BC} + j^2_{ABC}}$ would hold for every i. Hence, the inequality $\sum^1_{i=0} p_{(i)} \sqrt{(j^{(i)}_{BC})^2 + (j^{(i)}_{ABC})^2} > \sqrt{j^2_{BC} + j^2_{ABC}}$ would hold. This contradicts Lemma 5. This means that (3.140) holds for all cases, which completes the proof of Theorem 4.□

In the above, we have found that the upper limit of the bipartite entanglement between the qubits B and C is $\sqrt{j_{BC}^2 + j_{ABC}^2}$. We can realize this upper limit by substituting $\alpha = 0$ in Lemma 4. Therefore, we have proven that we can carry out the Step_{TtB} 1 and Step_{TtB} 2. Note that each of these two steps are realized by one DMT. Hence, we have also proven that we can reproduce the deterministic transformation from an EP-definite state $|\psi\rangle$ to an EP-indefinite state by performing only two DMTs. This corresponds to the second row of Table 3.1.

Next, let us see the relation between the result we have obtained and Main Theorem 1. Because the state $|\psi'\rangle$ is EP indefinite, we can leave out Condition 2 of Theorem 2. The condition of Main Theorem 1 is the existence of real numbers $J_{\text{ap}}/(K_{AB} - \zeta_A j_{ABC}^2)(K_{AC} - \zeta_B j_{ABC}^2)(K_{BC} - \zeta_A j_{ABC}^2) \leq \zeta \leq 1$, $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$ and $0 \leq \zeta_C \leq 1$ which satisfy that

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta \begin{pmatrix} \zeta_A\zeta_B & & & & \\ & \zeta_A\zeta_C & & & \\ & & \zeta_B\zeta_C & & \\ & & & \zeta_A\zeta_B\zeta_C & \\ & & & & \zeta_A\zeta_B\zeta_C \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}, \tag{3.348}
$$

where $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5)$ and $(K'_{AB}, K'_{AC}, K'_{BC}, j'_{ABC}, J'_5)$ are the sets of the K-parameters of the EP-definite state $|\psi\rangle$ and the EP-indefinite state $|\psi'\rangle$, respectively. Hence all of K_{AB}, K_{AC} and K_{BC} are greater than j_{ABC}^2 and at least one of K'_{AB}, K'_{AC} and K'_{BC} is equal to j'^2_{ABC}. Therefore, at least one of the main parameters ζ_A, ζ_B and ζ_C must be zero, and thereby the EP-indefinite state $|\psi'\rangle$ must be a full-separable state or a biseparable state whose bipartite entanglement is less than or equal to the upper limit. These are equivalent to the result which we have obtained, and thus we have proven Main Theorem 1 in Case $\mathfrak{B}$.

3.5.3 Case $\mathfrak{C}$

In this subsection, we prove Main Theorems in Case $\mathfrak{C}$.

Transferless DMT

In section 6.3.1, we show that if a state $|\psi\rangle$ has no bipartite entanglement between the qubits B and C, an A-DMT can make all entanglement parameters of the state $|\psi\rangle$ multiplied by a real number α which is from zero to one. This Theorem and the next Lemma are used instead of Step 1 of Case $\mathfrak{C}$.

Theorem 5 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the J-parameters of the states $|\psi\rangle$ and $|\psi'\rangle$ as $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j'_{AB}, j'_{AC}, j'_{BC}, j'_{ABC}, J'_5, Q'_e)$, respectively. The state $|\psi\rangle$ can be transformed into the state $|\psi'\rangle$ by an A-DMT, if $Q'_e = Q_e = j'_{BC} = j_{BC} = 0$ and if there is a real number $0 \leq \alpha \leq 1$ which satisfies the equations $j'_{AB} = \alpha j_{AB}$, $j'_{AC} = \alpha j_{AC}$ and $j'_{ABC} = \alpha j_{ABC}$.*

Proof: Let us define a measurement $\{M_{(i)}|i = 0, 1\}$ and its measurement parameters a, b, k and θ as in (A.32) and (A.33). Let the notation $\{|\psi^{(i)}\rangle\}$ stand for the results of performing the measurement $\{M_{(i)}|i = 0, 1\}$ on the qubit A. We refer to the set of

the J-parameters of the state $|\psi^{(i)}\rangle$ as $(j_{AB}^{(i)}, j_{AC}^{(i)}, j_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)}, Q_e^{(i)})$. Substituting the equation $j_{BC} = 0$ in (A.44), we find that if $k_{(i)} = k = 0$, then $j_{BC}^{(i)} = 0$. Hence, in order to prove Theorem 5, it suffices to show that there is a measurement $\{M_{(i)} | i = 0, 1\}$ which satisfies the expressions $k = 0$ and $0 \leq \alpha_{(0)} = \alpha_{(1)} \leq 1$, where $\alpha^{(i)}$ is defined in (A.45). Note that if $j_{BC} = j_{BC}^{(i)} = j'_{BC} = 0$, then $J_5 = J_5^{(i)} = J'_5 = 0$ and $Q_e = Q_e^{(i)} = Q'_e = 0$ hold, because if a state is EP indefinite J_5 and Q_e are zero.

From the equation (A.45) and $k = 0$, it follows that the equation $\alpha_{(0)} = \alpha_{(1)}$ is equivalent to the following equation:

$$\frac{ab}{p_{(0)}^2} = \frac{(1-a)(1-b)}{(1-p_{(0)})^2}. \tag{3.349}$$

Hence, in order to show the present Theorem, it suffices to show that $\alpha_{(0)}$ can take any value from zero to one under the condition of (3.349).

The equation (3.349) is equivalent to the following equation:

$$(1 - 2p_{(0)})ab = (1 - a - b)p_{(0)}^2. \tag{3.350}$$

We can interpret this equation as a relation between a and b which can be expressed as a hyperbola and a straight line. To see this, we perform substitution of the equations $k = 0$ and (A.34) in the probability $p_{(0)}$ of (3.350) and the following transformation:

$$
\begin{aligned}
[1 - 2(1 - \lambda_0^2)b - 2\lambda_0^2 a]ab &= (1 - a - b)[(1 - \lambda_0^2)b + \lambda_0^2 a]^2, \\
0 &= -ab + (1 - \lambda_0^2)^2 b^2 + \lambda_0^4 a^2 + 2(1 - \lambda_0^2)\lambda_0^2 ab + (1 - \lambda_0^2)^2 ab^2 - \lambda_0^4 a^3 + \lambda_0^4 a^2 b \\
0 &= (b - a)[-(1 - a)a\lambda_0^4 + b(1 - \lambda_0^2)^2 - b^2(1 - \lambda_0^2)^2].
\end{aligned}
$$

After the transformation of the expression in the last parentheses of (3.351), we obtain the following equation:

$$0 = (b - a)\left[\lambda_0^4 \left(a - \frac{1}{2}\right)^2 - (1 - \lambda_0^2)^2 \left(b - \frac{1}{2}\right)^2 + \frac{1 - 2\lambda_0^2}{4}\right]. \tag{3.352}$$

Then, the condition $\alpha_{(0)} = \alpha_{(1)}$ is equivalent to that $a = b$ or

$$\lambda_0^4 \left(a - \frac{1}{2}\right)^2 - (1 - \lambda_0^2)^2 \left(b - \frac{1}{2}\right)^2 = -\frac{1 - 2\lambda_0^2}{4}. \tag{3.353}$$

In the former case, after substitution of the equations $a = b$ and $k = 0$ and (A.34) in (3.349), we find that the equation $a = b$ is followed by $\alpha_{(0)} = \alpha_{(1)} = 1$. In the latter case, let us examine (3.353) in detail. Because $0 \leq \lambda_0 \leq 1$, we examine (3.353) in the following five cases: $\lambda_0 = 0$, $0 < \lambda_0 < 1/2$, $\lambda_0 = 1/2$, $1/2 < \lambda_0 < 1$ and $\lambda_0 = 1$.

First, if $0 < \lambda_0 < 1/\sqrt{2}$, (3.353) is equivalent to the following equation:

$$(1 - \lambda_0^2)^2 \left(b - \frac{1}{2}\right)^2 - \lambda_0^4 \left(a - \frac{1}{2}\right)^2 = \frac{1 - 2\lambda_0^2}{4} > 0. \tag{3.354}$$

This is expressed as hyperbolas in Fig. 3.14(a). Note that $\alpha_{(0)} = 1$ at $(a, b) = (1, 1)$
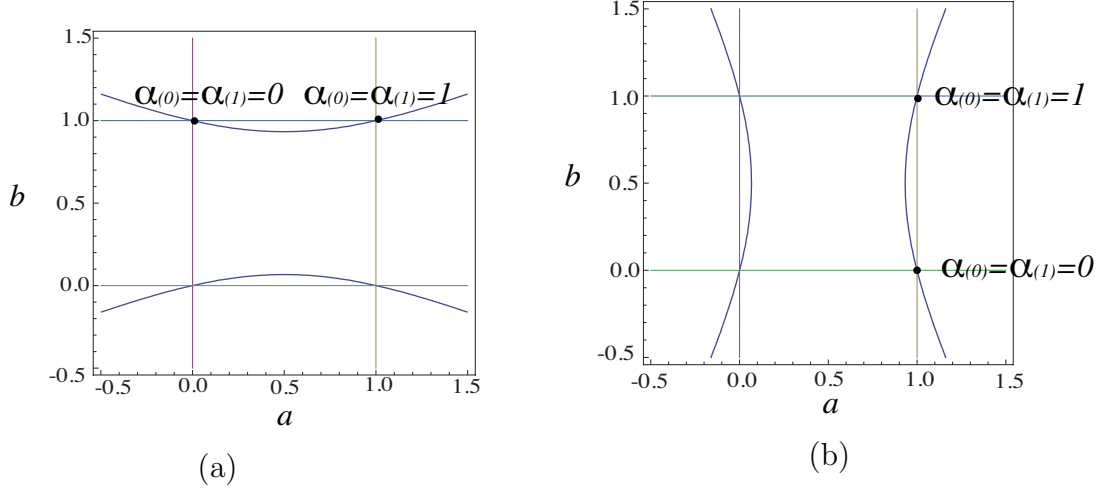
Figure 3.14: The hyperbolas where the measurement parameters a and b satisfy $\alpha_{(0)} = \alpha_{(1)}$ (a) for $\lambda_0 = 1/\sqrt{3}$ and (b) for $\lambda_0 = \sqrt{2}/\sqrt{3}$.

and $\alpha_{(0)} = 0$ at $(a, b) = (0, 1)$. The multiplication factor $\alpha_{(0)}$ is continuous with respect to the measurement parameters a and b, unless $b = -\lambda_{(0)}^2 a/(1 - \lambda_{(0)}^2)$. The line $b = -\lambda_{(0)}^2 a/(1 - \lambda_{(0)}^2)$ does not cross the upper hyperbola. Thus, if we move (a, b) from $(1,1)$ to $(0,1)$ along the upper hyperbola, $\alpha_{(0)} = \alpha_{(1)}$ takes any value from one to zero. Then, Theorem 5 holds for $0 < \lambda_0 < 1/\sqrt{2}$.

Second, if $1/\sqrt{2} < \lambda_0 < 1$, (3.353) is equivalent to the following equation:

$$-(1 - \lambda_0^2)^2 \left(b - \frac{1}{2}\right)^2 + \lambda_0^4 \left(a - \frac{1}{2}\right)^2 = -\frac{1 - 2\lambda_0^2}{4} > 0. \qquad (3.355)$$

This is expressed as hyperbolas in Fig. 3.14(b). In the same manner as in the case of $0 < \lambda_0 < 1/\sqrt{2}$, if we move (a, b) from $(1,1)$ to $(1,0)$ along the right hyperbola, then $\alpha_{(0)} = \alpha_{(1)}$ takes any values from one to zero. Thus, Theorem 5 holds for $1/\sqrt{2} < \lambda_0 < 1$.

Third, if $\lambda_0 = 0$, we have $j_{AB} = j_{AC} = j_{ABC} = J_5 = 0$. In addition, Theorem 5 assumes the equation $j_{BC} = 0$. In this case, therefore, the state $|\psi\rangle$ is full-separable. Thus, an arbitrary measurement is sufficient for our perpose, because any measurement on the qubit A leaves $|\psi\rangle$ full-separable. Therefore Theorem 5 also holds in this case.

Fourth, if $\lambda_0 = 1/\sqrt{2}$, (3.353) is equivalent to the equation $(b - 1/2)^2 = (a - 1/2)^2$. This is equivalent to that $(a = b) \vee (a + b = 1)$. If $a = b$, then $\alpha_{(0)} = \alpha_{(1)} = 1$. If $a + b = 1$, then $\lambda_0^2 = 1/2$. The equations $a + b = 1$, $k = 0$, $\lambda_0^2 = 1/2$, (3.349) and (A.34) give that $\alpha_{(0)} = \alpha_{(1)} = 2a(1 - a)$. Thus, in this case, the multiplication factor $\alpha_{(0)} = \alpha_{(1)}$ can take any values from zero to one. Thus, Theorem 5 also holds in this case.

Finally, if $\lambda_0 = 1$, then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ because $\sum_{k=0}^{4} \lambda_k^2 = 1$. Thus the state $|\psi\rangle$ is full-separable, and Theorem 5 holds in the same manner as in the case of $\lambda_0 = 0$.

Hence, Theorem 5 holds in all cases of λ_0. $\square$

Step 2 of Case $\mathfrak{C}$

The next Theorem 6 gives a necessary and sufficient condition of the possibility of a C-LOCC transformation from an EP-indefinite state whose j_{ABC} is not zero. In the proof of

Theorem 6, we also prove that an arbitrary C-LOCC transformation can be reproduced by three DMTs. This corresponds to Step 2 of Case $\mathfrak{C}$.

Theorem 6 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the K-parameters of the states $|\psi\rangle$ and $|\psi'\rangle$ as $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5, Q_e)$ and $(K'_{AB}, K'_{AC}, K'_{BC}, j'_{ABC}, J'_5, Q'_e)$, respectively. We assume that $|\psi\rangle$ is EP indefinite. We also assume that j_{ABC} is not zero. Then, a necessary and sufficient condition of the possibility of a C-LOCC transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is that the following two conditions are satisfied:*
Condition 1: There are real numbers $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$, $0 \leq \zeta_C \leq 1$ and $\zeta_{lower} \leq \zeta \leq 1$ which satisfy the following equation:

$$\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta \begin{pmatrix} \zeta_A\zeta_B & & & & \\ & \zeta_A\zeta_C & & & \\ & & \zeta_B\zeta_C & & \\ & & & \zeta_A\zeta_B\zeta_C & \\ & & & & \zeta_A\zeta_B\zeta_C \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}, \qquad (3.356)$$

where

$$\zeta_{lower} = \frac{j^2_{AB}j^2_{AC}j^2_{BC}}{(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}, \qquad (3.357)$$

and we refer to ζ, ζ_A, ζ_B and ζ_C as the sub parameter and the main parameters of A, B and C, respectively.
   Condition 2: Let us check whether the state $|\psi\rangle$ is $\tilde{\zeta}$-definite or not. When the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the condition is

$$Q_e = Q'_e \ \text{ and } \ \zeta = \tilde{\zeta}. \qquad (3.358)$$

where

$$\tilde{\zeta} \equiv \frac{K_{ap}(4J_{ap} - J^2_5) + \Delta_J J_{ap}}{K_{ap}(4J_{ap} - J^2_5) + \Delta_J(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_C j^2_{ABC})}. \qquad (3.359)$$

When the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, the condition is

$$|Q'_e| = sgn[(1 - \zeta)(\zeta - \zeta_{lower})], \qquad (3.360)$$

or in the other words,

$$Q'_e \begin{cases} = 0 & (\zeta = 1 \ or \ \zeta = \zeta_{lower}), \\ \neq 0 & (otherwise). \end{cases} \qquad (3.361)$$

Comment: Note that Condition 2 means that ifthe state $|\psi\rangle$ is $\tilde{\zeta}$-definite and the state $|\psi'\rangle$ is EP definite, the C-LOCC transformation from the EP-indefinite state $|\psi\rangle$ to EP-definite state $|\psi'\rangle$ is impossible. Because $|\psi\rangle$ is EP-indefinite, $4J_{ap} = J^2_5$. Thus, if tthe state $|\psi\rangle$ is $\tilde{\zeta}$-definite, $\Delta_J > 0$ has to hold. Thus, $\zeta = \tilde{\zeta}$ is equal to

$$\zeta = \frac{j^2_{AB}j^2_{AC}j^2_{BC}}{(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}. \qquad (3.362)$$

Because of (3.362), the equation (3.356) is equivalent to

$$j'^2_{AB} = \frac{\zeta_A \zeta_B j^2_{AB} j^2_{AC} j^2_{BC}}{(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}, \tag{3.363}$$

$$j'^2_{AC} = \frac{\zeta_A \zeta_C j^2_{AB} j^2_{AC} j^2_{BC}}{(K_{AB} - \zeta_C j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}, \tag{3.364}$$

$$j'^2_{BC} = \frac{\zeta_B \zeta_C j^2_{AB} j^2_{AC} j^2_{BC}}{(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})}. \tag{3.365}$$

Because of $K_{AB} - \zeta_C j^2_{ABC} \geq j^2_{AB}$, $K_{AC} - \zeta_B j^2_{ABC} \geq j^2_{AC}$, $K_{BC} - \zeta_A j^2_{ABC} \geq j^2_{BC}$ and (3.363)–(3.365), if $j_{AB} j_{AC} j_{BC} = 0$, at least one of j'_{AB}, j'_{AC} and j'_{BC} is zero. Thus, if $|\psi\rangle$ is EP-indefinite and $\tilde{\zeta}$-definite, the state $|\psi'\rangle$ has to be EP-indefinite.

Proof: First, we show that if the state $|\psi'\rangle$ is EP indefinite, Condition 1 is a necessary and sufficient condition for the possibility of a C-LOCC transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$. In this case, we can neglect the entanglement charge Q_e and the J-parameter J_5, because both of the states $|\psi\rangle$ and $|\psi'\rangle$ are EP indefinite. The necessity of Condition 1 is clear because of Lemma 2 and (A.40)–(A.42). Let us show the sufficiency. At first, we show the sufficiency in the case that at least one of the main parameters ζ_A, ζ_B and ζ_C is zero. We can assume that $\zeta_C = 0$ without loss of generality. In the present case, we can reproduce the change of entanglement parameters (3.356) as follows:
1. We operate a dissipationless C-DMT whose transfer parameter is zero.
2. We operate an A-DMT whose transfer parameter is equal to K'_{AB}/K_{AB}.
The first of the above is possible because of Lemma 4. The second of the above is possible because of Ref [14]. Next, we show the sufficiency of Condition 1 in the case that none of the main parameters ζ_A, ζ_B and ζ_C is zero. The state $|\psi'\rangle$ is EP indefinite, and thus at least one of K'_{AB}, K'_{AC} and K'_{BC} is equal to j'^2_{ABC}. Thus, at least one of (K_{BC}, ζ_A), (K_{AC}, ζ_B) and (K_{AB}, ζ_C) is equal to $(j^2_{ABC}, 1)$, because of $j^2_{ABC} \leq K_{AB}$, $j^2_{ABC} \leq K_{AC}$, $j^2_{ABC} \leq K_{BC}$, $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$ and $0 \leq \zeta_C \leq 1$. We can assume that $(K_{AB}, \zeta_C) = (j^2_{ABC}, 1)$ without loss of generality. Now we can reproduce the change of entanglement parameters (3.356) in the following three steps:
1. We operate an A-DMT whose main and sub parameters are ζ_A and one, respectively.
2. We operate a B-DMT whose main and sub parameters are ζ_B and one, respectively.
3. We operate a C-DMT which makes all entanglement parameters multiplied by ζ.
The first and second of the above are possible because of Lemma 4. The third of the above is possible because of Theorem 5.

Second, we prove the present theorem in the case where if the state $|\psi'\rangle$ is EP definite. Because of Theorem 1, a DMT from an EP-indefinite state to an EP-definite state is possible only if the EP-indefinite state is $\tilde{\zeta}$-indefinite. Because of Corollary 1, a DMT from a $\tilde{\zeta}$-definite state to a $\tilde{\zeta}$-indefinite state is impossible. Thus, a C-LOCC transformation from an EP-indefinite state to an EP-definite state is possible only if the EP-indefinite state is $\tilde{\zeta}$-indefinite. Because $(\zeta < 1) \Rightarrow (\Delta'_{\text{norm}} > \Delta_J)$ and $(\zeta_{\text{lower}} < \zeta) \Rightarrow j'_{BC} \sin \varphi_5 > j_{BC} \sin \varphi_5$, the equation $|Q'_e| = \text{sgn}[(1 - \zeta)(\zeta - \zeta_{\text{lower}})]$ has to hold. Thus, if the state $|\psi'\rangle$ is EP definite, Condition 2 is necessary. The necessity of Condition 1 is clear because of Lemma 2. Lastly, we prove the sufficiency of Conditions 1 and 2. Let us assume that the states $|\psi\rangle$ and $|\psi'\rangle$ satisfy Conditions 1 and 2 and that the state $|\psi\rangle$ is EP indefinite. We can assume that $j_{AB} = 0$ without losing generality. Then, we can reproduce the

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transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ as follows:

1. We operate a dissipationless A-DMT whose transfer parameter is equal to $\sqrt{\zeta_A}$.
2. We then operate a dissipationless B-DMT whose transfer parameter is equal to $\sqrt{\zeta_B}$.
3. We finally operate a C-DMT whose main and sub parameters are equal to ζ_C and ζ, respectively.

The first and second of the above are possible because of Lemma 5. The third of the above is possible because of Theorem 1. Thus, if Conditions 1 and 2 are satisfied, the C-LOCC transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is possible. $\square$

Note that we have also proven in the above that an arbitrary C-LOCC can be reproduced by three DMTs.

Step 3 of Case $\mathfrak{C}$

In section 6.3.3, we show that a necessary and sufficient condition of the possibility of a deterministic LOCC transformation from an EP-indefinite state is equivalent to that of a C-LOCC transformation from an EP-indefinite state.

Theorem 7 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the K-parameters of the states $|\psi\rangle$ and $|\psi'\rangle$ as $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5, Q_e)$ and $(K'_{AB}, K'_{AC}, K'_{BC}, j'_{ABC}, J'_5, Q'_e)$, respectively. We assume that the state $|\psi\rangle$ is EP indefinite. We also assume that $j_{ABC} \neq 0$. Then, a necessary and sufficient condition of the possibility of a deterministic LOCC transformation from the EP-indefinite state $|\psi\rangle$ to the state $|\psi'\rangle$ is that the following two conditions are satisfied:*

Condition 1: There are real numbers $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$, $0 \leq \zeta_C \leq 1$ and $\zeta_{lower} \leq \zeta \leq 1$ which satisfy the following equation:

$$\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta \begin{pmatrix} \zeta_A\zeta_B & & & & \\ & \zeta_A\zeta_C & & & \\ & & \zeta_B\zeta_C & & \\ & & & \zeta_A\zeta_B\zeta_C & \\ & & & & \zeta_A\zeta_B\zeta_C \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}, \qquad (3.366)$$

where

$$\zeta_{lower} = \frac{J_{ap}}{(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}, \qquad (3.367)$$

and we refer to ζ, ζ_A, ζ_B and ζ_C as the sub parameter and the main parameter A, B and C, respectively.

Condition 2: Let us check whether the state $|\psi\rangle$ is $\tilde{\zeta}$-definite or not. When the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the condition is

$$Q_e = Q'_e \ \ and \ \ \zeta = \tilde{\zeta}. \qquad (3.368)$$

where

$$\tilde{\zeta} \equiv \frac{K_{ap}(4J_{ap} - J_5^2) + \Delta_J J_{ap}}{K_{ap}(4J_{ap} - J_5^2) + \Delta_J(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_C j^2_{ABC})}. \qquad (3.369)$$

When the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, the condition is

$$|Q'e| = sgn[(1 - \zeta)(\zeta - \zeta_{lower})],\tag{3.370}$$

or in the other words,

$$Q'e \begin{cases} = 0 & (\zeta = 1 \ \ or \ \ \zeta = \zeta_{lower}), \\ \neq 0 & (otherwise). \end{cases}\tag{3.371}$$

Comment1: In the same manner as Theorem 6, Condition 2 means that if the state $|\psi\rangle$ is $\tilde{\zeta}$-definite and the state $|\psi'\rangle$ is EP definite, the C-LOCC transformation from the EP-indefinite state $|\psi\rangle$ to the EP-definite state $|\psi'\rangle$ is impossible.

Comment2: In general, if the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, at least one of $4J_{\text{ap}} > J_5^2$ and $\Delta_J > 0$ holds. In the present theorem, we have $4J_{\text{ap}} = J_5^2 = 0$, because $|\psi\rangle$ is EP-indefinite. Thus, if the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, $\Delta_J > 0$ has to hold. We use this fact for performing the proof of the present theorem.

Proof: To prove the present theorem, it suffices to show the following statement S': "An arbitrary deterministic LOCC transformation can be reproduced by a C-LOCC transformation." We show this statement by mathematical induction with respect to N, which is the number of times measurements are performed in the deterministic LOCC transformation. If $N = 1$, the statement S' clearly holds. We prove the statement S' for $N = k + 1$, assuming that the statement S' holds whenever $1 \leq N \leq k$.

Let the notation T_L stand for a deterministic LOCC transformation with $N = k + 1$. Let the notation $\{|\psi^{(i)}\rangle\}$ stand for the results of the first measurement in the deterministic LOCC transformation T_L, which we refer to as the measurement $\{M_{(i)}|i = 0, 1\}$. We can assume that the operator $M_{(i)}$ acts on the qubit A without loss of generality because the EP-indefinite state $|\psi\rangle$ is arbitrary except for its EP indefiniteness. Let the notation $(K_{AB}^{(i)}, K_{AC}^{(i)}, K_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)})$ stand for the set of K-parameters of the state $|\psi^{(i)}\rangle$. We define the measurement parameters a, b, k and θ as (A.32) and (A.33) corresponding to the measurement $M_{(i)}$. We also define multiplication factors $\{\alpha^{(i)}\}$ as in (A.45). Because of (A.40)–(A.42), the multiplication factor $\alpha^{(i)}$ satisfies that

$$\alpha^{(i)} = \frac{j_{AB}^{(i)}}{j_{AB}} = \frac{j_{AC}^{(i)}}{j_{AC}} = \frac{j_{ABC}^{(i)}}{j_{ABC}}.\tag{3.372}$$

Because of the assumption for $N = k$, a C-LOCC transformation $T_{(i)}^D$ can reproduce the transformation from the state $|\psi^{(i)}\rangle$ to the state $|\psi'\rangle$. The C-LOCC transformation $T_{(i)}^D$ consists of an A-DMT $T_A^{(i)}$, a B-DMT $T_B^{(i)}$ and a C-DMT $T_C^{(i)}$ (Fig. 3.15). We refer to the sets of the main parameter and the sub parameter of DMTs $T_A^{(i)}$, $T_B^{(i)}$ and $T_C^{(i)}$ as $(\zeta_A^{(i)}, \zeta^{A(i)})$, $(\zeta_B^{(i)}, \zeta^{B(i)})$ and $(\zeta_C^{(i)}, \zeta^{C(i)})$, respectively. In the same manner as in Theorem 2, the following equation has to hold:

$$\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta^{(i)} \begin{pmatrix} \zeta_A^{(i)}\zeta_B^{(i)} & & & & \\ & \zeta_A^{(i)}\zeta_C^{(i)} & & & \\ & & \zeta_B^{(i)}\zeta_C^{(i)} & & \\ & & & \zeta_A^{(i)}\zeta_B^{(i)}\zeta_C^{(i)} & \\ & & & & \zeta_A^{(i)}\zeta_B^{(i)}\zeta_C^{(i)} \end{pmatrix} \begin{pmatrix} (\alpha^{(i)})^2 K_{AB} \\ (\alpha^{(i)})^2 K_{AC} \\ K_{BC}^{(i)} \\ (\alpha^{(i)})^2 j^2_{ABC} \\ J_5^{(i)} \end{pmatrix}\tag{3.373}$$
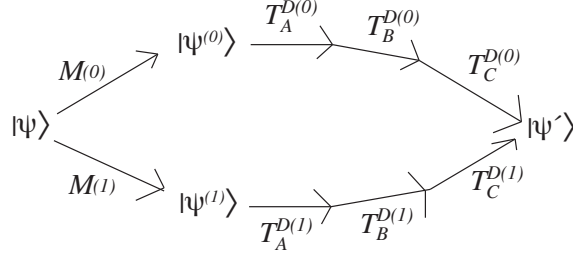
Figure 3.15: The deterministic transformations from the states $\{\left|\psi^{(i)}\right\rangle\}$ to the state $|\psi'\rangle$ which are reproduced by C-LOCC transformations.

where,

$$\zeta^{(i)} \equiv \zeta^{A(i)}\zeta^{B(i)}\zeta^{C(i)}. \tag{3.374}$$

First, let us show that $J_5' = 0$. Because the state $|\psi\rangle$ is EP indefinite, we have $J_5 = 0$. If we can prove $J_5' = 0$, the last row of (3.366) obviously holds and hence we can leave J_5 and J_5' out of the discussion. First, (3.373) gives that

$$J_5' = \zeta^{(i)}\zeta_A^{(i)}\zeta_B^{(i)}\zeta_C^{(i)} J_5^{(i)}. \tag{3.375}$$

On the other hand, the equation (A.26) gives that

$$J_5^{(i)} = 2j_{AB}^{(i)}j_{AC}^{(i)}(\lambda_2^{(i)}\lambda_3^{(i)} - \lambda_1^{(i)}\lambda_4^{(i)} \cos \varphi^{(i)}). \tag{3.376}$$

Because of the equations $j_{AB}^{(i)} = \alpha^{(i)}j_{AB}$ and $j_{AC}^{(i)} = \alpha^{(i)}j_{AC}$, we can transform (3.376) as follows:

$$J_5^{(i)} = 2(\alpha^{(i)})^2 j_{AB}j_{AC}(\lambda_2^{(i)}\lambda_3^{(i)} - \lambda_1^{(i)}\lambda_4^{(i)} \cos \varphi^{(i)}). \tag{3.377}$$

Then, (3.377) and (3.375) give the following equation:

$$J_5' = \Gamma \cdot 2j_{AB}j_{AC}(\lambda_2^{(i)}\lambda_3^{(i)} - \lambda_1^{(i)}\lambda_4^{(i)} \cos \varphi^{(i)}), \tag{3.378}$$

where the product $\Gamma \equiv (\alpha^{(i)})^2\zeta^{(i)}\zeta_A^{(i)}\zeta_B^{(i)}\zeta_C^{(i)}$ is independent of i because $\Gamma = j_{ABC}'^2/j_{ABC}^2$. From (A.47) and (3.378), we obtain that

$$J_5' = \sum_{i=0}^{1} p_{(i)}J_5' = \Gamma \cdot 2j_{AB}j_{AC}(\lambda_2\lambda_3 - \lambda_1\lambda_4 \cos \varphi) = \Gamma J_5. \tag{3.379}$$

Hence $J_5' = 0$ because $J_5 = 0$. We thereby leave the J-parameters J_5 and J_5' out of the discussion hereafter.

Second, we prove the statement S' in the case of $j_{ABC}' = 0$. In this case, the equation $0 = j_{ABC}'/j_{ABC} = \alpha^{(i)}\sqrt{\zeta^{(i)}\zeta_A^{(i)}\zeta_B^{(i)}\zeta_C^{(i)}}$ holds. This equation means that at least one of $\alpha^{(i)}$, $\zeta^{(i)}$, $\zeta_A^{(i)}$, $\zeta_B^{(i)}$ and $\zeta_C^{(i)}$ is zero for each i. Thus, substituting $\alpha^{(i)} = 0$, $\zeta^{(i)} = 0$, $\zeta_A^{(i)} = 0$, $\zeta_B^{(i)} = 0$ or $\zeta_C^{(i)} = 0$ in (3.373), we find that if the deterministic LOCC transformation T_L is executable, the state $|\psi'\rangle$ is a full-separable state or a biseparable state. Let us consider the case where the remaining entanglement of the state $|\psi'\rangle$ is j_{AB}'. This assumption means
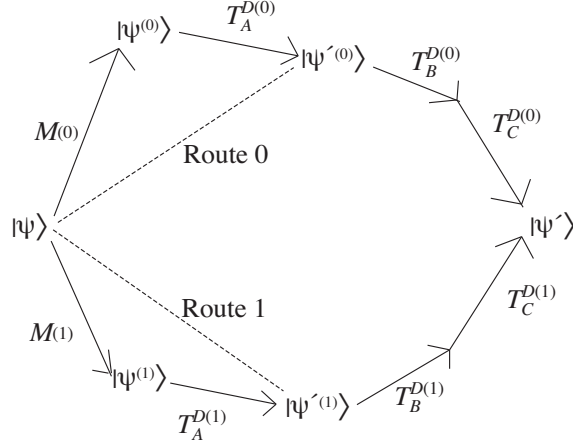
Figure 3.16: The concept of the proof of Theorem 7 in the case of $j_{AB} = 0 \wedge j'_{ABC} \neq 0$. Either of Routes 0 or 1 can be realized by a C-LOCC transformation.

$j'_{AC} = j'_{BC} = j'_{ABC} = J'_5 = 0$. Note that j'_{AB} may be equal to zero. Then, Theorem 4 in the section 6.2 guarantees that if the deterministic LOCC transformation T_L is executable, the inequality $j'_{AB} \leq \sqrt{K_{AB}}$ must hold. We already know from Lemma 4 that if $j'_{AB} \leq \sqrt{K_{AB}}$, the deterministic LOCC transformation T_L can be reproduced by performing a C-dissipationless DMT and an A-dissipationless DMT whose transfer parameters are equal to zero and $j'_{AB}/\sqrt{K_{AB}}$, respectively. This means that the deterministic transformation T_L is reproduced by a C-LOCC transformation, and thus we have proven the statement S' in the case where the remaining entanglement of the state $|\psi'\rangle$ is j'_{AB}. In the same manner, we can prove the statement S' in the case where the remaining entanglement is j'_{AC} or j'_{BC}. Therefore, we have proven the statement S' in the case of $j'_{ABC} = 0$.

Third, we prove the statement S' in the case of $j'_{ABC} \neq 0$. At least one of the J-parameters j_{AB}, j_{AC} and j_{BC} is zero because the state $|\psi\rangle$ is EP indefinite. First, we prove the statement S' in the case of $j'_{ABC} \neq 0 \wedge j_{AB} = 0$ by showing that the transformation from the state $|\psi\rangle$ to a state $|\psi'^{(i)}\rangle$, which is transformed from the state $|\psi^{(i)}\rangle$ by the DMT $T_A^{(i)}$ on the qubit A, can be reproduced by a DMT for at least one of i (Fig. 3.16). The sets of the J-parameters of $\{|\psi^{(i)}\rangle\}_{i=0,1}$ are given by $(j_{AB}^{(0)} = 0, \alpha^{(0)} j_{AC}, j_{BC}^{(0)}, \alpha^{(0)} j_{ABC}, Q_{\mathrm{e}}^{(0)} = 0)$ and $(j_{AB} = 0, \alpha^{(1)} j_{AC}, j_{BC}^{(1)}, \alpha^{(1)} j_{ABC}, Q_{\mathrm{e}}^{(1)} = 0)$, respectively, because the first measurement $M_{(i)}$ is operated on the qubit A. Let $(j_{AB}'^{(i)}, j_{AC}'^{(i)}, j_{BC}'^{(i)}, j_{ABC}'^{(i)}, Q_{\mathrm{e}}'^{(i)})$ stand for the set of the J-parameters of the state $|\psi'^{(i)}\rangle$. Because the state $|\psi'^{(i)}\rangle$ is transformed from the

state $\left|\psi^{(i)}\right\rangle$ by the C-LOCC transformation $T_A^{(i)}$, the following equations hold:

$$
\begin{align}
(j_{AB}'^{(i)})^2 &= (\alpha_A^{(i)})^2(j_{AB}^{(i)})^2 = (\alpha_A^{(i)}\alpha^{(i)})^2 j_{AB}^2 = 0, \tag{3.380} \\
(j_{AC}'^{(i)})^2 &= (\alpha_A^{(i)})^2(j_{AC}^{(i)})^2 = (\alpha_A^{(i)}\alpha^{(i)})^2 j_{AC}^2, \tag{3.381} \\
(j_{ABC}'^{(i)})^2 &= (\alpha_A^{(i)})^2(j_{ABC}^{(i)})^2 = (\alpha_A^{(i)}\alpha^{(i)})^2 j_{ABC}^2, \tag{3.382} \\
(j_{BC}'^{(i)})^2 &= (j_{BC}^{(i)})^2 + \beta_A^{(i)}[1 - (\alpha_A^{(i)})^2](j_{ABC}^{(i)})^2 \\
&= (j_{BC}^{(i)})^2 + \beta_A^{(i)}[1 - (\alpha_A^{(i)})^2](\alpha^{(i)})^2 j_{ABC}^2, \tag{3.383} \\
Q_{\mathrm{e}}'^{(i)} &= 0, \tag{3.384}
\end{align}
$$

where we used the fact $j_{AB} = 0$, while $\alpha_A^{(i)}$ and $\beta_A^{(i)}$ are the transfer and dissipation parameters, respectively. Let us show that the changes of the entanglement parameters (3.380)–(3.383) can be reproduced by a C-LOCC transformation for at least one of i.

We prepare to prove this statement through (3.393). The inequality

$$
j_{BC} j_{ABC}^{(i)} = j_{BC}\alpha^{(i)} j_{ABC} \leq j_{BC}^{(i)} j_{ABC} \tag{3.385}
$$

holds for at least one of i, because the expressions (A.50) and $j_{BC} \leq \sum_i p_i j_{BC}^{(i)}$ gives that $j_{BC}\sum_i p_{(i)}\alpha^{(i)} j_{ABC} \leq j_{ABC}\sum_i p_{(i)} j_{BC}^{(i)}$. On the other hand, the equations (3.382) and (3.383) and the inequalities $0 \leq \alpha_A^{(i)} \leq 1$ and $0 \leq \beta_A^{(i)} \leq 1$ give the following inequality:

$$
\frac{j_{BC}^{(i)}}{\alpha^{(i)} j_{ABC}} \leq \frac{\sqrt{(j_{BC}^{(i)})^2 + \beta_A^{(i)}[1 - (\alpha_A^{(i)})^2](\alpha^{(i)})^2 j_{ABC}^2}}{\alpha^{(i)}\alpha_A^{(i)} j_{ABC}} = \frac{j_{BC}'^{(i)}}{j_{ABC}'^{(i)}}. \tag{3.386}
$$

The inequalities (3.385) and (3.386) yield the following inequalities for at least one of i:

$$
\frac{j_{BC}}{j_{ABC}} \leq \frac{j_{BC}^{(i)}}{\alpha^{(i)} j_{ABC}} \leq \frac{j_{BC}'^{(i)}}{j_{ABC}'^{(i)}}. \tag{3.387}
$$

We show the right-hand side to be independent of i as follows. We can reproduce the transformation from the state $\left|\psi'^{(i)}\right\rangle$ to the state $\left|\psi'\right\rangle$ by performing DMTs $T_B^{(i)}$ and $T_C^{(i)}$ successively, and thus the following equation holds:

$$
\frac{j_{ABC}'}{j_{ABC}'^{(i)}} = \frac{j_{BC}'}{j_{BC}'^{(i)}} = \alpha_B^{(i)}\alpha_C^{(i)}. \tag{3.388}
$$

From this equation, we find that $j_{BC}'^{(i)}/j_{ABC}'^{(i)} = j_{BC}'/j_{ABC}'$ is independent of i. Thus,

$$
\frac{j_{BC}}{j_{ABC}} \leq \frac{j_{BC}'^{(i)}}{j_{ABC}'^{(i)}} \tag{3.389}
$$

holds for both i. On the other hand, the inequality $(j_{BC}^{(i)})^2 + (j_{ABC}^{(i)})^2 \leq j_{ABC}^2 + j_{BC}^2$ holds for at least one of i, which has been shown in the proof of Theorem 4. We can assume that the inequality

$$
(j_{BC}^{(0)})^2 + (j_{ABC}^{(0)})^2 \leq j_{ABC}^2 + j_{BC}^2 \tag{3.390}
$$

holds without loss of generality. The transformation $T_A^{(i)}$ is an A-dissipative C-LOCC transformation, and hence

$$
\begin{aligned}
(j_{BC}'^{(i)})^2 + (\alpha^{(i)}\alpha_A^{(i)}j_{ABC})^2 &= (j_{BC}^{(i)})^2 + \beta_A^{(i)}\left[1 - (\alpha_A^{(i)})^2\right](\alpha^{(i)})^2 j_{ABC}^2 + (\alpha^{(i)}\alpha_A^{(i)}j_{ABC})^2 \\
&\leq (j_{BC}^{(i)})^2 + (j_{ABC}^{(i)})^2
\end{aligned}
\tag{3.391}
$$

holds for every i. Because of (3.390) and (3.391),

$$
(j_{BC}'^{(0)})^2 + (\alpha^{(0)}\alpha_A^{(0)}j_{ABC})^2 \leq j_{ABC}^2 + j_{BC}^2.
\tag{3.392}
$$

The expressions (3.382), (3.389) and (3.392) give the following inequality:

$$
(\alpha^{(0)}\alpha_A^{(0)})^2(j_{BC}^2 + j_{ABC}^2) \leq (j_{BC}'^{(0)})^2 + (\alpha^{(0)}\alpha_A^{(0)}j_{ABC})^2 \leq j_{ABC}^2 + j_{BC}^2.
\tag{3.393}
$$

Now, we have completed the preparation for reproducing (3.380)–(3.383).

Let us show that the transformation from the state $|\psi\rangle$ to the state $\left|\psi'^{(0)}\right\rangle$ is reproduced by performing the following two steps:
1. we multiply all entanglement parameters by a real number $\tilde{\alpha}$;
2. we perform a dissipationless A-DMT whose transfer parameter is α',
where the transfer parameters $\tilde{\alpha}$ and α' are defined as follows:

$$
\tilde{\alpha}^2 = \frac{(j_{BC}'^{(0)})^2 + (j_{ABC}'^{(0)})^2}{j_{BC}^2 + j_{ABC}^2},
\tag{3.394}
$$

$$
\alpha'^2 = \frac{(\alpha^{(0)}\alpha_A^{(0)})^2}{\tilde{\alpha}^2} = \frac{j_{BC}^2 + j_{ABC}^2}{(j_{BC}'^{(0)})^2 + (j_{ABC}'^{(0)})^2}\left(\frac{j_{ABC}'^{(0)}}{j_{ABC}}\right)^2.
\tag{3.395}
$$

Theorem 5 and Lemma 4 guarantee that these two steps are possible if the inequalities $0 \leq \tilde{\alpha} \leq 1$ and $0 \leq \alpha' \leq 1$ hold. Indeed, the inequalities (3.393) and (3.394) guarantee the inequalities $0 \leq (\alpha^{(0)}\alpha_A^{(0)})^2 \leq \tilde{\alpha}^2 \leq 1$. The equation (3.395) and the inequalities $(\alpha^{(0)}\alpha_A^{(0)})^2 \leq \tilde{\alpha}^2 \leq 1$ also guarantee the inequalities $0 \leq \alpha' \leq 1$. Therefore, the two steps indeed reproduce the transformation from the state $|\psi\rangle$ to the state $\left|\psi'^{(0)}\right\rangle$ because of (3.380)–(3.382), (3.394), (3.395) and the following equations:

$$
\tilde{\alpha}^2\alpha'^2 = (\alpha^{(0)}\alpha_A^{(0)})^2 = \left(\frac{j_{ABC}'^{(0)}}{j_{ABC}}\right)^2,
\tag{3.396}
$$

$$
\tilde{\alpha}^2 j_{BC}^2 + \tilde{\alpha}^2(1 - \alpha'^2)j_{ABC}^2 = \tilde{\alpha}^2(j_{BC}^2 + j_{ABC}^2) - \left(\frac{j_{ABC}'^{(0)}}{j_{ABC}}\right)^2 j_{ABC}^2 = (j_{BC}'^{(0)})^2.
\tag{3.397}
$$

We have proven that the two steps reproduce the change of the entanglement parameters as well. To summarize the above, we reproduce Route 0 of Fig. 3.16 by the two steps. We have thereby proven the statement S' in the case of $j_{AB} = 0$. In the same manner, we can prove the statement S' in the case of $j_{AC} = 0$.

The rest is the case of $j_{BC} = 0 \wedge j_{ABC}' \neq 0$. In this case, we can assume that $j_{AB} \neq 0$ and $j_{AC} \neq 0$; otherwise we can use the proofs above. The proof of the statement S' in this case is provided for the following two cases:

Case 1 $k \neq 0$;

Case 2 $k = 0$.

In the Case 1, the equations (A.44), $j_{ABC} \neq 0$ and $j_{BC} = 0$ give that the expression $j_{BC}^{(i)} \neq 0$ holds for all i. Thus, the following equation holds:

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ J'_{ABC} \\ J'_5 \end{pmatrix} = \zeta^{A(i)}\zeta^{B(i)}\zeta^{C(i)} \begin{pmatrix} \zeta_A^{(i)}\zeta_B^{(i)} & & & & \\ & \zeta_A^{(i)}\zeta_C^{(i)} & & & \\ & & \zeta_B^{(i)}\zeta_C^{(i)} & & \\ & & & \zeta_A^{(i)}\zeta_B^{(i)}\zeta_C^{(i)} & \\ & & & & \zeta_A^{(i)}\zeta_B^{(i)}\zeta_C^{(i)} \end{pmatrix} \begin{pmatrix} (\alpha^{(i)})^2 K_{AB} \\ (\alpha^{(i)})^2 K_{AC} \\ K_{BC}^{(i)} \\ (\alpha^{(i)})^2 j_{ABC}^2 \\ J_5^{(i)} \end{pmatrix},
$$

(3.398)

where $(\zeta_A^{(i)}, \zeta^{A(i)})$, $(\zeta_B^{(i)}, \zeta^{B(i)})$ and $(\zeta_C^{(i)}, \zeta^{C(i)})$ are the sets of main and sub parameters of the DMTs $T_A^{(i)}$, $T_B^{(i)}$ and $T_C^{(i)}$, respectively.

Here, the product $(\alpha^{(i)})^2 \zeta_A^{(i)}\zeta_B^{(i)}\zeta_C^{(i)} \zeta^{A(i)}\zeta^{B(i)}\zeta^{C(i)}$ must be independent of i because $j_{ABC}'^2/j_{ABC}^2 = (\alpha^{(i)})^2 \zeta_A^{(i)}\zeta_B^{(i)}\zeta_C^{(i)} \zeta^{A(i)}\zeta^{B(i)}\zeta^{C(i)}$. Similarly, the product $(\alpha^{(i)})^2 \zeta_A^{(i)}\zeta_C^{(i)} \zeta^{A(i)}\zeta^{B(i)}\zeta^{C(i)}$ must be independent of i because $K'_{AC}/K_{AC} = (\alpha^{(i)})^2 \zeta_A^{(i)}\zeta_C^{(i)} \zeta^{A(i)}\zeta^{B(i)}\zeta^{C(i)}$. Hence, the main parameter $\zeta_B^{(i)}$ must be independent of i. In the same manner, we can show that the main parameter $\zeta_C^{(i)}$ must be also independent of i. Note that the DMT $T_A^{(i)}$ is a DMT between EP-definite states and that the step 2-1 of Lemma 6, (A.52) and (A.54) guarantees that the final state of a DMT between EP-definite states is $\tilde{\zeta}$-definite. Thus, the final states of $T_A^{(i)}$ are $\tilde{\zeta}$-definite. Thus, the sub parameter $\zeta^{C(i)}$ is equal to the ζ-specifying $\tilde{\zeta}$ of the final states of $T_B^{(i)}$. Hence, the sub parameter $\zeta^{C(i)}$ is a monotonously increasing function of the main parameter $\zeta_C^{(i)}$, because the ζ-specifying parameter of the final states of the DMT $T_B^{(i)}$ is a monotonously increasing function of the main parameter $\zeta_C^{(i)}$. Thus, $\zeta^{C(i)}\zeta_C^{(i)}$ is a monotonously increasing function of the main parameter $\zeta_C^{(i)}$. This means that if the main parameter $\zeta_C^{(i)}$ is specified, $\zeta^{C(i)}\zeta_C^{(i)}$ is also determined unieq) uly. Thus, $\zeta^{C(i)}\zeta_C^{(i)}$ is independent of i because the main parameter $\zeta_C^{(i)}$ is independent of i. Hence, the entanglement parameters of the initial state of the DMT $T_C^{(i)}$ is independent of i, because $\zeta_C^{(i)}$, $\zeta^{C(i)}\zeta_C^{(i)}$ and $|\psi''\rangle$ are independent of i. In the same manner, the entanglement parameters of the initial state of the DMT $T_B^{(i)}$ is independent of i. Let us refer to the initial state of $T_B^{(i)}$ as $|\psi''\rangle$ (Fig. 3.17).

Therefore, we only have to prove the following statement S_2: "A deterministic LOCC transformation which consists of the measurement $M_{(i)}$ and the DMT $T_A^{(i)}$ can be reproduced by an A-DMT whose DM is a two-choice measurement." The reason why we have to prove the statement S_2 is that a C-LOCC transformation consists of DMTs whose DMs are two-choice measurements. We prove the statement S_2 by showing that an A-DMT whose DM is a two-choice measurement can realize the change of the entanglement parameters which is caused by performing the measurement $M_{(i)}$ and the DMT $T_A^{(i)}$ successively. Let the notation $|\psi''\rangle$ stand for the state which is transformed from the state $|\psi^{(i)}\rangle$ by the DMT $T_A^{(i)}$; we just proved that $|\psi''\rangle$ must be independent of i (Fig. 3.17). We denote the set of entanglement parameters of the state $|\psi''\rangle$ in Fig. 3.17 by $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5, Q''_e)$. We prove the statement S_2 by showing that
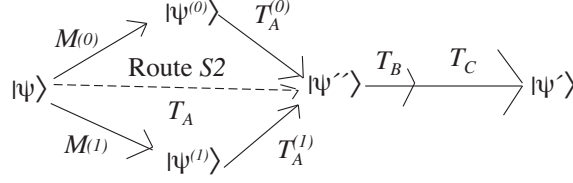
Figure 3.17: The concept of the statement S_2. The Route S_2 is realized by an A-DMT whose DM is a two-choice measurement.

$(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5, Q''_e)$ satisfy Conditions 1 and 2 of Theorem 1.

Let us prove that the sets of J-parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5, Q''_e)$ satisfy Condition 1 of Theorem 1 by examining the change of the J-parameters j_{AB}, j_{AC}, j_{BC}, j_{ABC} and J_5. First, we examine the change of the J-parameters j_{AB}, j_{AC} and j_{ABC}. From (A.40)–(A.42), we have the following three equations:

$$j''^2_{AB} = (\alpha_A^{(i)} \alpha^{(i)})^2 j_{AB}^2, \tag{3.399}$$

$$j''^2_{AC} = (\alpha_A^{(i)} \alpha^{(i)})^2 j_{AC}^2, \tag{3.400}$$

$$j''^2_{ABC} = (\alpha_A^{(i)} \alpha^{(i)})^2 j_{ABC}^2, \tag{3.401}$$

where $\alpha_A^{(i)}$ is the transfer parameter of the DMT $T_A^{(i)}$.

Second, we examine the change of the parameter J_5. We have already shown that $J'_5 = 0$. Thus, the equation $J''_5 = 0$ holds, because of $0 = J'_5 = \zeta_B^{(i)} \zeta_C^{(i)} \zeta^{B(i)} \zeta^{C(i)} J''_5$. Because the state $|\psi\rangle$ is EP-indefinite, the equation $J_5 = 0$ holds. Hence,

$$J''_5 = (\alpha_A^{(i)} \alpha^{(i)})^2 J_5. \tag{3.402}$$

Third, we examine the change of the parameter j_{BC} and prove that the sets of the J-parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5)$ satisfy Condition 1 of Theorem 1. Because of Lemma 5, at least one of $K_{BC}^{(0)}$ and $K_{BC}^{(1)}$ is less than or equal to K_{BC}. Because both of $T_A^{(i)}$ are A-DMTs, the inequalities $K''_{BC} \leq K_{BC}^{(i)}$ hold. Thus, we obtain $K''_{BC} \leq K_{BC}$. Because of Lemma 2, at least one of $j_{BC}^{(i)}$ is more than or equal to j_{BC}. Because both of $T_A^{(i)}$ are A-DMTs, the inequalities $j_{BC}^{(i)} \leq j''_{BC}$ hold. Thus, we obtain $j_{BC} \leq j''_{BC}$. Because of $K''_{BC} \leq K_{BC}$ and $j_{BC} \leq j''_{BC}$, we can find a parameter $0 \leq \beta_A \leq 1$ which satisfies

$$j''^2_{BC} = j_{BC}^2 + \beta_A (1 - (\alpha^{(i)} \alpha_A^{(i)})^2) j_{ABC}^2. \tag{3.403}$$

Note that (3.399)–(3.403) mean that the sets of the J-parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5)$ satisfy Condition 1 of Theorem 1.

Next, let us prove that the sets of the J-parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5, Q''_e)$ satisfy Condition 2 of Theorem 1. First, we prove this in the case where the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite by examining the change of the entanglement charge Q''_e. In other words, we prove that the entanglement charge Q''_e is zero if and only if β_A is zero or one. Because of (A.19) and (A.30), the entanglement charge Q''_e is zero if

and only if at least one of Δ''_J and $\sin\varphi''_5$ is zero. First we prove that $\sin\varphi''_5 > 0$ if and only if $\beta_A > 0$. Because of $J''_5 = 0$, (3.399) and (3.400), we obtain

$$j''_{BC}\cos\varphi''_5 = \frac{J''_5}{2j''_{AB}j''_{AC}} = 0. \tag{3.404}$$

From (3.403), (3.404) and $j_{BC} = 0$, we obtain

$$j''^2_{BC}\sin^2\varphi''_5 = j''^2_{BC} - j''^2_{BC}\cos^2\varphi''_5 = \beta_A(1 - (\alpha^{(i)}\alpha^{(i)}_A)^2)j^2_{ABC}. \tag{3.405}$$

Thus, $j''_{BC}\sin\varphi''_5 > 0$ if and only if $\beta_A > 0$. Because the state $|\psi''\rangle$ is EP definite, $j''_{BC} > 0$ holds, and thus $\sin\varphi''_5 > 0$ if and only if $\beta_A > 0$. Next we prove that $\Delta''_J > 0$ if and only if $1 > \beta$. From (3.399)–(3.403), we obtain

$$
\begin{aligned}
\Delta''_{\text{norm}} &\equiv \frac{\Delta''_J}{(\alpha^{(i)}\alpha^{(i)}_A)^2} = K_5^2 - 4K_{AB}K_{AC}K''_{BC} \\
&= K_5^2 - 4K_{AB}K_{AC}(j^2_{BC} + \beta_A(1 - (\alpha^{(i)}\alpha^{(i)}_A)^2)j^2_{ABC} + (\alpha^{(i)}\alpha^{(i)}_A)^2 j^2_{ABC}) \\
&= \Delta_J + 4K_{AB}K_{AC}j^2_{ABC}(1 - \beta_A)(1 - (\alpha^{(i)}\alpha^{(i)}_A)^2) \\
&= 4K_{AB}K_{AC}j^2_{ABC}(1 - \beta_A)(1 - (\alpha^{(i)}\alpha^{(i)}_A)^2).
\end{aligned}
\tag{3.406}
$$

Thus, $\Delta''_J > 0$ if and only if $1 > \beta$. Hence, the entanglement charge Q''_e is zero if and only if the dissipation parameter β_A is zero or one. Thus, the entanglement charge Q''_e and the dissipation parameter β_A satisfy Condition 2 of Theorem 1.

Now, we have proven that if the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, the sets of the J-parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ and $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5, Q''_e)$ satisfy Conditions 1 and 2 of Theorem 1. Thus, we can reproduce the transformation from the state $|\psi\rangle$ to the state $|\psi''\rangle$ by an A-DMT whose transfer parameter and dissipative parameter are $\alpha^{(i)}\alpha^{(i)}_A$ and β_A, respectively. Hence, we have completed the proof of the statement S_2 in the case that the $|\psi\rangle$ is $\tilde{\zeta}$-indefinite.

Next, we show the statement S_2 in the case that the state $|\psi\rangle$ is $\tilde{\zeta}$-definite. In this case, no A-DMT transforms the EP-indefinite state $|\psi\rangle$ into the EP-definite state $|\psi'\rangle$. Thus, we only have to show that if the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the deterministic LOCC transformation which consists of the measurement $M_{(i)}$ and the DMT $T_A^{(i)}$ from the EP-indefinite state $|\psi\rangle$ to the EP-definite state $|\psi''\rangle$ is impossible. In order to prove this, we show that if the deterministic LOCC transformation is executable, the state $|\psi^{(i)}\rangle$ is $\tilde{\zeta}$-definite.

We prove that the state $|\psi^{(i)}\rangle$ is $\tilde{\zeta}$-definite by reduction to absurdity. Let us assume that the state $|\psi^{(i)}\rangle$ were $\tilde{\zeta}$-indefinite. If $|\psi^{(i)}\rangle$ were $\tilde{\zeta}$-indefinite, both of $\sin\varphi^{(i)}_5$ and $\Delta^{(i)}_J$ would be zero. Because of (3.404) and because the transformation $T_A^{(i)}$ is a DMT, the equation $j^{(i)}_{BC}\cos\varphi^{(i)}_5 = j''_{BC}\cos\varphi''_5 = 0$ would hold. Because of $\sin\varphi^{(i)}_5 = 0$ and $j^{(i)}_{BC}\cos\varphi^{(i)}_5 = 0$, the equation $j^{(i)}_{BC} = 0$ would hold. This contradicts the fact that the state $|\psi^{(i)}\rangle$ is EP-definite. Note that the state $|\psi^{(i)}\rangle$ is EP-definite because of the assumption $k \neq 0$. Hence, the state $|\psi^{(i)}\rangle$ is $\tilde{\zeta}$-definite.

Next, we will derive the equations

$$1 - \tilde{\beta}''_A = \frac{\Delta''_{\text{norm}}}{\Delta''_{\text{norm}} + 4K_{AB}K_{AC}j''^2_{BC}\sin^2\varphi''_5}$$

$$= \frac{\Delta^{(i)}_{\text{norm}}}{\Delta^{(i)}_{\text{norm}} + 4K_{AB}K_{AC}(j^{(i)}_{BC})^2\sin^2\varphi^{(i)}_5} = 1 - \tilde{\beta}^{(i)}_A \qquad (3.407)$$

and

$$\frac{Q''_{\text{e}}\sqrt{\Delta_J}}{\sqrt{\Delta''_{\text{norm}}}} = 0. \qquad (3.408)$$

for $Q''_{\text{e}} \neq 0$. We will use them to prove thatthe state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the deterministic LOCC transformation from the state $|\psi\rangle$ to the state $|\psi''\rangle$ is impossible. Now we assume that $Q''_{\text{e}} \neq 0$ until the end of derivation of (3.408). Let us derive (3.407) and (3.408). Because the state $|\psi^{(i)}\rangle$ is $\tilde{\zeta}$-definite, and because the DMT $T^{(i)}_A$ is an A-DMT, we obtain

$$1 - \tilde{\beta}''_A = \frac{\Delta''_J}{\Delta''_J + 4K''_{AB}K''_{AC}j''^2_{BC}\sin^2\varphi''_5}$$

$$= \frac{\Delta^{(i)}_J + 4K^{(i)}_{AB}K^{(i)}_{AC}(j^{(i)}_{ABC})^2(1-\beta^{(i)}_A)(1-(\alpha^{(i)})^2)}{\Delta^{(i)}_J + 4K^{(i)}_{AB}K^{(i)}_{AC}(j^{(i)}_{ABC})^2(1-(\alpha^{(i)})^2) + 4K^{(i)}_{AB}K^{(i)}_{AC}(j^{(i)}_{BC})^2\sin^2\varphi^{(i)}_5}$$

$$= \frac{\Delta^{(i)}_J \frac{\Delta^{(i)}_J + 4K^{(i)}_{AB}K^{(i)}_{AC}(j^{(i)}_{ABC})^2(1-(\alpha^{(i)})^2) + 4K^{(i)}_{AB}K^{(i)}_{AC}(j^{(i)}_{BC})^2\sin^2\varphi^{(i)}_5}{\Delta^{(i)}_J + 4K^{(i)}_{AB}K^{(i)}_{AC}(j^{(i)}_{BC})^2\sin^2\varphi^{(i)}_5}}{\Delta^{(i)}_J + 4K^{(i)}_{AB}K^{(i)}_{AC}(j^{(i)}_{ABC})^2(1-(\alpha^{(i)})^2) + 4K^{(i)}_{AB}K^{(i)}_{AC}(j^{(i)}_{BC})^2\sin^2\varphi^{(i)}_5}$$

$$= \frac{\Delta^{(i)}_J}{\Delta^{(i)}_J + 4K^{(i)}_{AB}K^{(i)}_{AC}(j^{(i)}_{BC})^2\sin^2\varphi^{(i)}_5} = 1 - \tilde{\beta}^{(i)}_A. \qquad (3.409)$$

Now, we define

$$\Delta''_{\text{norm}} \equiv \frac{\Delta''_J}{(\alpha^{(i)}\alpha^{(i)}_A)^4} = K^2_5 - 4K_{AB}K_{AC}K''_{BC}, \qquad (3.410)$$

$$\Delta^{(i)}_{\text{norm}} \equiv \frac{\Delta^{(i)}_J}{(\alpha^{(i)})^4} = K^2_5 - 4K_{AB}K_{AC}K^{(i)}_{BC}. \qquad (3.411)$$

The equation (3.409) can be transformed to (3.407) with

$$1 - \tilde{\beta}''_A = \frac{\Delta''_{\text{norm}}}{\Delta''_{\text{norm}} + 4K_{AB}K_{AC}j''^2_{BC}\sin^2\varphi''_5}$$

$$= \frac{\Delta^{(i)}_{\text{norm}}}{\Delta^{(i)}_{\text{norm}} + 4K_{AB}K_{AC}(j^{(i)}_{BC})^2\sin^2\varphi^{(i)}_5} = 1 - \tilde{\beta}^{(i)}_A, \qquad (3.412)$$

$$\frac{(j^{(i)}_{BC})^2\sin^2\varphi^{(i)}_5}{j''^2_{BC}\sin^2\varphi''_5} = \frac{\Delta^{(i)}_{\text{norm}}}{\Delta''_{\text{norm}}}. \qquad (3.413)$$

Because a DMT between $\tilde{\zeta}$-definite states conserves the entanglement charge, the expression $Q^{(0)}_{\text{e}} = Q^{(1)}_{\text{e}} = Q''_{\text{e}} \neq 0$ holds. Because we have assumed $\sin\theta^{(0)} \geq 0$, the equation

$Q_{\mathrm{e}}^{(0)} = Q_{\mathrm{e}}^{(1)} = Q_{\mathrm{e}}'' \neq 0$ is equivalent to the following equations:

$$(\lambda_0^{(0)})^2 = \frac{K_5 + Q_{\mathrm{e}}''\sqrt{\Delta_{\mathrm{norm}}^{(0)}}}{2K_{BC}^{(0)}}(\alpha^{(0)})^2 = \frac{p}{b}(\alpha^{(0)})^2\frac{K_5 + t\sqrt{\Delta_J}}{2K_{BC}}, \tag{3.414}$$

$$(\lambda_0^{(1)})^2 = \frac{K_5 + Q_{\mathrm{e}}''s\sqrt{\Delta_{\mathrm{norm}}^{(1)}}}{2K_{BC}^{(1)}}(\alpha^{(1)})^2 = \frac{1-p}{1-b}(\alpha^{(1)})^2\frac{K_5 + t\sqrt{\Delta_J}}{2K_{BC}}, \tag{3.415}$$

where s and t are determined as follows: if $\sin\varphi^{(1)} \geq 0$, s is $+1$; if $\sin\varphi^{(1)} < 0$, s is -1; if $Q_{\mathrm{e}} = 1$, t is $+1$; if $Q_{\mathrm{e}} = -1$, t is -1; if $Q_{\mathrm{e}} = 0$, t can be either $+1$ or -1. Because of (3.414), (3.415) and $K_{BC}^{(i)}/K_{BC} = (K_5^2 - \Delta_{\mathrm{norm}}^{(i)})/(K_5^2 - \Delta_J)$, we obtain the expression of the probability p in terms of Δ_J and $\Delta_{\mathrm{norm}}^{(i)}$:

$$p = \frac{Q_{\mathrm{e}}''t\sqrt{\Delta_J} - s\sqrt{\Delta_{\mathrm{norm}}^{(1)}}}{\sqrt{\Delta_{\mathrm{norm}}^{(0)}} - s\sqrt{\Delta_{\mathrm{norm}}^{(1)}}}. \tag{3.416}$$

Because of (A.49) and $j_{BC} = 0$, we have

$$pj_{BC}^{(0)}\sin\varphi_5^{(0)} + s(1-p)j_{BC}^{(1)}\sin\varphi_5^{(1)} = 0. \tag{3.417}$$

From (3.417), we obtain

$$p = \frac{-sj_{BC}^{(1)}\sin\varphi_5^{(1)}}{j_{BC}^{(0)}\sin\varphi_5^{(0)} - sj_{BC}^{(1)}\sin\varphi_5^{(1)}}. \tag{3.418}$$

From (3.413), (3.416), (3.418) and $j_{BC} = 0$, we arrive at (3.408)

Next, let us prove that if the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the deterministic LOCC transformation from the state $|\psi\rangle$ to the state $|\psi''\rangle$ is impossible by reduction to absurdity. Let us assume that the state $|\psi\rangle$ were $\tilde{\zeta}$-definite and that the deterministic LOCC transformation from $|\psi\rangle$ to $|\psi''\rangle$ were executable. Because $|\psi\rangle$ is EP-indefinite, $\sin\varphi$ is zero. Because the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, at least one of $\sin\varphi$ and Δ_J would not be zero. Thus, $\Delta_J > 0$ would hold, but the equation (3.408) contradicts $Q_{\mathrm{e}}'' \neq 0$ and $\Delta_J > 0$. Thus, if the equations (3.408) and $Q_{\mathrm{e}}'' \neq 0$ hold, the deterministic LOCC transformation from $|\psi\rangle$ to $|\psi''\rangle$ is impossible. We have already proven that if the entanglement charge Q_{e}'' is not zero, then (3.408) holds. Thus, we only have to prove $Q_{\mathrm{e}}'' \neq 0$. Because $\Delta_{\mathrm{norm}}^{(i)} \geq \Delta_J$ holds for at least one of i and because of (A.54) and $\Delta_J > 0$, the inequality $\Delta_J'' > 0$ holds. Because $|\psi''\rangle$ is EP definite, the inequality $j_{BC}'' > 0$ holds. Because of (3.404) and $j_{BC} = 0$, the inequality $j_{BC}''\sin\varphi_5'' > 0$ holds, and thus $\sin\varphi_5'' > 0$. Because of $\sin\varphi_5'' > 0$ and (A.30), the inequality $\sin\varphi'' > 0$ also holds. Thus, $\sin\varphi'' > 0$ and $\Delta_J'' > 0$ hold, and thus $Q_{\mathrm{e}}'' \neq 0$ holds. Thus, if the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the deterministic LOCC transformation which consists of the measurement $M_{(i)}$ and the DMT $T_A^{(i)}$ from the state $|\psi\rangle$ to the state $|\psi''\rangle$ is impossible. Now, we complete the proof of the statement S_2. Thus, we complete the proof of the present theorem in the Case 1.

In the Case 2, $k = 0$, the first measurement $M_{(i)}$ only makes all entanglement parameters multiplied by $\alpha^{(i)}(= \sqrt{ab}/p_{(0)}$ or $= \sqrt{(1-a)(1-b)}/(1-p_{(0)}))$. Note that $j_{BC} = 0$

holds now. We can assume that $\alpha^{(0)}$ is less than $\alpha^{(1)}$ without loss of generality. Then, (A.50) is followed by the inequalities $0 \leq \alpha^{(0)} \leq 1$. Theorem 5 in the section 6.3.1 tells us that we can take a DMT $T'_{\alpha^{(0)}}$ which makes all entanglement parameters multiplied by $\alpha^{(0)}$. The DMT $T'_{\alpha^{(0)}}$ realizes the transformation from the state $|\psi\rangle$ to the state $|\psi^{(0)}\rangle$. The assumption for $N = k$ guarantees that a C-LOCC transformation can realize a transformation from the state $|\psi^{(0)}\rangle$ to the state $|\psi'\rangle$. Hence, the statement S' also holds in the Case 2.$\square$

Note that Main Theorem 2 also has been proven, because of the statement S'' and because an arbitrary C-LOCC transformation can be reproduced by three DMTs.

3.5.4 Case $\mathfrak{D}$

Lastly, we carry out Steps 1–3 for the Case $\mathfrak{D}$, where $j_{ABC} = 0$ for the initial state. We present Steps 1–3 here again.

Step 1 We give a necessary and sufficient condition of the possibility of a two-choice DMT which transforms an arbitrary state $|\psi\rangle$ to another arbitrary state $|\psi'\rangle$.

Step 2 We give a necessary and sufficient condition of the possibility of a C-LOCC transformation from an arbitrary state $|\psi\rangle$ to another arbitrary state $|\psi'\rangle$. We also prove that an arbitrary C-LOCC transformation can be reproduced by performing an A-DMT, a B-DMT and a C-DMT, successively.

Step 3 We show that an executable deterministic LOCC transformation from an arbitrary state $|\psi\rangle$ to an arbitrary state $|\psi'\rangle$ can be reproduced by a C-LOCC transformation. Conversely, a C-LOCC transformation can be reproduced by a deterministic LOCC transformation, because a C-LOCC transformation is also a deterministic LOCC transformation. Then, we find that the condition given in Step 2 is also a necessary and sufficient condition of the possibility of a deterministic LOCC transformation and that an arbitrary deterministic LOCC transformation can be reproduced by performing an A-DMT, a B-DMT and a C-DMT, successively.

We carry out these Step 1–3 in the following Theorem 8.

Theorem 8 *Let the notations $|\psi\rangle$ and $|\psi'\rangle$ stand for three-qubit pure states. We refer to the sets of the K-parameters of the states $|\psi\rangle$ and $|\psi'\rangle$ as $(K_{AB}, K_{AC}, K_{BC}, j_{ABC}, J_5, Q_e)$ and $(K'_{AB}, K'_{AC}, K'_{BC}, j'_{ABC}, J'_5, Q'_e)$, respectively. We assume that $j_{ABC} = 0$. Then, a necessary and sufficient condition of the possibility of a deterministic LOCC transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is that the following two conditions are satisfied: Condition 1: There are real numbers $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$, $0 \leq \zeta_C \leq 1$ and $\zeta_{lower} \leq \zeta \leq 1$ which satisfy the following equation:*

$$
\begin{pmatrix} K'_{AB} \\ K'_{AC} \\ K'_{BC} \\ j'^2_{ABC} \\ J'_5 \end{pmatrix} = \zeta \begin{pmatrix} \zeta_A\zeta_B & & & & \\ & \zeta_A\zeta_C & & & \\ & & \zeta_B\zeta_C & & \\ & & & \zeta_A\zeta_B\zeta_C & \\ & & & & \zeta_A\zeta_B\zeta_C \end{pmatrix} \begin{pmatrix} K_{AB} \\ K_{AC} \\ K_{BC} \\ j^2_{ABC} \\ J_5 \end{pmatrix}, \qquad (3.419)
$$

where

$$\zeta_{lower} = \frac{J_{ap}}{(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_A j^2_{ABC})}, \qquad (3.420)$$

and we refer to ζ, ζ_A, ζ_B and ζ_C as the sub parameter, the main parameter of A, B and C, respectively.

Condition 2: If the state $|\psi'\rangle$ is EP definite, we check whether the state $|\psi\rangle$ is $\tilde{\zeta}$-definite or not. When the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, the condition is

$$Q_e = Q'_e \ \ and \ \zeta = \tilde{\zeta}, \qquad (3.421)$$

where

$$\tilde{\zeta} \equiv \frac{K_{ap}(4J_{ap} - J_5^2) + \Delta_J J_{ap}}{K_{ap}(4J_{ap} - J_5^2) + \Delta_J(K_{AB} - \zeta_C j^2_{ABC})(K_{AC} - \zeta_B j^2_{ABC})(K_{BC} - \zeta_C j^2_{ABC})}. \qquad (3.422)$$

When the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, the condition is

$$|Q'_e| = sgn[(1 - \zeta)(\zeta - \zeta_{lower})], \qquad (3.423)$$

or in the other words,

$$Q'_e \begin{cases} = 0 & (\zeta = 1 \ or \ \zeta = \zeta_{lower}), \\ \neq 0 & (otherwise). \end{cases} \qquad (3.424)$$

Proof: First, we simplify (3.419). Let us show that we can leave j_{ABC}, J_5 and Q_e out of the discussion hereafter. First, $j'_{ABC} = 0$ follows from $j_{ABC} = 0$, because an arbitrary measurement makes the J-parameter j_{ABC} only multiplied by a constant. Next, because of (A.2)–(A.6) and the equation $j_{ABC} = 0$, the equation $J_5 = 2j_{AB}j_{AC}j_{BC}$ holds. Thus, in order to examine the change of the J-parameter J_5, it suffices to examine the change of the J-parameters j_{AB}, j_{AC} and j_{BC}. Because of $j_{ABC} = 0 \Leftrightarrow \lambda_0 = 0 \vee \lambda_4 = 0$ and because if there is a zero in $\{\lambda_k | k = 0, ..., 4\}$ then $\sin\varphi = 0$, the entanglement charge Q_e is equal to zero. In the same manner, the entanglement charge Q'_e is also zero because of $j'_{ABC} = 0$. Thus, $Q_e = Q'_e = 0$ holds. This equation satisfies Condition 2 of Theorem 8. Let us show this. The state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, because $j_{ABC} = 0$:

$$\begin{aligned} \Delta_J = K_5^2 - 4K_{ap} = J_5^2 - 4J_{ap} &= 4j^2_{AB}j^2_{AC}j^2_{BC}\sin^2\varphi_5 \\ &= 4j^2_{AB}j^2_{AC}\lambda_1^2\lambda_4^2\sin^2\varphi = 0, \end{aligned} \qquad (3.425)$$

where we use $\sin\varphi = 0$. Note that $\sin\varphi = 0$ holds when there is a zero in $\{\lambda_i | i = 0, ..., 4\}$ and that $j_{ABC} = \lambda_0\lambda_4 = 0$. Because the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite and because of $|Q'_e| = sgn[(1 - \zeta)(\zeta - \zeta_{lower})]$ and $((\zeta_{lower} \le \zeta \le 1) \wedge (j_{ABC} = 0)) \Rightarrow \zeta = 1$, the equation $Q'_e = 0$ satisfies Condition 2. Hence, in order to prove the present theorem, it suffices to show that a necessary and sufficient condition is that there are real numbers α_A, α_B and α_C which are from zero to one and which satisfy the following equation:

$$\begin{pmatrix} j'^2_{AB} \\ j'^2_{AC} \\ j'^2_{BC} \end{pmatrix} = \begin{pmatrix} \alpha_A^2\alpha_B^2 & & \\ & \alpha_A^2\alpha_C^2 & \\ & & \alpha_B^2\alpha_C^2 \end{pmatrix} \begin{pmatrix} j^2_{AB} \\ j^2_{AC} \\ j^2_{BC} \end{pmatrix}. \qquad (3.426)$$

Note that (3.426) and $J_5 = 2j_{AB}j_{AC}j_{BC}$ give $J_5' = \alpha_A^2\alpha_B^2\alpha_C^2 J_5$, and that $j_{ABC} = 0 \Rightarrow (K_{AB} = j_{AB}^2) \wedge (K_{AC} = j_{AC}^2) \wedge (K_{BC} = j_{BC}^2)$.

Third, we perform Step 1 of Case $\mathfrak{D}$. In other words, we show that a necessary and sufficient condition of the possibility of a two-choice A-DMT from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is that there is a real number α_A which is from zero to one and which satisfies the following equation:

$$
\begin{pmatrix} j_{AB}'^2 \\ j_{AC}'^2 \\ j_{BC}'^2 \end{pmatrix} = \begin{pmatrix} \alpha_A^2 & & \\ & \alpha_A^2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} j_{AB}^2 \\ j_{AC}^2 \\ j_{BC}^2 \end{pmatrix}. \tag{3.427}
$$

Substitution of the equations $\alpha = \alpha_A$ and $j_{ABC} = j_{ABC}' = 0$ in Lemma 4 gives that (3.427) is clearly a sufficient condition. Substitution of $j_{ABC} = 0$ in (A.51) gives that $j_{BC} \leq j_{BC}' \leq j_{BC}$; we thereby have $j_{BC} = j_{BC}'$ for an arbitrary A-DM. An A-DM makes the J-parameters j_{AB} and j_{AC} multiplied by the transfer parameter $\alpha^{(0)} = \alpha^{(1)} =: \alpha_A$. Hence, (3.427) is also a necessary condition. We thereby obtained a necessary and sufficient condition of the possibility of an arbitrary two-choice DMT.

Fourth, we perform Step 2 of Case $\mathfrak{D}$. In other words, we prove that a necessary and sufficient condition of the possibility of a C-LOCC from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is that there are real numbers α_A, α_B and α_C which are from zero to one and which satisfy (3.426). We easily see that a transformation in the form of (3.426) can be reproduced by performing an A-dissipationless DMT, a B-dissipationless DMT and C-dissipationless DMT whose transfer parameters are α_A, α_B and α_C, respectively. Thus, the existence of the real numbers α_A, α_B and α_C which are from zero to one and which satisfy (3.426) is a sufficient condition of the possibility of a C-LOCC from the state $|\psi\rangle$ to the state $|\psi'\rangle$. In order to prove the necessity, we show that we can take the transfer parameters α_A, α_B and α_C for an arbitrary C-LOCC transformation. An arbitrary C-LOCC transformation T_{CL} consists of A-DMTs $\{T_{A1}, ..., T_{Ak}\}$, B-DMTs $\{T_{B1}, ..., T_{Bm}\}$ and C-DMTs $\{T_{C1}, ..., T_{Cn}\}$. We refer to the transfer parameters of the DMTs as $\{\alpha_{A1}, ..., \alpha_{Ak}\}$, $\{\alpha_{B1}, ..., \alpha_{Bm}\}$ and $\{\alpha_{C1}, ..., \alpha_{Cn}\}$, respectively. Then, the C-LOCC transformation can be reproduced by performing an A-dissipationless DMT, a B-dissipationless DMT and a C-dissipationless DMT, whose transfer parameters are $\prod_{i=1}^{k} \alpha_{Ai}$, $\prod_{i=1}^{m} \alpha_{Bi}$ and $\prod_{i=1}^{n} \alpha_{Ci}$, respectively; note that we do not need to consider the dissipation parameters because $j_{ABC} = 0$. We have thereby proven that the existence of the real numbers α_A, α_B and α_C which are from zero to one and which satisfy (3.426) is a necessary and sufficient condition for a C-LOCC transformation in the case $j_{ABC} = 0$.

Fifth, we define the notations used to carry out Step 3. We refer to the first measurement of the deterministic LOCC transformation as $\{M_{(i)}|i = 0, 1\}$. Let the notation $\{|\psi^{(i)}\rangle\}$ stand for the results of the measurement $\{M_{(i)}|i = 0, 1\}$. Let the notation $(j_{AB}^{(i)}, j_{AC}^{(i)}, j_{BC}^{(i)}, j_{ABC}^{(i)}, J_5^{(i)})$ stand for the set of the J-parameters of the state $|\psi^{(i)}\rangle$. We define the measurement parameters $a_{(i)}$, $b_{(i)}$, $k_{(i)}$, $\theta_{(i)}$ and $\alpha^{(i)}$ for $\{M_{(i)}|i = 0, 1\}$ as (A.32) and (A.33), and (A.45), respectively.

Finally, we perform Step 3 of Case $\mathfrak{D}$. In other words, we show the following statement S'': "An arbitrary deterministic LOCC transformation can be reproduced by a C-LOCC transformation for $j_{ABC} = 0$." By carrying out Step 3, we show that the condition which we have obtained in Step 2 is also a necessary and sufficient condition for a deterministic LOCC transformation.

Now, we can classify the sets of the initial and final states as follows:

Case 8-1 At least one of the J-parameters j_{AB}, j_{BC} and j_{AC} is zero.

Case 8-2 None of the J-parameters j_{AB}, j_{BC} and j_{AC} is zero, and at least one of the J-parameters j'_{AB}, j'_{BC} and j'_{AC} is zero.

Case 8-3 All of the J-parameters j_{AB}, j_{BC}, j_{AC}, j'_{AB}, j'_{BC} and j'_{AC} are nonzero.

Note that we already have $j_{ABC} = j'_{ABC}$ in the present Case $\mathfrak{D}$.

In Case 8-1 we derive the necessary and sufficient condition directly. In other cases, Case 8-2 and Case 8-3, we prove the statement S'' separately.

In Case 8-1, we first note that only the biseparable states are allowed as the initial states in this case. We can assume that the only nonzero J-parameter is j_{AC} without loss of generality. We proved at the end of Sec. 2 (Fig. 3.5(b)) that there is no state which has only two kinds of the bipartite entanglement. The set of full-separable states and biseparable states which have the same kinds of bipartite entanglements is a totally ordered set [14]. Thus, if we prove that we cannot transform a full-separable state or a biseparable state into other type states with LOCC transformations, then we can derive the necessary and sufficient condition, which reduces to the following: there is an executable deterministic LOCC transformation from $|\psi\rangle$ to $|\psi'\rangle$ if and only if $j_{AC} \geq j'_{AC}$.

Let us prove that we cannot transform a full-separable state or a biseparable state into other types of states with LOCC transformations. In order to prove this, it suffices to show that if $j_{AB} = 0$, the bipartite entanglement between the qubits A and B is always zero after an LOCC transformation. An LOCC transformation is a set of measurements. Thus it suffices to prove the following statement S_0: "A measurement transforms an arbitrary state whose j_{AB} and j_{ABC} are zero, only into a state whose j_{AB} and j_{ABC} are also zero." Then we have $j'_{AB} = j'_{ABC} = 0$ after all measurements. Let us show the above statement. We already proved above that j_{ABC} is zero after a measurement, we therefore prove the statement only for j_{AB}.

Let the notation $\{M'_{(i)}\}$ stand for an arbitrary measurement. We take a state $|\psi''\rangle$ as an arbitrary three-qubit pure state and take states $\{|\psi''^{(i)}\rangle\}$ as the results of the measurement $\{M'_{(i)}\}$, which is performed on the qubit A of the state $|\psi''\rangle$. The notations $(j''_{AB}, j''_{AC}, j''_{BC}, j''_{ABC}, J''_5)$ and $(j''^{(i)}_{AB}, j''^{(i)}_{AC}, j''^{(i)}_{BC}, j''^{(i)}_{ABC}, J''^{(i)}_5)$ stand for the sets of the J-parameters of the states $|\psi''\rangle$ and $|\psi''^{(i)}\rangle$, respectively, where we assume $j''_{ABC} = j''^{(i)}_{ABC} = 0$. For each measurement $\{M'_{(i)}\}$, we define the measurement parameters $a'_{(i)}$, $b'_{(i)}$, $k'_{(i)}$, $\theta'_{(i)}$ and $\alpha'^{(i)}$ as (A.32), (A.33), and (A.45), respectively. Then, the equations (A.40)–(A.44) give the equations $\alpha'^{(i)} j''_{AC} = j''^{(i)}_{AC}$, $\alpha'^{(i)} j''_{AB} = j''^{(i)}_{AB}$ and $j''^{(i)}_{BC} = j''_{BC} b'_{(i)} / p'_{(i)}$. Hence, if j''_{AB}, j''_{AC} or j''_{BC} is zero, then $j''^{(i)}_{AB}$, $j''^{(i)}_{AC}$ or $j''^{(i)}_{BC}$ also must be zero, respectively. This means the statement S_0 is true.

Thus if the initial state $|\psi\rangle$ is biseparable, the final state $|\psi'\rangle$ is also biseparable, because $|\psi'\rangle$ is the results of LOCC transformations from the state $|\psi\rangle$. Thus, in Case 8-1, the necessary and sufficient condition is $j_{AC} \geq j'_{AC}$. Note that this condition is equivalent to the equation (3.426), because $j_{AB} = j_{BC} = j'_{AC} = j'_{BC} = 0$ in Case 8-1.

In Case 8-2, where none of j_{AB}, j_{BC} and j_{AC} is zero and at least one of j'_{AB}, j'_{BC} and j'_{AC} is zero, the initial state is EP definite while the final state is EP indefinite.

Then Theorem 3 guarantees that the final EP-indefinite state $|\psi'\rangle$ is a full-separable state or a biseparable state. In the case where the state $|\psi'\rangle$ is full-separable, the transfer parameters α_A, α_B and α_C which are all zero satisfy (3.426). In the case where the state $|\psi'\rangle$ is a biseparable state, we can assume that $j'_{AB} \neq 0$ and $j'_{AC} = j'_{BC} = 0$ without loss of generality. Then, Theorem 4 and $j_{ABC} = 0$ give that $j'_{AB} \leq j_{AB}$. Hence, $\alpha_A = j'_{AB}/j_{AB}$, $\alpha_B = 1$ and $\alpha_C = 0$ are indeed from zero to one and satisfy (3.426). Thus, we have proven the statement S'' in the Case 8-2 too.

In Case 8-3, where all of the J-parameters j_{AB}, j_{BC}, j_{AC}, j'_{AB}, j'_{BC} and j'_{AC} are nonzero, the existence of the transfer parameters α_A, α_B and α_C which are from zero to one and which satisfy (3.426) is equivalent to the following statement SD: There are real numbers α_{AB}, α_{AC} and α_{BC} which are from zero to one and which satisfy the inequalities

$$\alpha_{AB} \geq \alpha_{AC}\alpha_{BC}, \ \ \alpha_{BC} \geq \alpha_{AB}\alpha_{AC}, \ \ \alpha_{AC} \geq \alpha_{BC}\alpha_{AB} \tag{3.428}$$

as well as the equations

$$\alpha_{AB} = j'_{AB}/j_{AB}, \ \ \alpha_{AC} = j'_{AC}/j_{AC}, \ \ \alpha_{BC} = j'_{BC}/j_{BC}. \tag{3.429}$$

In order to see this equivalence, it suffices to note that the transfer parameters α_A, α_B and α_C can be expressed as $\alpha_C^2 = \alpha_{AC}\alpha_{BC}/\alpha_{AB}, \alpha_A^2 = \alpha_{AB}\alpha_{AC}/\alpha_{BC}$ and $\alpha_B^2 = \alpha_{AB}\alpha_{BC}/\alpha_{AC}$.

We show the statement S'' by using the equivalence between the statement SD and the existence of the transfer parameters α_A, α_B and α_C. We show the statement S'' by mathematical induction with respect to the number of measurement times N. For $N = 1$, the statement S'' clearly holds. Let us prove the statement S'' for $N = k + 1$, assuming that the statement S'' holds whenever $1 \leq N \leq k$.

First, let us define parameters which are necessary for the proof. We refer to the first measurement of the deterministic LOCC transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ as $\{M_{(i)}|i = 0, 1\}$ and refer to the ith result of the measurement $\{M_{(i)}|i = 0, 1\}$ as $|\psi^{(i)}\rangle$. Because of the assumption for $N = k$, the transformation from the state $|\psi^{(i)}\rangle$ to the state $|\psi\rangle$ can be reproduced by a C-LOCC transformation. Thus we can define real parameters $\alpha_{AB}^{(i)}$, $\alpha_{AC}^{(i)}$ and $\alpha_{BC}^{(i)}$ as $\alpha_{AB}^{(i)} = j'_{AB}/j_{AB}^{(i)}, \alpha_{AC}^{(i)} = j'_{AC}/j_{AC}^{(i)}$ and $\alpha_{BC}^{(i)} = j'_{BC}/j_{BC}^{(i)}$, which are from zero to one and which satisfy the following equations:

$$\alpha_{AB}^{(i)} \geq \alpha_{AC}^{(i)}\alpha_{BC}^{(i)}, \tag{3.430}$$

$$\alpha_{BC}^{(i)} \geq \alpha_{AB}^{(i)}\alpha_{AC}^{(i)}, \tag{3.431}$$

$$\alpha_{AC}^{(i)} \geq \alpha_{BC}^{(i)}\alpha_{AB}^{(i)}. \tag{3.432}$$

We define a real parameter $\gamma^{(i)}$ as $\gamma^{(i)} = j_{BC}^{(i)}/j_{BC}$ and define $\alpha^{(i)}$ as the multiplication factor of the measurement $\{M_{(i)}|i = 0, 1\}$.

Next, we carry out the proof of the statement S'' for $N = k + 1$. In order to show this, it suffices to prove the following inequalities because of the statement SD:

$$\alpha_{AB} \geq \alpha_{AC}\alpha_{BC}, \ \ \alpha_{BC} \geq \alpha_{AB}\alpha_{AC}, \ \ \alpha_{AC} \geq \alpha_{BC}\alpha_{AB}, \tag{3.433}$$

$$0 \leq \alpha_{AB} \leq 1, \ \ 0 \leq \alpha_{AC} \leq 1, \ \ 0 \leq \alpha_{BC} \leq 1, \tag{3.434}$$

where $\alpha_{AB} = j'_{AB}/j_{AB}$, $\alpha_{AC} = j'_{AC}/j_{AC}$ and $\alpha_{BC} = j'_{BC}/j_{BC}$. Note that the following equations hold:

$$\alpha_{AB} = \alpha^{(i)}\alpha_{AB}^{(i)}, \ \ \alpha_{AC} = \alpha^{(i)}\alpha_{AC}^{(i)}, \ \ \alpha_{BC} = \gamma^{(i)}\alpha_{BC}^{(i)}. \tag{3.435}$$

Because of (3.435), the inequalities (3.433) is equivalent to the following inequalities:

$$\alpha^{(i)}\alpha_{AB}^{(i)} \geq \alpha^{(i)}\alpha_{AC}^{(i)}\gamma^{(i)}\alpha_{BC}^{(i)}, \tag{3.436}$$

$$\gamma^{(i)}\alpha_{BC}^{(i)} \geq (\alpha^{(i)})^2\alpha_{AB}^{(i)}\alpha_{AC}^{(i)}, \tag{3.437}$$

$$\alpha^{(i)}\alpha_{AC}^{(i)} \geq \alpha^{(i)}\alpha_{BC}^{(i)}\gamma^{(i)}\alpha_{AB}^{(i)}. \tag{3.438}$$

Let us first prove (3.434) and then (3.436)–(3.438). Because of (3.435), the quantities $\alpha^{(i)}\alpha_{AB}^{(i)}$, $\alpha^{(i)}\alpha_{AC}^{(i)}$ and $\gamma^{(i)}\alpha_{BC}^{(i)}$ are independent of i. All of $\alpha_{AB}^{(i)}$, $\alpha_{AC}^{(i)}$ and $\alpha_{BC}^{(i)}$ are from zero to one. The real numbers $\alpha^{(i)}$ and $\gamma^{(i)}$ are from zero to one for at least one of i, which can be seen by substituting $j_{ABC} = 0$ in (A.51) and the inequality (A.50). Thus the products $\alpha^{(i)}\alpha_{AB}^{(i)}$, $\alpha^{(i)}\alpha_{AC}^{(i)}$ and $\gamma^{(i)}\alpha_{BC}^{(i)}$ are from zero to one for both i. Note that $\alpha^{(i)}\alpha_{AB}^{(i)}$, $\alpha^{(i)}\alpha_{AC}^{(i)}$ and $\gamma^{(i)}\alpha_{BC}^{(i)}$ are independent of i. Now, we have proven (3.434).

Next, we prove (3.436)–(3.438). The products $\alpha^{(i)}\alpha_{AB}^{(i)}$, $\alpha^{(i)}\alpha_{AC}^{(i)}$ and $\gamma^{(i)}\alpha_{BC}^{(i)}$ are independent of i and the inequalities (3.436)–(3.438) are equivalent for $i = 0$ and $i = 1$. At least one of $\alpha^{(0)}$ and $\alpha^{(1)}$ and at least one of $\gamma^{(0)}$ and $\gamma^{(1)}$ are less than or equal to one. We can assume that the real number $\gamma^{(0)}$ is less than or equal to one without loss of generality. Then, (3.430) and (3.432) are followed by (3.436) and (3.438).

The rest is (3.437). If this were invalid, (3.431) would be followed by the inequality $\gamma^{(i)} < (\alpha^{(i)})^2$ for all i. The inequality $\gamma^{(0)} < (\alpha^{(0)})^2$ would mean that

$$\frac{b}{p_{(0)}} < \frac{ab - k^2}{p_{(0)}^2}, \tag{3.439}$$

and $\gamma^{(1)} < (\alpha^{(1)})^2$ would mean that

$$\frac{1-b}{1-p_{(0)}} < \frac{(1-a)(1-b) - k^2}{(1-p_{(0)})^2}. \tag{3.440}$$

The inequality (3.439) would give that $p_{(0)} < a$. Then, the inequality $1 - p_{(0)} > 1 - a$ would hold, but this contradicts (3.440). Hence, (3.437) holds so does the statement S''.□

Note that Main Theorem 2 also has been proven in Theorem 8, because of the statement S'' and because an arbitrary C-LOCC transformation can be reproduced by three DMTs.

Now, we have completed the proof of Main Theorems.

Finally, we prove Table 3.1. Main Theorem 2 gives the first row of Table 3.1. In the section 6.2, we have obtained the second row of Table 3.1. In Case 8-1 of the proof of Theorem 8, we have proven the third and fourth rows of Table 3.1. The fifth and the last rows have been proven in Ref. [24]. We thereby completed Table 3.1.

3.6 The judgement protocol

In the present section, we give the protocol of determining whether a deterministic LOCC transformation from an arbitrary state $|\psi\rangle$ to an another arbitrary state $|\psi'\rangle$ is executable or not.

1 Examine whether $j_{ABC} \neq 0$. If $j_{ABC} \neq 0$, proceed to 2-A. If $j_{ABC} = 0$, proceed to 2-B.

2-A Because $j_{ABC} \neq 0$, we have $K_{AB} = j_{AB}^2 + j_{ABC}^2 \neq 0$, $K_{AC} = j_{AC}^2 + j_{ABC}^2 \neq 0$ and $K_{BC} = j_{BC}^2 + j_{ABC}^2 \neq 0$. Thus the fractions K'_{AB}/K_{AB}, K'_{AC}/K_{AC}, K'_{BC}/K_{BC} and j'_{ABC}/j_{ABC} are definite. Examine whether $j'_{ABC}/j_{ABC} = 0$. If $j'_{ABC}/j_{ABC} \neq 0$, proceed to 3-A-A. If $j'_{ABC}/j_{ABC} = 0$, proceed to 3-A-B.

3-A-A Examine whether the expressions $K'_{AB}/K_{AB} \neq 0$, $K'_{AC}/K_{AC} \neq 0$ and $K'_{BC}/K_{BC} \neq 0$ hold. Unless all of these three expressions hold, the deterministic transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is impossible because of $j'_{ABC} \neq 0$ and (3.8)–(3.10). If all of the three expressions hold, we can define real parameters ζ, ζ_A, ζ_B and ζ_C as follows:

$$\zeta_A = \frac{j'^2_{ABC} K_{BC}}{j^2_{ABC} K'_{BC}}, \tag{3.441}$$

$$\zeta_B = \frac{j'^2_{ABC} K_{AC}}{j^2_{ABC} K'_{AC}}, \tag{3.442}$$

$$\zeta_C = \frac{j'^2_{ABC} K_{AB}}{j^2_{ABC} K'_{AB}}, \tag{3.443}$$

$$\zeta = \frac{j'^2_{ABC}}{j^2_{ABC} \zeta_A \zeta_B \zeta_C}. \tag{3.444}$$

Examine whether the state $|\psi'\rangle$ is EP definite. If the state $|\psi'\rangle$ is EP definite, proceed to 3-A-A-A. If the state $|\psi'\rangle$ is EP indefinite, proceed to 3-A-A-B.

3-A-A-A Examine whether the state $|\psi\rangle$ is $\tilde{\zeta}$-definite. If the state $|\psi\rangle$ is $\tilde{\zeta}$-definite, proceed to 3-A-A-A-A. If the state $|\psi\rangle$ is $\tilde{\zeta}$-indefinite, proceed to 3-A-A-A-B.

3-A-A-A-A The deterministic LOCC transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is executable, if and only if the parameters ζ, ζ_A, ζ_B and ζ_C satisfy $\zeta_{\text{lower}} \leq \zeta \leq 1$, $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$, $0 \leq \zeta_C \leq 1$, (3.72), $\zeta = \tilde{\zeta}$ and $Q_{\text{e}} = Q'_{\text{e}}$, where ζ_{lower} is defined by (3.73).

3-A-A-A-B The deterministic LOCC transformation from state $|\psi\rangle$ to state $|\psi'\rangle$ is executable, if and only if the parameters ζ, ζ_A, ζ_B and ζ_C satisfy $\zeta_{\text{lower}} \leq \zeta \leq 1$, $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$, $0 \leq \zeta_C \leq 1$, (3.72) and $|Q'_{\text{e}}| = \text{sgn}[(1 - \zeta)(\zeta - \zeta_{\text{lower}})]$.

3-A-A-B The deterministic LOCC transformatioinn from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is executable, if and only if the parameters ζ, ζ_A, ζ_B and ζ_C satisfy $\zeta_{\text{lower}} \leq \zeta \leq 1$, $0 \leq \zeta_A \leq 1$, $0 \leq \zeta_B \leq 1$, $0 \leq \zeta_C \leq 1$ and (3.72).

3-A-B Because $j'_{ABC} = 0$, if there are ζ, ζ_A, ζ_B and ζ_C which are from zero to one and satisfy (3.72), then at least one of ζ, ζ_A, ζ_B and ζ_C is equal to zero because of (3.72) and the expression $j_{ABC} \neq 0$. Therefore, at most one of K'_{AB}, K'_{AC} and K'_{BC} is nonzero. Hence, the deterministic LOCC transformation from the state $|\psi\rangle$ to the state $|\psi'\rangle$ is executable if and only if $j'_{ABC} = 0$, $J'_5 = 0$, at least two of the three parameters K'_{AB}, K'_{AC} and K'_{BC} are zero, and the remaining one is less than or equal to the corresponding K-parameter of K_{AB}, K_{AC} and K_{BC}. For example, if $K'_{AB} \neq 0$, then the inequality $K'_{AB} \leq K_{AB}$ is the condition.

2-B Examine whether the initial state is EP definite. If the initial state is EP definite, proceed to 3-B-A. If the initial state is EP indefinite, proceed to 3-B-B.

3-B-A In this case, $\zeta = 1$ holds, because of $j_{ABC}^2 = 0$, $\zeta_{\text{lower}} \leq \zeta \leq 1$ and $j_{AB}^2 j_{AC}^2 j_{BC}^2 \neq 0$. The equation $J_5 = \sqrt{2 j_{AB} j_{AC} j_{BC}}$ holds because $j_{ABC} = 0$. Then, the deterministic LOCC transformation from the EP-definite state $|\psi\rangle$ to the state $|\psi'\rangle$ is executable if and only if there are ζ_A, ζ_B and ζ_C which are from zero to one and satisfy the following equations:

$$
\begin{align}
K'_{AB} &= \zeta_A \zeta_B K_{AB}, & (3.445) \\
K'_{AC} &= \zeta_A \zeta_C K_{AC}, & (3.446) \\
K'_{BC} &= \zeta_B \zeta_C K_{BC}, & (3.447) \\
j'^2_{ABC} &= 0, & (3.448) \\
J'_5 &= \zeta_A \zeta_B \zeta_C J_5. & (3.449)
\end{align}
$$

The above equations are equivalent to the following equations:

$$
\begin{align}
K'_{AB} K'_{AC} K_{BC} &= \zeta_A^2 K_{AB} K_{AC} K'_{BC}, & (3.450) \\
K'_{AB} K_{AC} K'_{BC} &= \zeta_B^2 K_{AB} K'_{AC} K_{BC}, & (3.451) \\
K_{AB} K'_{AC} K'_{BC} &= \zeta_C^2 K'_{AB} K_{AC} K_{BC}, & (3.452) \\
j'^2_{ABC} &= 0, & (3.453) \\
J'_5 &= \zeta_A \zeta_B \zeta_C J_5. & (3.454)
\end{align}
$$

Hence, the deterministic LOCC transformation from the EP-definite state $|\psi\rangle$ to the state $|\psi'\rangle$ is executable if and only if the following inequalities hold:

$$
\begin{align}
K'_{AB} K'_{AC} K_{BC} &\leq K_{AB} K_{AC} K'_{BC}, & (3.455) \\
K'_{AB} K_{AC} K'_{BC} &\leq K_{AB} K'_{AC} K_{BC}, & (3.456) \\
K_{AB} K'_{AC} K'_{BC} &\leq K'_{AB} K_{AC} K_{BC}, & (3.457) \\
j'^2_{ABC} &= 0, & (3.458) \\
\sqrt{K_{\text{ap}}} J'_5 &= \sqrt{K'_{AB} K'_{AC} K'_{BC}} J_5. & (3.459)
\end{align}
$$

3-B-B In this case, the state $|\psi\rangle$ is EP indefinite, and hence $J_5 = 0$ holds. Therefore the deterministic transformation from the EP-indefinite state $|\psi\rangle$ to the state $|\psi'\rangle$ is impossible if $J'_5 \neq 0$, because of (3.72). In the case of $J'_5 = 0$, we define real parameters ζ'_A, ζ'_B and ζ'_C as $\zeta'_A = \sqrt{\zeta}\zeta_A$, $\zeta'_B = \sqrt{\zeta}\zeta_B$, $\zeta'_C = \sqrt{\zeta}\zeta_C$. Then, the deterministic LOCC transformation from the EP-indefinite state $|\psi\rangle$ to the state $|\psi'\rangle$ is executable if and only if there are ζ'_A, ζ'_B and ζ'_C which are from zero to one and satisfy the following equations:

$$
\begin{align}
K'_{AB} &= \zeta'_A \zeta'_B K_{AB}, & (3.460) \\
K'_{AC} &= \zeta'_A \zeta'_C K_{AC}, & (3.461) \\
K'_{BC} &= \zeta'_B \zeta'_C K_{BC}, & (3.462) \\
j'^2_{ABC} &= 0, & (3.463) \\
J'_5 &= 0. & (3.464)
\end{align}
$$

The above equations are equivalent to the following equations:

$$K'_{AB}K'_{AC}K_{BC} = \zeta'^2_A K_{AB}K_{AC}K'_{BC}, \tag{3.465}$$
$$K'_{AB}K_{AC}K'_{BC} = \zeta'^2_B K_{AB}K'_{AC}K_{BC}, \tag{3.466}$$
$$K_{AB}K'_{AC}K'_{BC} = \zeta'^2_C K'_{AB}K_{AC}K_{BC}, \tag{3.467}$$
$$j'^2_{ABC} = 0, \tag{3.468}$$
$$J'_5 = 0. \tag{3.469}$$

Hence, the deterministic LOCC transformation is executable if and only if the following inequalities hold:

$$K'_{AB}K'_{AC}K_{BC} \leq K_{AB}K_{AC}K'_{BC}, \tag{3.470}$$
$$K'_{AB}K_{AC}K'_{BC} \leq K_{AB}K'_{AC}K_{BC}, \tag{3.471}$$
$$K_{AB}K'_{AC}K'_{BC} \leq K'_{AB}K_{AC}K_{BC}, \tag{3.472}$$
$$j'^2_{ABC} = 0, \tag{3.473}$$
$$J'_5 = 0. \tag{3.474}$$

3.7 Conclusion

In the present paper, we have given four important results. First, we have introduced the entanglement charge Q_e. This new entanglement parameter Q_e has features which the electric charge has. The set of the six parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ is a perfect set for LU-equivalence; arbitrary three qubit pure states are LU-equivalent if and only if their entanglement parameters $(j_{AB}, j_{AC}, j_{BC}, j_{ABC}, J_5, Q_e)$ are the same. The entanglement charge Q_e satisfies a conservation law partially. Deterministic LOCC transformations between $\tilde{\zeta}$-definite states conserve the entanglement charge Q_e. When we transform a $\tilde{\zeta}$-indefinite state into a $\tilde{\zeta}$-definite state, we can choose the value of the entanglement charge. Once the value is determined, we cannot change it anymore (Fig. 3.6). In this sense, we can regard $\tilde{\zeta}$-indefinite states as charge-definite states.

Second, we have given a necessary and sufficient condition of the possibility of deterministic LOCC transformations of three-qubit pure states. We have revealed that we need *six* entanglement parameters in order to describe deterministic transformations of three-qubit pure states. In other words, we have revealed that three-qubit pure states are a partially ordered set parametrized by the six entanglement parameters.

Third, we have given the minimum times of measurements to reproduce an arbitrary executable deterministic LOCC transformation. We can realize the minimum times by performing DMTs. We can also determine the order of measurements; we can determine which qubit is measured first, second and third.

Fourth, we have clarified the rules of the change of the entanglement parameters. The rules indicate the transfer of entanglement. When one qubit is measured, the entanglement moves from the tripartite entanglement to the bipartite entanglement between the other two qubits. For example, if the qubit A is measured, the tripartite entanglement j_{ABC} among the qubits A, B and C is squeezed into the bipartite entanglement j_{BC} between the qubits B and C. This implies that the tripartite entanglement is a higher entity than the bipartite entanglements.

Is there entanglement transfer for a stochastic LOCC transformation? For this question, the present thesis has given a partial answer. Let us see the inequalities which is given in Lemma 2:

$$j_{BC} \leq \sum_{i=1}^{2} p_{(i)} j_{BC}^{(i)} \leq \sqrt{j_{BC}^2 + \left[1 - \left(\sum_{k=1}^{2} p_i \alpha^{(i)}\right)^2\right] j_{ABC}^2}. \tag{3.475}$$

The left inequality means that the bipartite entanglement j_{BC} between the qubits B and C increases when the qubit A is measured. The right inequality is equivalent to the following inequality:

$$\left(\sum_{i=1}^{2} p_{(i)} j_{BC}^{(i)}\right)^2 + \left(\sum_{i=1}^{2} p_{(i)} j_{ABC}^{(i)}\right)^2 \leq j_{BC}^2 + j_{ABC}^2, \tag{3.476}$$

because $j_{ABC}^{(i)} = \alpha_{(i)} j_{ABC}$. We can interpret the left-hand side of (3.476) as the sum of the bipartite entanglement j_{BC} between the qubits B and C and the tripartite entanglement j_{ABC} among the qubits A, B and C *after* a measurement. On the other hand, the right-hand side is the sum *before* a measurement. Thus, (3.476) means that a measurement decreases the sum. Note that the bipartite entanglement j_{BC} of the qubits B and C increases, whereas the tripartite entanglement j_{ABC} among the qubits A, B and C decreases. To summarize the above, a kind of dissipative entanglement transfer also occurs for a two-choice measurement which are not a DM. It is expected that the transfer occurs for an n-choice measurement too. Indeed, the left inequality of (3.475) also holds for an n-choice measurement. However, the right inequality of (3.475) for an n-choice measurement has not been proven yet.

In the present thesis, we have exhaustively analyzed deterministic LOCC transformations of three-qubit pure states. This is the first step of the extension of Nielsen's work [14] to multipartite entanglements.

Appendix A

Equation List

In the present Appendix, we list up equations which are often used.

The general Schmidt decomposition is given by

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\varphi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle. \tag{A.1}$$

This corresponds to (3.1). The definitions of the entanglement parameters j_{AB}, j_{AC}, j_{BC}, j_{ABC} and J_5:

$$j_{AB} = \lambda_0 \lambda_3, \tag{A.2}$$

$$j_{AC} = \lambda_0 \lambda_2, \tag{A.3}$$

$$j_{BC} = |\lambda_1 \lambda_4 e^{i\varphi} - \lambda_2 \lambda_3|, \tag{A.4}$$

$$j_{ABC} = \lambda_0 \lambda_4, \tag{A.5}$$

$$J_5 = \lambda_0^2 (j_{BC}^2 + \lambda_2^2 \lambda_3^2 - \lambda_1^2 \lambda_4^2). \tag{A.6}$$

These correspond to (3.2)–(3.6). The definitions of the entanglement parameters K_{AB}, K_{AC} and K_{BC}:

$$K_{AB} = j_{AB}^2 + j_{ABC}^2, \tag{A.7}$$

$$K_{AC} = j_{AC}^2 + j_{ABC}^2, \tag{A.8}$$

$$K_{BC} = j_{BC}^2 + j_{ABC}^2. \tag{A.9}$$

These correspond to (3.8)–(A.9).

The definition of abbreviations J_{ap}, K_{ap} and K_5:

$$J_{\mathrm{ap}} \equiv j_{AB}^2 j_{AC}^2 j_{BC}^2, \quad K_{\mathrm{ap}} \equiv K_{\mathrm{ap}}, \quad K_5 \equiv j_{ABC}^2 + J_5. \tag{A.10}$$

This corresponds to (3.11). The ambiguity of J-parameters is

$$(\lambda_0^\pm)^2 = \frac{J_5 + j_{ABC}^2 \pm \sqrt{\Delta_J}}{2(j_{BC}^2 + j_{ABC}^2)} \tag{A.11}$$

$$(\lambda_2^\pm)^2 = \frac{j_{AC}^2}{(\lambda_0^\pm)^2}, \tag{A.12}$$

$$(\lambda_3^\pm)^2 = \frac{j_{AB}^2}{(\lambda_0^\pm)^2}, \tag{A.13}$$

$$(\lambda_4^\pm)^2 = \frac{j_{ABC}^2}{(\lambda_0^\pm)^2}, \tag{A.14}$$

$$(\lambda_1^\pm)^2 = 1 - (\lambda_0^\pm)^2 - \frac{j_{AB}^2 + j_{AC}^2 + j_{ABC}^2}{(\lambda_0^\pm)^2}, \tag{A.15}$$

$$\cos\varphi^\pm = \frac{(\lambda_1^\pm)^2(\lambda_4^\pm)^2 + (\lambda_2^\pm)^2(\lambda_3^\pm)^2 - j_{BC}^2}{2\lambda_1^\pm \lambda_2^\pm \lambda_3^\pm \lambda_4^\pm}, \tag{A.16}$$

where

$$\Delta_J \equiv K_5^2 - 4K_{\mathrm{ap}} \geq 0, \tag{A.17}$$

$$0 \leq \varphi^\pm \leq \pi. \tag{A.18}$$

These correspond to (3.12)–(3.18). The definition of Q_{e}.

$$Q_{\mathrm{e}} = \mathrm{sgn}\left[\sin\varphi\left(\lambda_0^2 - \frac{j_{ABC}^2 + J_5}{2(j_{BC}^2 + j_{ABC}^2)}\right)\right] \tag{A.19}$$

This corresponds to (3.20). The equations which determine the general Schmidt coefficients from J-parameters and Q_{e} when Q_{e} is not zero:

$$\lambda_0^2 = \frac{J_5 + j_{ABC}^2 \overset{..}{\mp} Q_{\mathrm{e}}\sqrt{\Delta_J}}{2(j_{BC}^2 + j_{ABC}^2)} = \frac{K_5 \overset{..}{\mp} Q_{\mathrm{e}}\sqrt{\Delta_J}}{2K_{BC}} \tag{A.20}$$

$$\lambda_2^2 = \frac{j_{AC}^2}{\lambda_0^2}, \tag{A.21}$$

$$\lambda_3^2 = \frac{j_{AB}^2}{\lambda_0^2}, \tag{A.22}$$

$$\lambda_4^2 = \frac{j_{ABC}^2}{\lambda_0^2}, \tag{A.23}$$

$$\lambda_1^2 = 1 - \lambda_0^2 - \frac{j_{AB}^2 + j_{AC}^2 + j_{ABC}^2}{\lambda_0^2}, \tag{A.24}$$

$$\cos\varphi = \frac{\lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 - j_{BC}^2}{2\lambda_1\lambda_2\lambda_3\lambda_4}, \tag{A.25}$$

where $\overset{..}{\mp}$ is $+$ or $-$ when $\{\lambda_i, \varphi | i = 0, ..., 4\}$ is positive-decomposition coefficients or negative decomposition coefficients, respectively. These correspond to (3.22)–(3.27).

Another expression of J_5:

$$\begin{aligned} J_5 &= 2\lambda_0^2\lambda_2\lambda_3(\lambda_2\lambda_3 - \lambda_1\lambda_4\cos\varphi) \\ &= 2j_{AB}j_{AC}(\lambda_2\lambda_3 - \lambda_1\lambda_4\cos\varphi) \\ &= 2j_{AB}j_{AC}j_{BC}\frac{\lambda_2\lambda_3 - \lambda_1\lambda_4\cos\varphi}{|\lambda_2\lambda_3 - \lambda_1\lambda_4 e^{i\varphi}|}. \end{aligned} \tag{A.26}$$

This corresponds to (3.28). The inequalities which the J_5 satisfies:

$$0 \leq \left| \frac{\lambda_2 \lambda_3 - \lambda_1 \lambda_4 \cos \varphi}{|\lambda_2 \lambda_3 - \lambda_1 \lambda_4 e^{i\varphi}|} \right| \leq 1. \tag{A.27}$$

This corresponds to (3.29). The definition of the EP:

$$\frac{J_5}{2 j_{AB} j_{AC} j_{BC}} = \cos \varphi_5. \tag{A.28}$$

This corresponds to (3.31). The relation between φ_5 and φ:

$$j_{BC} \sin \varphi_5 = \overset{..}{\mp} \lambda_1 \lambda_4 \sin \varphi. \tag{A.29}$$

This corresponds to (3.32). The expression which is derived from (A.29):

$$\sin \varphi_5 = 0 \Leftrightarrow \sin \varphi = 0. \tag{A.30}$$

This corresponds to (3.33). The definition of a, b, k, $0 \leq \theta \leq 2\pi$:

$$M^\dagger M = \begin{pmatrix} a & k e^{-i\theta} \\ k e^{i\theta} & b \end{pmatrix}. \tag{A.31}$$

This corresponds to (3.45). The definition of the parameters a, b, k and θ for a two-choice measurement:

$$M^\dagger_{(0)} M_{(0)} = \begin{pmatrix} a_{(0)} & k_{(0)} e^{-i\theta_{(0)}} \\ k_{(0)} e^{i\theta_{(0)}} & b_{(0)} \end{pmatrix} = \begin{pmatrix} a & k e^{-i\theta} \\ k e^{i\theta} & b \end{pmatrix}, \tag{A.32}$$

$$M^\dagger_{(1)} M_{(1)} = \begin{pmatrix} a_{(1)} & k_{(1)} e^{-i\theta_{(1)}} \\ k_{(1)} e^{i\theta_{(1)}} & b_{(1)} \end{pmatrix} = \begin{pmatrix} 1 - a & -k e^{-i\theta} \\ -k e^{i\theta} & 1 - b \end{pmatrix}. \tag{A.33}$$

These equations correspond to (3.93) and (3.94). The expression of $p_{(i)}$ in terms of $a_{(i)}$, $b_{(i)}$, $k_{(i)}$, $0 \leq \theta_{(i)} \leq 2\pi$:

$$p_{(i)} = \lambda_0^2 a_{(i)} + (1 - \lambda_0^2) b_{(i)} + 2 \lambda_0 \lambda_1 k_{(i)} \cos (\theta_{(i)} - \varphi). \tag{A.34}$$

This corresponds to (3.47). The changes of the coefficients of the general Schmidt decomposition by a measurement on the qubit A:

$$\lambda_0^{(i)} = \frac{\lambda_0 \sqrt{a_{(i)} b_{(i)} - k_{(i)}^2}}{\sqrt{p_{(i)}} \sqrt{b_{(i)}}}, \tag{A.35}$$

$$\lambda_1^{(i)} e^{i\varphi^{(i)}} = \frac{\lambda_0 k_{(i)} e^{i\theta_{(i)}} + \lambda_1 e^{i\varphi} b_{(i)}}{\sqrt{p_{(i)}} \sqrt{b_{(i)}}}, \tag{A.36}$$

$$\lambda_2^{(i)} = \frac{\lambda_2 \sqrt{b_{(i)}}}{\sqrt{p_{(i)}}}, \tag{A.37}$$

$$\lambda_3^{(i)} = \frac{\lambda_3 \sqrt{b_{(i)}}}{\sqrt{p_{(i)}}}, \tag{A.38}$$

$$\lambda_4^{(i)} = \frac{\lambda_4\sqrt{b_{(i)}}}{\sqrt{p_{(i)}}}. \tag{A.39}$$

These correspond to (3.49)–(3.53). The changes of the parameters j_{AB}, j_{AC}, j_{ABC}, j_{BC}, K_{AB} and K_{AC} by a measurement on the qubit A:

$$j_{AB}^{(i)} = \alpha^{(i)} j_{AB} = \frac{\sqrt{a_{(i)}b_{(i)} - k_{(i)}^2} j_{AB}}{p_{(i)}}, \tag{A.40}$$

$$j_{AC}^{(i)} = \alpha^{(i)} j_{AC} = \frac{\sqrt{a_{(i)}b_{(i)} - k_{(i)}^2} j_{AC}}{p_{(i)}}, \tag{A.41}$$

$$j_{ABC}^{(i)} = \alpha^{(i)} j_{ABC} = \frac{\sqrt{a_{(i)}b_{(i)} - k_{(i)}^2} j_{ABC}}{p_{(i)}}, \tag{A.42}$$

$$K_{AB}^{(i)} = (\alpha^{(i)})^2 K_{AB}, \quad K_{AC}^{(i)} = (\alpha^{(i)})^2 K_{AC} \tag{A.43}$$

$$
\begin{aligned}
p_{(i)} j_{BC}^{(i)} &= |\lambda_0\lambda_4 k_{(i)} e^{i\theta_{(i)}} - (\lambda_2\lambda_3 - \lambda_1\lambda_4 e^{i\varphi})b_{(i)}| \\
&= |k_{(i)} j_{ABC} e^{i\theta_{(i)}} - j_{BC} b_{(i)} e^{-i\tilde{\varphi}_5}| \\
&= |k_{(i)} j_{ABC} e^{i(\theta_{(i)}+\tilde{\varphi}_5)} - j_{BC} b_{(i)}|.
\end{aligned} \tag{A.44}
$$

These correspond to (3.54)–(3.56), (3.59) and (3.58). The definition of the multiplication factor $\alpha_{(i)}$:

$$\alpha^{(i)} = \frac{\sqrt{a_{(i)}b_{(i)} - k_{(i)}^2}}{p_{(i)}}. \tag{A.45}$$

This corresponds to (3.57). The change of the quantity $j_{BC}\cos\varphi_5$ by a measurement on the qubit A:

$$p_{(i)} j_{BC}^{(i)}\cos\varphi_5^{(i)} = b_{(i)} j_{BC}\cos\varphi_5 - k_{(i)} j_{ABC}\cos\theta_{(i)}. \tag{A.46}$$

This corresponds to (3.62). The change of the average of the quantity $j_{BC}\cos\varphi_5$ by a measurement on qubit A in the two expressions:

$$\sum_i p_{(i)}(\lambda_2^{(i)}\lambda_3^{(i)} - \lambda_1^{(i)}\lambda_4^{(i)}\cos\varphi^{(i)}) = \lambda_2\lambda_3 - \lambda_1\lambda_4\cos\varphi, \tag{A.47}$$

$$\sum_i p_{(i)} j_{BC}^{(i)}\cos\varphi_5^{(i)} = j_{BC}\cos\varphi_5. \tag{A.48}$$

These correspond to (3.66) and (3.67). The change of average of the quantity $\lambda_1\lambda_4\sin\varphi$:

$$\sum_{i=1}^n p_{(i)}\lambda_1^{(i)}\lambda_4^{(i)}\sin\varphi^{(i)} = \sum_{i=1}^n b_{(i)}\lambda_1^{(i)}\lambda_4^{(i)}\sin\varphi^{(i)} + k_{(i)}\lambda_0^{(i)}\lambda_4^{(i)}\sin\theta_{(i)} = \lambda_1\lambda_4\sin\varphi. \tag{A.49}$$

This corresponds to (3.68) The change of the average of $\alpha^{(i)}$:

$$\sum_{k=0}^1 p_{(i)}\alpha^{(i)} \le 1. \tag{A.50}$$

This corresponds to (3.118). The change of j_{BC} caused by a measurement on the qubit A:

$$j_{BC} \leq \sum_{i=0}^{1} p_{(i)} j_{BC}^{(i)} \leq \sqrt{j_{BC}^2 + \left[1 - \left(\sum_{k=0}^{1} p_{(i)} \alpha^{(i)} \right)^2 \right] j_{ABC}^2}. \tag{A.51}$$

This corresponds to (3.95). The inequalities of Corollary 1:

$$j_{BC}'^2 \sin^2 \varphi_5' \geq j_{BC}^2 \sin^2 \varphi_5, \tag{A.52}$$

$$j_{BC}^2 \leq K_{BC}' \equiv j_{BC}'^2 + j_{ABC}'^2 \leq K_{BC} \equiv j_{BC}^2 + j_{ABC}^2, \tag{A.53}$$

$$\Delta'_{\text{norm}} \equiv \frac{\Delta'_J}{\alpha^4} = K_5^2 - 4 K_{AB} K_{AC} K_{BC}' \geq \Delta_J, \tag{A.54}$$

They correspond to (3.106)–(3.108).

Appendix B

The proof that Q_{e} is a tripartite parameter

In the present appendix, we show that Q_{e} defined in (3.20) is a tripartite parameter; in other words, we show that Q_{e} is invariant with respect to permutations of the qubits A, B and C.

First, we perform the proof in the case of $Q_{\mathrm{e}} = 0$. Because of (A.11) and (A.19), the equation $Q_{\mathrm{e}} = 0$ holds if and only if $\Delta_J = 0 \vee \sin\varphi = 0$. Because of (A.30) and (A.28), the expression $\Delta_J = 0 \vee \sin\varphi = 0$ is equivalent to $\Delta_J = 0 \vee |J_5| = 2j_{AB}j_{AC}j_{BC}$. Thus, $Q_{\mathrm{e}} = 0$ is equivalent to $\Delta_J = 0 \vee |J_5| = 2j_{AB}j_{AC}j_{BC}$. Therefore, if we can show that the expression $\Delta_J = 0 \vee |J_5| = 2j_{AB}j_{AC}j_{BC}$ is invariant with respect to permutations of A, B and C, we can also show that Q_{e} is invariant with respect to the permutations. The parameters J_5 and $j_{AB}j_{AC}j_{BC}$ are invariant with respect to the permutations of A, B and C [20]. This fact and (A.17) give that Δ_J is also invariant with respect to the permutations of A, B and C. Hence, the quantities Δ_J, J_5 and $j_{AB}j_{AC}j_{BC}$ are invariant with respect to the permutations of A, B and C, and thus if $Q_{\mathrm{e}} = 0$, then Q_{e} is invariant with respect to permutations of A, B and C. Namely, if $Q_{\mathrm{e}} = 0$, then Q_{e} is a tripartite parameter.

Second, we perform the proof in the case of $Q_{\mathrm{e}} = \pm 1$. In order to show this, we only have to show that Q_{e} is invariant with respect to the permutation of A and B, because if we can prove the invariance with respect to the permutation of A and B we can also prove the invariance with respect to the permutation of A and C or B and C in the same manner.

Let us derive the generalized Schmidt decomposition whose order of the qubits is BAC and see the expression of Q_{e} in the new decomposition, which we refer to as Q_{e}^B. The generalized Schmidt decomposition of $|\psi\rangle$ is expressed as

$$|\psi\rangle = \lambda_0 |0_A 0_B 0_C\rangle + \lambda_1 e^{i\varphi} |1_A 0_B 0_C\rangle + \lambda_2 |1_A 0_B 1_C\rangle + \lambda_3 |1_A 1_B 0_C\rangle + \lambda_4 |1_A 1_B 1_C\rangle. \quad \text{(B.1)}$$

We can assume that (B.1) is a positive decomposition. Let us permute A and B of (B.1):

$$|\psi\rangle = \lambda_0 |0_B 0_A 0_C\rangle + \lambda_1 e^{i\varphi} |0_B 1_A 0_C\rangle + \lambda_2 |0_B 1_A 1_C\rangle + \lambda_3 |1_B 1_A 0_C\rangle + \lambda_4 |1_B 1_A 1_C\rangle. \quad \text{(B.2)}$$

In order to put (B.2) in the form of the generalized Schmidt decomposition, let us define

pure states $|0'_A\rangle$, $|1'_A\rangle$, $|0'_B\rangle$, $|1'_B\rangle$, $|0'_C\rangle$ and $|1'_C\rangle$ as follows:

$$|0'_A\rangle = -e^{-i\tilde{\varphi}_5}|1_A\rangle, \quad |1'_A\rangle = |0_A\rangle, \tag{B.3}$$

$$|0'_B\rangle = -e^{i\tilde{\varphi}_5}|1_A\rangle, \quad |1'_A\rangle = |0_A\rangle, \tag{B.4}$$

$$|0'_C\rangle = \frac{\lambda_3|0_C\rangle + \lambda_4|1_C\rangle}{\sqrt{\lambda_3^2 + \lambda_4^2}}, \quad |1'_C\rangle = \frac{\lambda_4|0_C\rangle - \lambda_3|1_C\rangle}{\sqrt{\lambda_3^2 + \lambda_4^2}}, \tag{B.5}$$

where $\tilde{\varphi}_5$ is defined in (3.60). The equation (B.2) is expressed in terms of $|0'_A\rangle$, $|1'_A\rangle$, $|0'_B\rangle$, $|1'_B\rangle$, $|0'_C\rangle$ and $|1'_C\rangle$ as follows:

$$|\psi\rangle = \sqrt{\lambda_3^2 + \lambda_4^2}\,|0'_B 0'_A 0'_C\rangle + \frac{-e^{i\tilde{\varphi}_5}(\lambda_1\lambda_3 e^{i\varphi} + \lambda_2\lambda_4)}{\sqrt{\lambda_3^2 + \lambda_4^2}}\,|1'_B 0'_A 0'_C\rangle$$

$$+ \frac{|\lambda_2\lambda_3 - \lambda_1\lambda_4 e^{i\varphi}|}{\sqrt{\lambda_3^2 + \lambda_4^2}}\,|1'_B 0_A 1_C\rangle + \frac{\lambda_0\lambda_3}{\sqrt{\lambda_3^2 + \lambda_4^2}}\,|1'_A 1'_B 0_C\rangle + \frac{\lambda_0\lambda_4}{\sqrt{\lambda_3^2 + \lambda_4^2}}\,|1'_B 1'_A 1'_C\rangle. \tag{B.6}$$

Note that (B.6) is the generalized Schmidt decomposition whose order of the qubits is BAC. The coefficients of (B.6) correspond to the coefficient of (B.1); $\sqrt{\lambda_3^2 + \lambda_4^2}$ corresponds to λ_0, $-e^{i\tilde{\varphi}_5}(\lambda_1\lambda_3 e^{i\varphi} + \lambda_2\lambda_4)/\sqrt{\lambda_3^2 + \lambda_4^2}$ corresponds to $\lambda_1 e^{i\varphi}$, and so on. Let us refer to Q_e for (B.6) as Q_e^B. Because of the definition of Q_e and (B.6),

$$Q_\mathrm{e}^B = \mathrm{sgn}\left\{\mathrm{Im}\left[\frac{-e^{i\tilde{\varphi}_5}(\lambda_1\lambda_3 e^{i\varphi} + \lambda_2\lambda_4)}{|(\lambda_1\lambda_3 e^{i\varphi} + \lambda_2\lambda_4)|}\right]\left(\lambda_3^2 + \lambda_4^2 - \frac{K_5}{2K_{AC}}\right)\right\} \tag{B.7}$$

$$= \mathrm{sgn}\left\{\mathrm{Im}\left[\frac{-e^{i\tilde{\varphi}_5}(\lambda_1\lambda_3 e^{i\varphi} + \lambda_2\lambda_4)}{|(\lambda_1\lambda_3 e^{i\varphi} + \lambda_2\lambda_4)|}\right]\right\}\mathrm{sgn}\left(\lambda_3^2 + \lambda_4^2 - \frac{K_5}{2K_{AC}}\right) \tag{B.8}$$

holds. Then, we can complete the proof by showing that $Q_\mathrm{e} = Q_\mathrm{e}^B$. Because (B.1) is a positive decomposition and (A.20), the following two equations hold:

$$\mathrm{sgn}\left\{\mathrm{Im}\left[\frac{-e^{i\tilde{\varphi}_5}(\lambda_1\lambda_3 e^{i\varphi} + \lambda_2\lambda_4)}{|(\lambda_1\lambda_3 e^{i\varphi} + \lambda_2\lambda_4)|}\right]\right\} = \mathrm{sgn}\left\{\mathrm{Im}[-j_{BC}e^{i\tilde{\varphi}_5}(\lambda_1\lambda_3 e^{i\varphi} + \lambda_2\lambda_4)]\right\}$$

$$= \mathrm{sgn}[\lambda_1\lambda_3\sin\varphi(-\lambda_2\lambda_3 + \lambda_1\lambda_4\cos\varphi) - \lambda_1\lambda_4\sin\varphi(\lambda_1\lambda_3\cos\varphi + \lambda_2\lambda_4)]$$

$$= \mathrm{sgn}[-\lambda_1\lambda_2(\lambda_3^2 + \lambda_4^2)\sin\varphi] = -1. \tag{B.9}$$

$$\mathrm{sgn}\left(\lambda_3^2 + \lambda_4^2 - \frac{K_5}{2K_{AC}}\right) = \mathrm{sgn}\left(\frac{K_{AB}}{\lambda_0^2} - \frac{K_5}{2K_{AC}}\right) = \mathrm{sgn}\left(\frac{2K_{AB}K_{BC}}{K_5 + Q_\mathrm{e}\sqrt{\Delta_J}} - \frac{K_5}{2K_{AC}}\right)$$

$$= \mathrm{sgn}\left[\frac{4K_\mathrm{ap} - K_5^2 - Q_\mathrm{e}K_5\sqrt{\Delta_J}}{2K_{AC}(K_5 + Q_\mathrm{e}\sqrt{\Delta_J})}\right]$$

$$= \mathrm{sgn}\left[\frac{-(\sqrt{\Delta_J} + Q_\mathrm{e}K_5)\sqrt{\Delta_J}}{2K_{AC}(K_5 + Q_\mathrm{e}\sqrt{\Delta_J})}\right]$$

$$= \mathrm{sqn}\left[\frac{-Q_\mathrm{e}\sqrt{\Delta_J}}{2K_{AC}}\right] = -Q_\mathrm{e}. \tag{B.10}$$

Because of (B.8), (B.9) and (B.10), we obtain $Q_\mathrm{e} = Q_\mathrm{e}^B$. $\square$

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