Non-exponential decay near the continuum threshold

Part 2: exceptional points

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21–22 February

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Part 2: exceptional points

Relevant works:

S. Garmon, and G. Ordonez, J. Math. Phys. **58**, 062101 (2017)

S. Garmon, G. Ordonez and N. Hatano, Phys. Rev. Research **3**, 033029 (2021)

S. Garmon, T. Sawada, K. Noba, and G. Ordonez,
J. Phys.: Conf. Ser. **2038**, 012011 (2021).

S. Garmon, M. Gianfreda and N. Hatano, Phys. Rev. A **92**, 022125 (2015)

Background:
Is quantum mechanics a Hermitian theory?

Undergraduate quantum mechanics class:

Quantum theory is Hermitian

However, quantum decay is often described by the exponential lifetime

$$E_R - i\frac{\Gamma}{2}$$

Hermitian?

Note: decay describes how the particle in the initial state is transferred to the environment.

Background:
Is quantum mechanics a Hermitian theory?

Undergraduate quantum mechanics class:

Quantum theory is Hermitian

However, quantum decay is often described by the exponential lifetime

$$E_R - i\frac{\Gamma}{2}$$

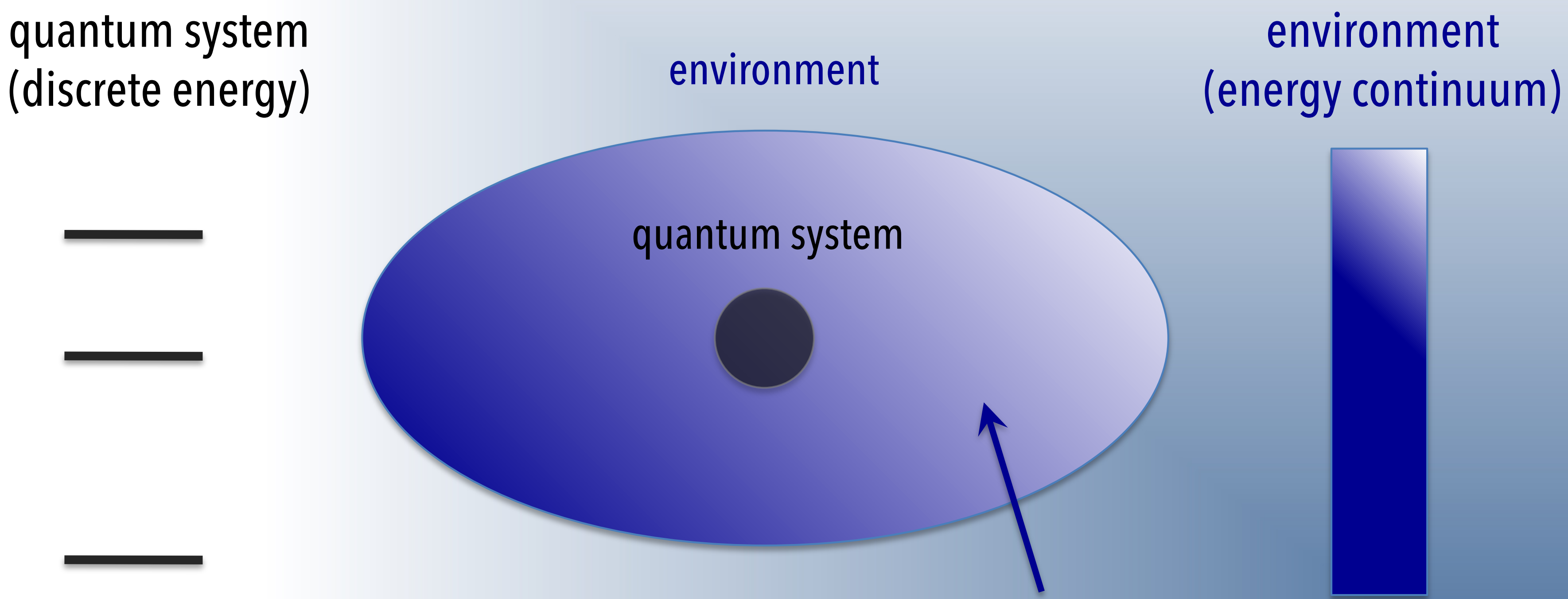
Hermitian?

Interpretation: only isolated systems are obviously Hermitian

(But no system is ever completely isolated)

Background: Open quantum systems

Open quantum systems describe how quantum systems are influenced by their surrounding environment.



details of continuum determined by microscopic details of “quantum bath”

Background: Open quantum systems

It is essential to distinguish between two different types of decay processes:

- **Markovian decay**: irreversible process that generally follows exponential dissipation

$$E_R - i\frac{\Gamma}{2}$$

- **Non-Markovian decay**: reversible (non-exponential) decay — may be useful for quantum info processing

Examples: branch-point effect, many-body effects, etc.

Non-Hermitian description of open quantum systems

Non-Hermitian theories are applied to the study of open systems from many perspectives in recent decades.

Many interesting features of non-Hermitian theories:

- Exceptional points (coalescing eigenstates)
- Non-Hermitian skin effect
- Non-trivial topological effects

Exceptional points: definition

An ***exceptional point*** **(EP)** is a branch point in parameter space at which two or more eigenstates *coalesce*

- Hamiltonian cannot be diagonalized at the EP

$$R^{-1}HR \sim \begin{pmatrix} E_{EP} & c \\ 0 & E_{EP} \end{pmatrix}$$

T. Kato, Perturbation Theory for Linear Operators (Springer, 1976)

G. Bhamathi and E. C. G. Sudarshan, Int. J. Mod. Phys. B **10**, 1531 (1996).

K. Kanki, S. G., S. Tanaka, and T. Petrosky,
J. Math. Phys. **58**, 092101 (2017).

Exceptional points: definition

An ***exceptional point*** **(EP)** is a branch point in parameter space at which two or more eigenstates *coalesce*

- Hamiltonian cannot be diagonalized at the EP

- This is distinct from the usual concept of degeneracy in quantum systems

- Exceptional points can only appear in non-Hermitian systems

Exceptional points: Puiseux expansion

Eigenvalues in the vicinity of the exceptional points can be expanded in characteristic Puiseux expansion:

$$z_{1,2} = z_{EP} \pm \alpha(\varepsilon - \varepsilon_{EP})^{1/2} + O(\varepsilon - \varepsilon_{EP})$$

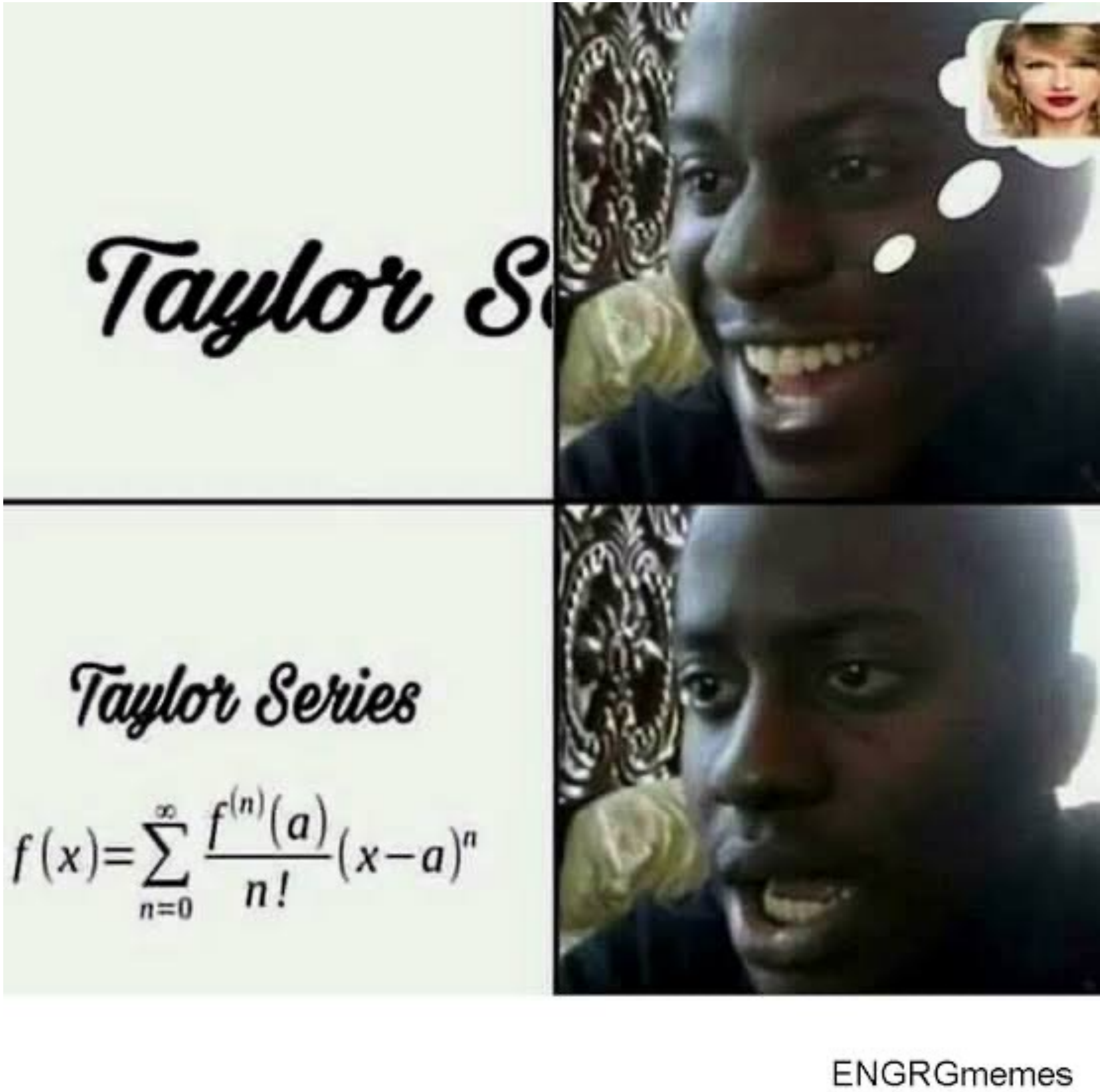
ε_{EP} is a branchpoint in ε space

coalesced eigenvalue

Exceptional points: Puiseux expansion

Eigenvalues in the vicinity of the exceptional points can be expanded in characteristic Puiseux expansion:

$$z_j = z_{EP} \pm \alpha_j(\varepsilon - \varepsilon_{EP})^{1/N} + O\left((\varepsilon - \varepsilon_{EP})^{2/N}\right)$$


Taylor S
Taylor Series
$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$
ENGRGmemes

Exceptional points: divergence of the norm

Another EP important property: norm of the coalescent eigenstates diverges

$$N_{1,2} \to \pm\infty \qquad \text{as} \qquad \varepsilon \to \varepsilon_{EP}$$

However, the divergent contributions must cancel when calculating physical quantities

Exceptional points: non-Hermitian systems

Exceptional points can only occur in non-Hermitian systems

- Open quantum systems (implicit non-Hermiticity)

  Non-Hermiticity is associated with the energy continuum that describes the environment (structured reservoir)

- Explicitly non-Hermitian systems

  Example: PT-symmetric, pseudo-Hermitian systems

  Many studies of EPs in recent years, particularly in optics.

Coupled mode theory

Coupled-mode theory: assumes relevant system properties can be described in terms of a few key modes

$$H = \begin{bmatrix} \varepsilon_1 - i\gamma_1 & \Omega \\ \Omega & \varepsilon_2 - i\gamma_2 \end{bmatrix}$$

macroscopic approximation for environmental influence

$$E_{\pm} = \frac{\varepsilon_1 + \varepsilon_2 + i(\gamma_1 + \gamma_2)}{2} \pm \frac{1}{2}\sqrt{\big(\varepsilon_1 + \varepsilon_2 + i(\gamma_1 + \gamma_2)\big)^2 + 4\Omega^2}$$

Coupled mode theory

Coupled-mode theory: assumes relevant system properties can be described in terms of a few key modes

$$H = \begin{bmatrix} \varepsilon - i\gamma & \Omega \\ \Omega & \varepsilon + i\gamma \end{bmatrix}$$

we can write a special case:
PT-symmetric two-level model

$$E_{\pm} = \varepsilon \pm \sqrt{\Omega^2 - \gamma^2}$$

Open quantum systems

Open quantum systems: microscopic description of the environment in terms of an energy continuum

$$H = \sum_{j=1}^{n} \varepsilon_j d_j^\dagger d_j + \int_{-\infty}^{\infty} dk\, E_k c_k^\dagger c_k + \sum_{j=1}^{n} \int_{-\pi}^{\pi} dk\, V_k \left(c_k^\dagger d_j + d_j^\dagger c_k \right)$$

<u>Energy continuum</u> E_k

Well defined range $E_k \in [E_{\text{th}}, \infty]$

and density of states $\rho(E) = \dfrac{dk}{dE}$

Open quantum systems

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<u>Energy continuum</u> E_k

Well defined range $E_k \in [E_{\mathrm{th}}, \infty]$

"structured reservoir"

and density of states $\rho(E) = \dfrac{dk}{dE}$

P. Lambropoulos, G. M. Nikolopoulos, T. R. Nielsen, and S. Bay, Rep. Prog. Phys. **63**, 455 (2000).

Open quantum systems

Open quantum systems: microscopic description of the environment in terms of an energy continuum

$$H = \sum_{j=1}^{n} \varepsilon_j d_j^\dagger d_j + \int_{-\infty}^{\infty} dk\, E_k c_k^\dagger c_k + \sum_{j=1}^{n} \int_{-\pi}^{\pi} dk\, V_k \left(c_k^\dagger d_j + d_j^\dagger c_k \right)$$

Interaction between discrete system and continuum gives rise to resonance states with complex eigenvalues

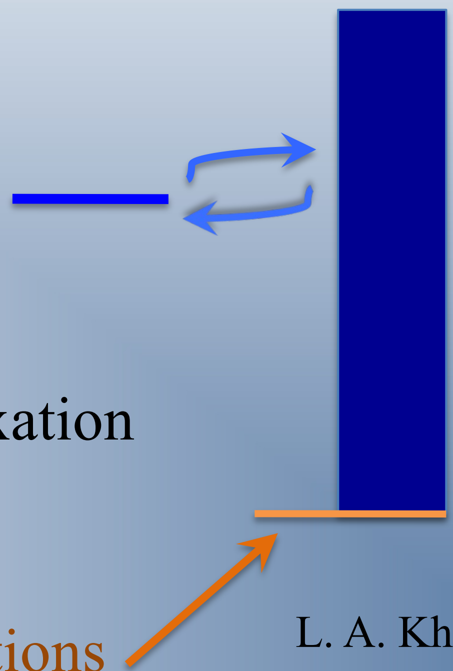
$$z_R = \varepsilon - i\gamma$$

Complex eigenvalue of resonance state

Complex eigenvalues can occur in open quantum systems due to interaction with the energy continuum

resonance eigenvalue: $E_R - i\frac{\Gamma}{2}$

$$P(t) = P_0 e^{-\Gamma t}$$

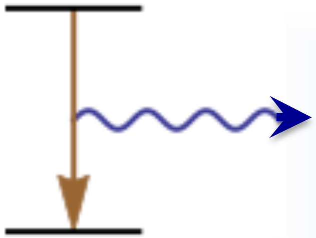


Examples: nuclear decay, atomic relaxation

However:
<u>continuum threshold</u> introduces deviations from exponential decay (branch-point effect)

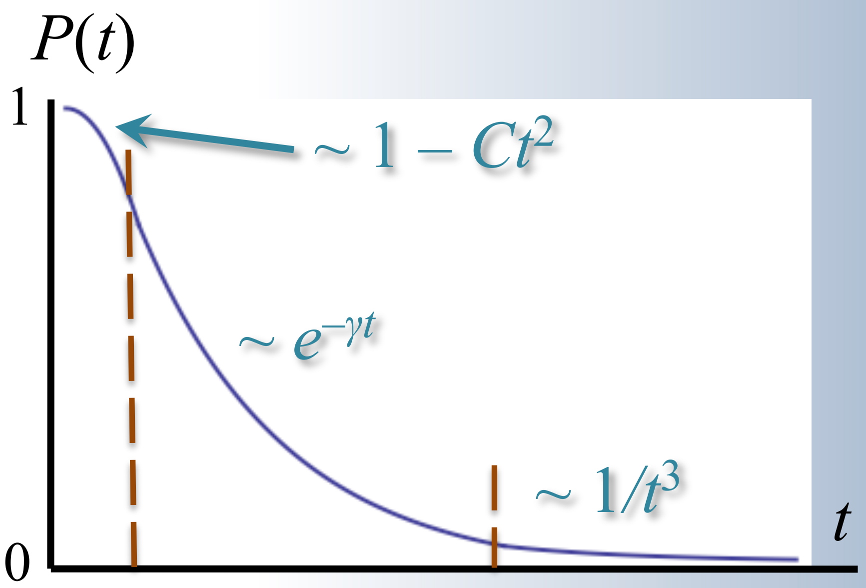
L. A. Khalfin, Sov. Phys. -JETP, **6**, 1053 (1958).

Exponential decay and deviations



Exponential decay is usually dominant form of dissipation in quantum mechanics

- Deviations from exponential decay occur at least on **extremely short** and **extremely long** time scales



Survival Probability:

$$P(t) = \left|\langle\psi_0|e^{-iHt}|\psi_0\rangle\right|^2$$

Modification of exponential decay near EPs

Exponential decay can be modified in the vicinity of exceptional points

Consider <u>two coalescing resonance states</u> (we call this: **EP2B**)

$$A(t) = \langle q | e^{-iHt} | q \rangle = \frac{1}{2\pi i} \int_{\mathcal{C}_E} dE\, e^{-iEt} \left\langle q \middle| \frac{1}{E - H} \middle| q \right\rangle$$

degenerate eigenvalues gives a double pole

Modification of exponential decay near EPs

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Consider <u>two coalescing resonance states</u> (we call this: **EP2B**)

$$A(t) = \langle q | e^{-iHt} | q \rangle = \frac{1}{2\pi i} \int_{\mathcal{C}_E} dE\, e^{-iEt} \left\langle q \left| \frac{1}{E-H} \right| q \right\rangle$$

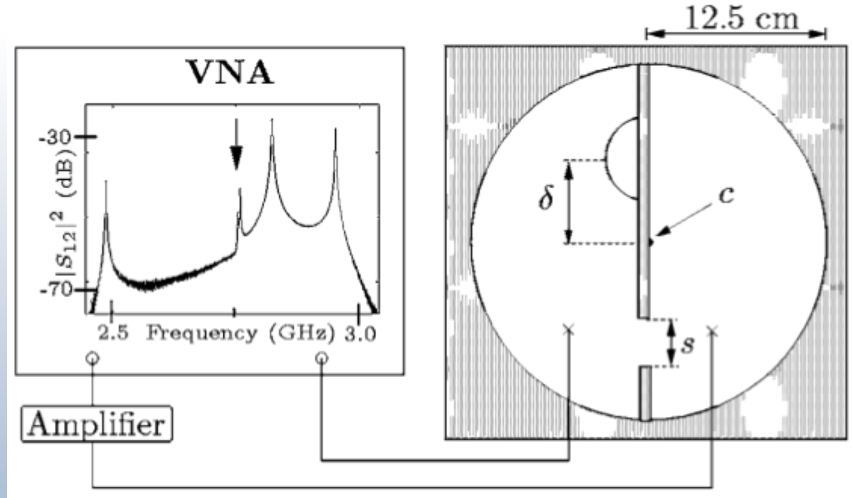
$A(t) \sim (1 + Ct) e^{-iE_R t - \gamma t/2}$ $\qquad$ $P(t) \sim (1 + C_1 t + C_2 t^2) e^{-\gamma t}$

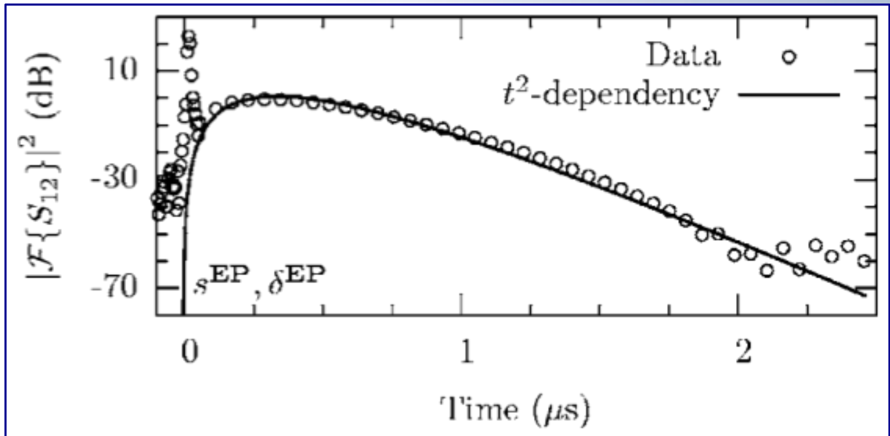
M. L. Goldberger and K. M Watson, Phys. Rev. **136**, B1472 (1964).

J. S. Bell and C. J. Goebel, Phys. Rev. **138**, B1198 (1965).

Experiment: EP2B power law-exponential decay

The power-law exponential decay has been verified in a microwave cavity experiment





$$\lim_{(s,\delta)\to\mathbf{EP}} |\tilde{\mathbb{G}}_{12}(t)|^2 \approx \frac{\sigma^2}{\mathcal{E}_+\mathcal{E}_-} t^2 e^{-\bar{\gamma}t}$$

B. Dietz, *et al*, Phys. Rev. E **75**, 027201 (2007).

S. Bittner, *et al*, Phys. Rev. E **89**, 032909 (2014).

Non-exponential decay on long timescale

In a true quantum system, the decay should eventually become non-exponential on the longest timescale.

Confirmed this still occurs at an EP2B in the following work:

S. Garmon and G. Ordonez J. Math. Phys. **58**, 062101 (2017)

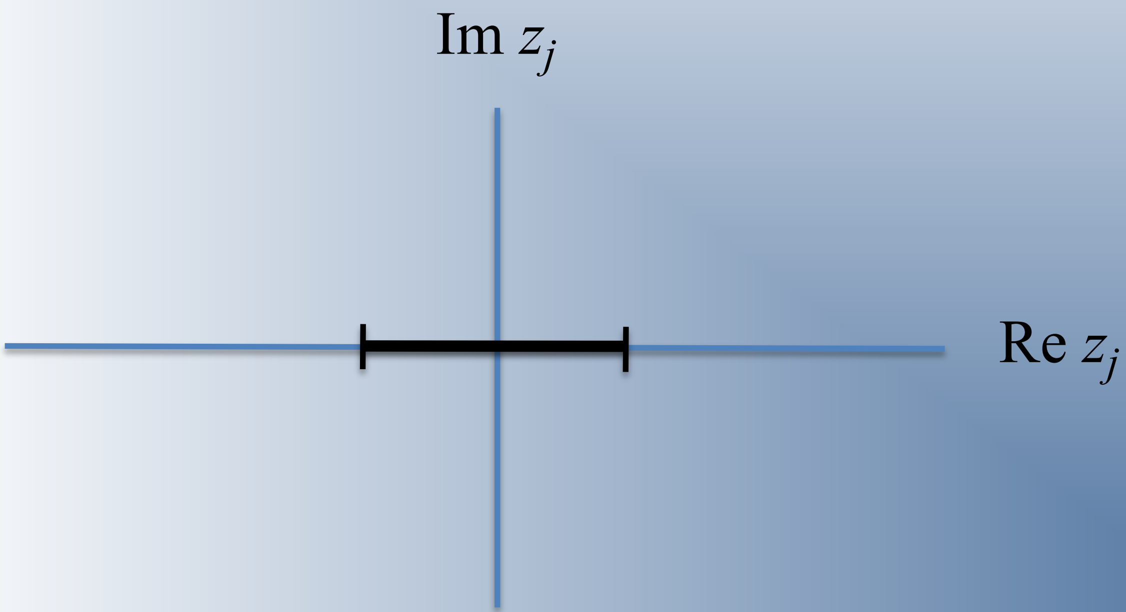
But this would still be very difficult to observe in experiment in most cases.

However, <u>non-exponential</u> (**non-Markovian**) <u>dynamics are very pronounced at an **EP2A**</u>.

Definition for A-type exceptional points

Recall: resonance appears due to interaction with the continuum

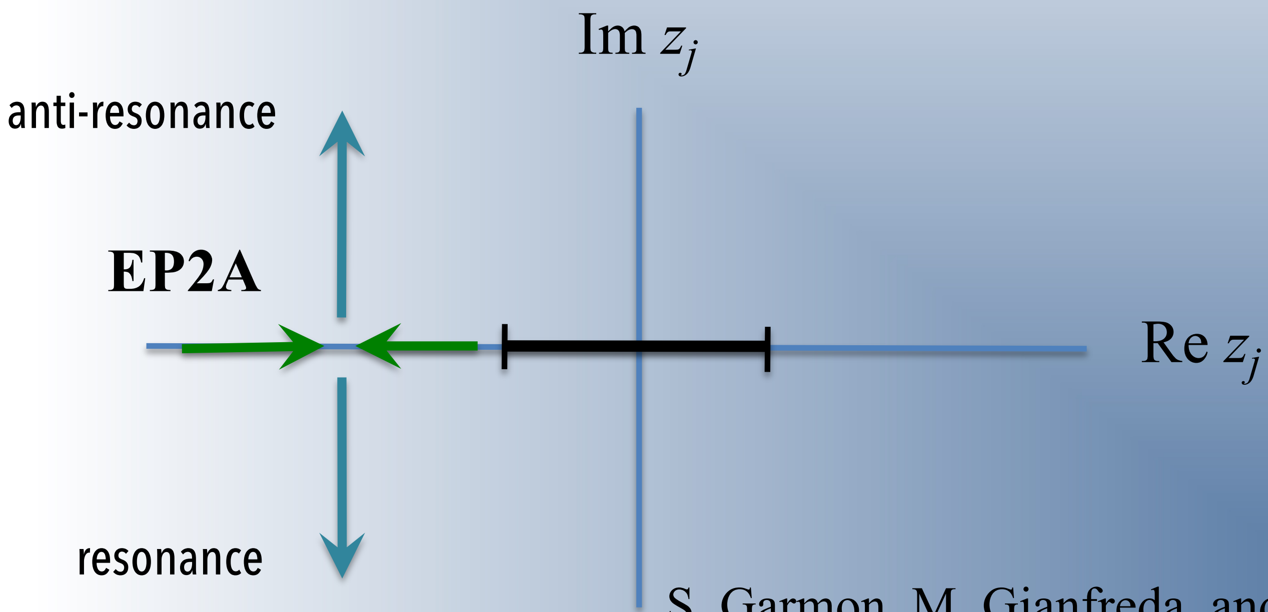
Hence, it's natural to expect resonance would originate near the threshold



Definition for A-type exceptional points

Recall: resonance appears due to interaction with the continuum

Indeed we find two virtual states coalesce before forming a resonance/anti-resonance pair.

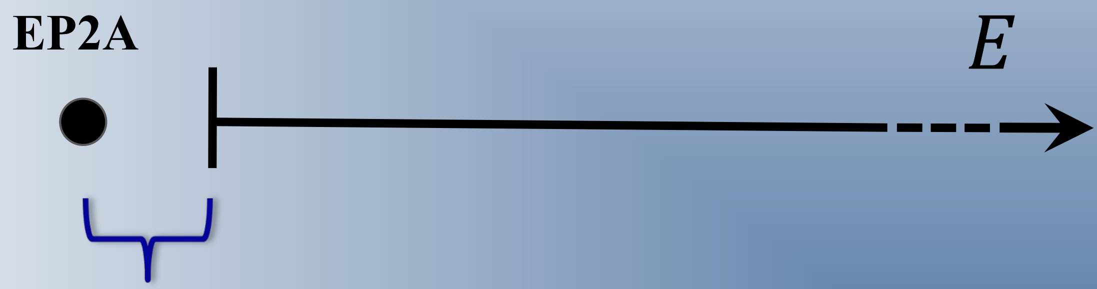


S. Garmon, M. Gianfreda, and N. Hatano, Phys. Rev. A **92**, 022125 (2015).

EP2A and non-Markovian dynamics

Due to proximity to the threshold, we find that non-Markovian dynamics are dramatically enhanced near the EP2A

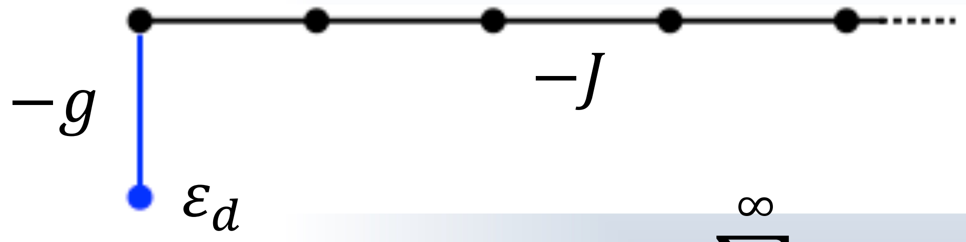
Further, the characteristic timescale for the dynamics is fixed by the gap between the exceptional point and the threshold



threshold gap: Δ_{EP}

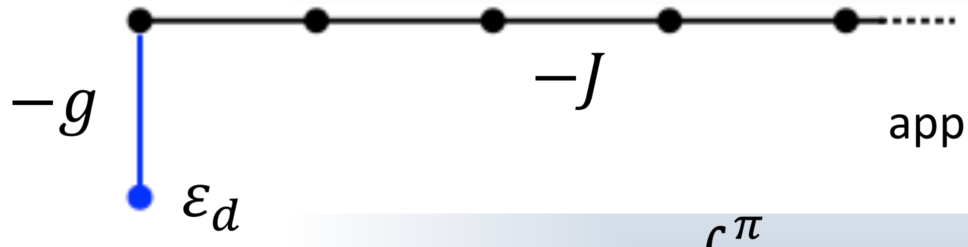
$$T_{EP} = \frac{1}{\Delta_{EP}}$$

EP2A simple model: Hamiltonian and spectrum



$$H = \varepsilon_d d^\dagger d - J \sum_{n=1}^{\infty} \left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n \right) - g\left(c_1^\dagger d + d^\dagger c_1 \right)$$

EP2A simple model: Hamiltonian and spectrum



apply half-range Fourier transform:

$$H = \varepsilon_d d^\dagger d + \int_0^\pi dk\, E_k c_k^\dagger c_k + g \int_0^\pi dk\, V_k \left(c_k^\dagger d + d^\dagger c_k\right)$$

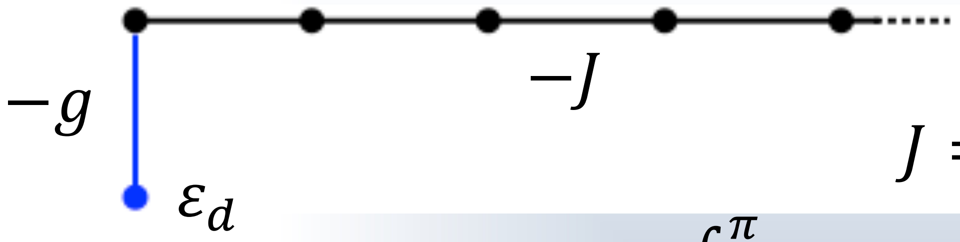
Continuum eigenvalue:

$$E_k = -2J \cos k$$

Two discrete eigenvalues:

$$\boxed{E_\pm = \frac{\varepsilon_d (2 - g^2) \pm g^2 \sqrt{\varepsilon_d^2 - 4(1 - g^2)}}{2(1 - g^2)}}$$

EP2A simple model: Hamiltonian and spectrum



$J = 1$

$$H = \varepsilon_d d^\dagger d + \int_0^\pi dk\, E_k c_k^\dagger c_k + g \int_0^\pi dk\, V_k \left(c_k^\dagger d + d^\dagger c_k \right)$$

Continuum eigenvalue: $E_k = -2\cos k \; = \lambda - 1/\lambda$

Two discrete eigenvalues:

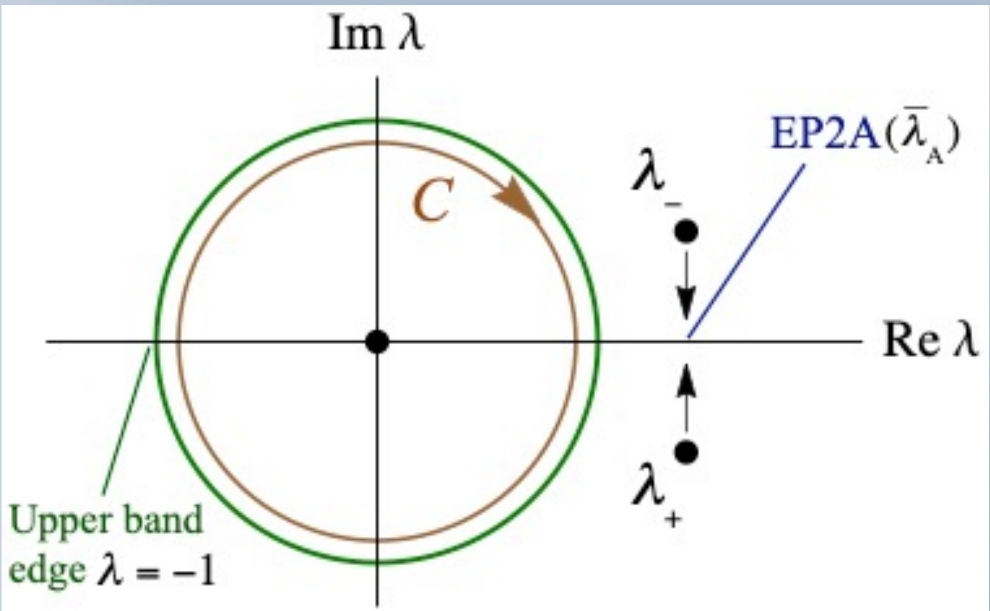
$$\boxed{\lambda_\pm = \frac{-\varepsilon_d \mp g^2 \sqrt{\varepsilon_d^2 - 4(1-g^2)}}{2(1-g^2)}}$$

We will reparameterize the eigenvalues: $\lambda = e^{ik}$

Survival probability formalism

In this formalism, survival probability at the impurity site is written $P(t) = |A(t)|^2$ with $A(t) = \langle d | e^{-iHt} | d \rangle =$

$$\frac{1}{2\pi i}\sum_{j=\pm}\int_C d\lambda\left(-\lambda+\frac{1}{\lambda}\right)exp\left[i\left(-\lambda+\frac{1}{\lambda}\right)t\right]\frac{\lambda_j}{\lambda-\lambda_j}\langle d|\psi_j\rangle$$



N. Hatano and G. Ordonez,
J. Math. Phys. **55**, 122106 (2014)

Survival probability: expansion near the EP2A

$$A(t) = \frac{1}{2\pi i}\sum_{j=\pm}\int_C d\lambda\left(-\lambda+\frac{1}{\lambda}\right) exp\left[i\left(-\lambda+\frac{1}{\lambda}\right)t\right]\frac{\lambda_j}{\lambda-\lambda_j}\langle d|\psi_j\rangle$$

To evaluate this near the threshold, we expand the norm and the eigenvalue: $\varepsilon_d = \varepsilon_{EP} + \delta$

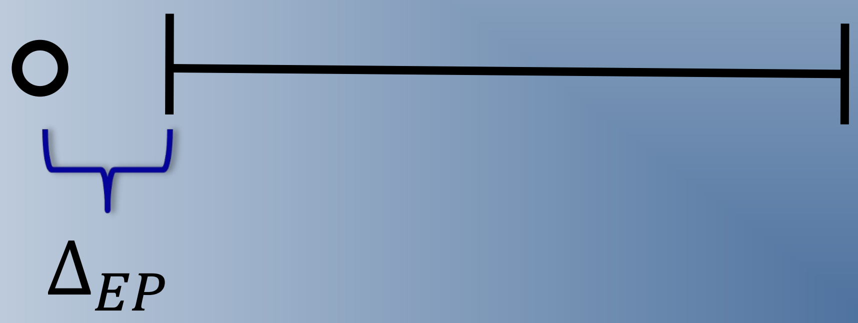
$$\langle d|\psi_\pm\rangle^2 \approx \frac{1}{2}\left(\pm\frac{i}{\sqrt{\lambda_{EP}\delta}}+1\right)+O\left(\delta^{1/2}\right)$$

$$\lambda_\pm \approx \lambda_{EP}\left(1 \mp i\sqrt{\lambda_{EP}\delta}+\frac{\lambda_{EP}\delta}{2}\right)+O\left(\delta^{3/2}\right)$$

Survival probability: EP2A near the band edge

When the EP2A is very close to the band edge (e.g. small coupling g) the survival amplitude follows

$$A(t) \approx e^{-iE_{EP}t}\left(1 + \sqrt{\frac{it\Delta_{EP}}{\pi}} + 2it\Delta_{EP}\right)$$



S. Garmon and G. Ordonez,
J. Math. Phys. **58**, 062101 (2017).

Survival probability: EP2A near the band edge

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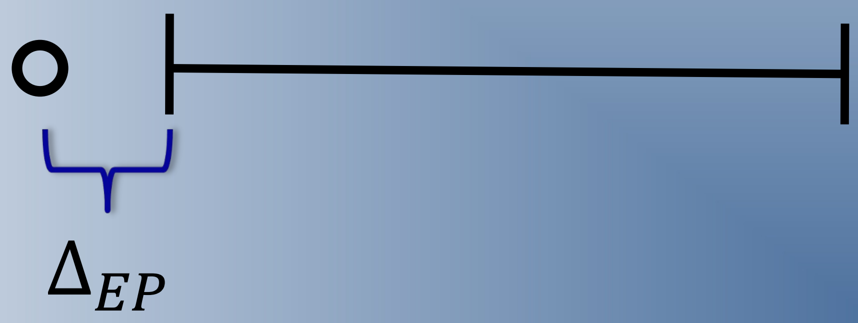
$$A(t) \approx e^{-iE_{EP}t}\left(1+\sqrt{\frac{it\Delta_{EP}}{\pi}}+2it\Delta_{EP}\right)$$

origin:

$$te^{-iE_{EP}t} \times \frac{1}{\sqrt{t}}$$

S. Garmon, et al, Forstchr Phys. **61**, 261 (2013)



Survival probability: EP2A near the band edge

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$$A(t) \approx e^{-iE_{EP}t}\left(1 + \sqrt{\frac{it\Delta_{EP}}{\pi}} + 2it\Delta_{EP}\right)$$

For the survival probability

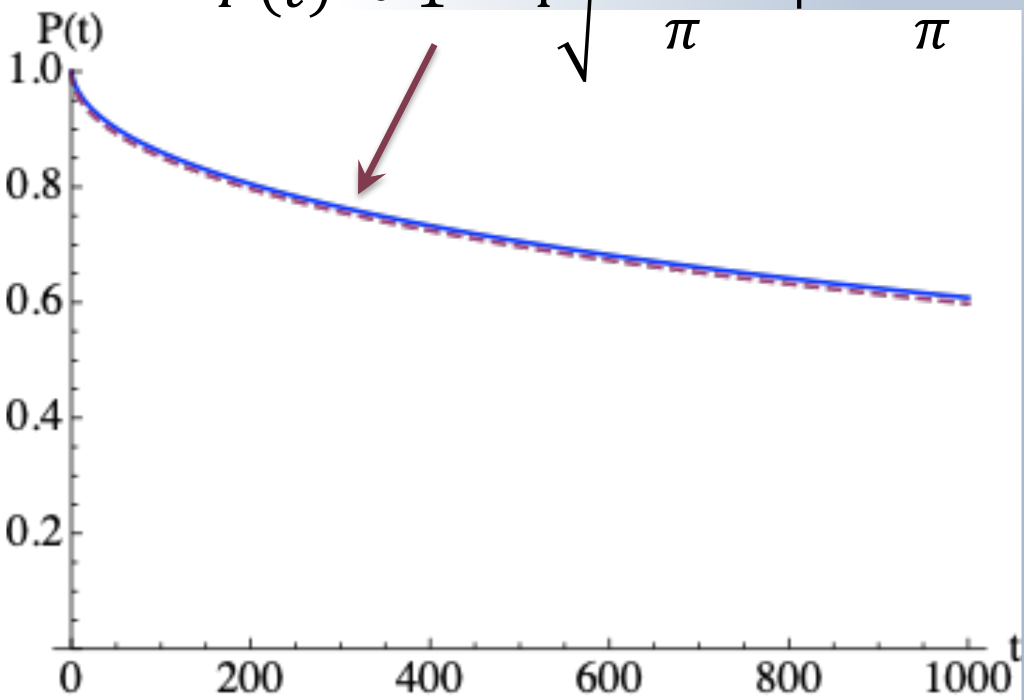
$$P(t) \approx 1 - 4\sqrt{\frac{2t\Delta_{EP}}{\pi}} + \frac{16t\Delta_{EP}}{\pi}$$

S. Garmon and G. Ordonez,
J. Math. Phys. **58**, 062101 (2017).

Dynamics for an EP2A near the threshold

Choose parameters close to the threshold:

$$P(t) \approx 1 - 4\sqrt{\frac{2t\Delta_{EP}}{\pi}} + \frac{16t\Delta_{EP}}{\pi}$$



$g = 0.1 \quad \varepsilon_d = -1.9879974$

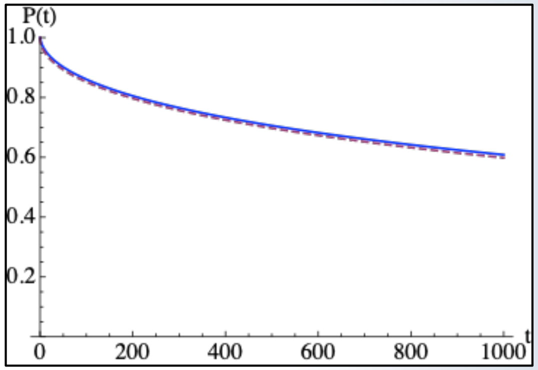
$-\varepsilon_d \approx -1.98997$

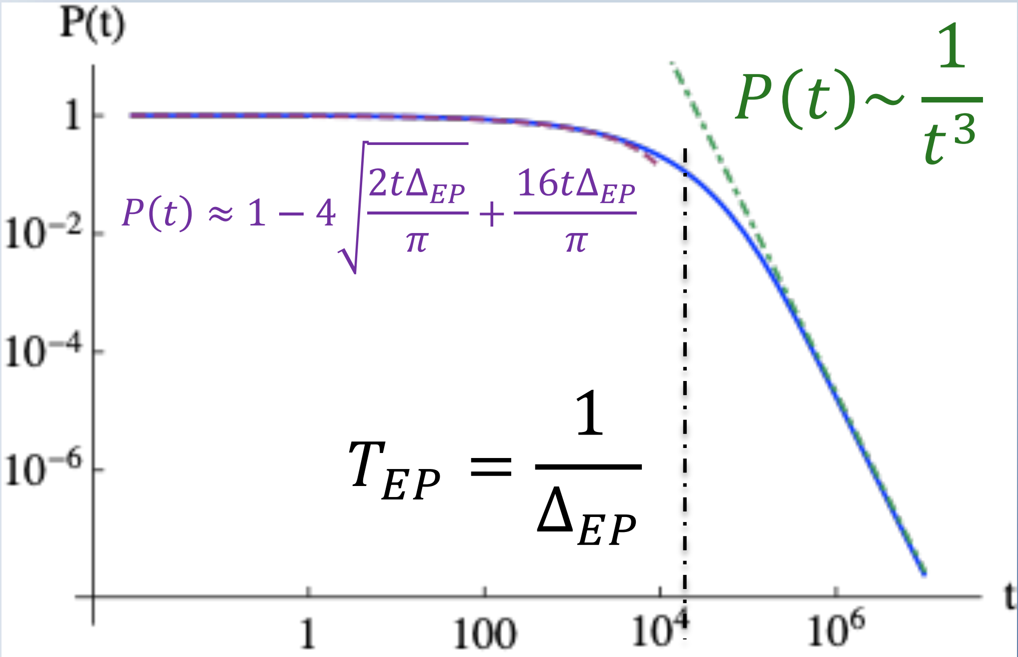
$\Delta_{EP} \approx 0.0100251$

S. Garmon and G. Ordonez,
J. Math. Phys. **58**, 062101 (2017)

Dynamics for an EP2A near the threshold

Choose parameters close to the threshold:





S. Garmon and G. Ordonez,
J. Math. Phys. **58**, 062101 (2017)

Exceptional point directly at threshold

One might consider if it is possible for an EP to appear directly at threshold.

Indeed, several examples appear in the literature…

W. D. Heiss and R. G. Nazmitdinov, Eur. Phys. J. D **63**, 369 (2011).

S. Tanaka, S. Garmon, K. Kanki, and T. Petrosky, Phys. Rev. A **94**, 022105 (2016).

S. Garmon, G. Ordonez, and N. Hatano, Phys. Rev. Research **3**, 033029 (2021)

Quantum emitter dynamics near a photonic band gap

An EP directly at threshold turns out to play a key role in a well-known problem in photonic band gap materials

JOURNAL OF MODERN OPTICS, 1994, VOL. 41, NO. 2, 353–384

Spontaneous and induced atomic decay in photonic band structures

A. G. KOFMAN, G. KURIZKI and B. SHERMAN

PHYSICAL REVIEW A VOLUME 50, NUMBER 2 AUGUST 1994

Spontaneous emission near the edge of a photonic band gap

Sajeev John and Tran Quang
Department of Physics, University of Toronto, 60 St. George Street, Toronto, Ontario, Canada M5S 1A7

(Received 19 November 1993; revised manuscript received 3 February 1994)

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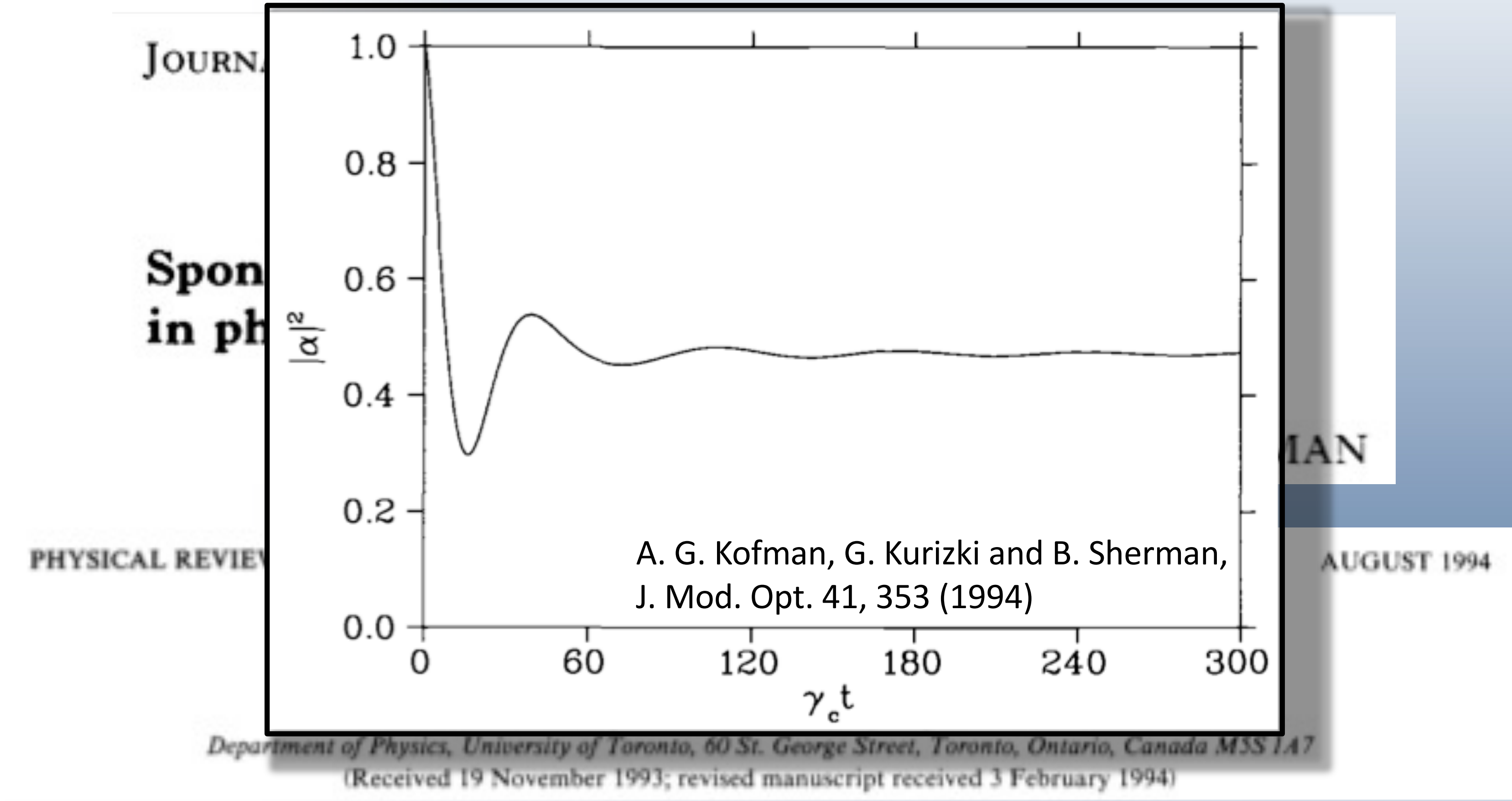


Generic 1-D model for EP at threshold

Let us consider a quantum emitter coupled to a generic 1-D continuum

$$H = \sum_{j=1}^{n} \varepsilon_j d_j^\dagger d_j + \int_{-\infty}^{\infty} dk\, E_k c_k^\dagger c_k + \sum_{j=1}^{n} \int_{-\pi}^{\pi} dk\, V_k \left(c_k^\dagger d_j + d_j^\dagger c_k \right)$$

Energy continuum E_k

Well defined range $E_k \in [E_{\mathrm{th}}, \infty]$

and density of states $\rho(E) = \dfrac{dk}{dE}$

S. Garmon, G. Ordonez, and N. Hatano,
Phys. Rev. Research **3**, 033029 (2021)

Two conditions for threshold EP

$$H = \sum_{j=1}^{n} \varepsilon_j d_j^\dagger d_j + \int_{-\infty}^{\infty} dk\, E_k c_k^\dagger c_k + \sum_{j=1}^{n} \int_{-\pi}^{\pi} dk\, V_k \left(c_k^\dagger d_j + d_j^\dagger c_k \right)$$

Assume the following two conditions are satisfied:

$$\rho(E) = \frac{dk}{dE} \sim \frac{1}{\sqrt{E_{th} - E}}$$

and

$V_k(E = E_{th})$ is non-singular

Then a generic, triple-level convergence will occur at threshold.

S. Garmon, G. Ordonez, and N. Hatano,
Phys. Rev. Research **3**, 033029 (2021)

Triple-level convergence at threshold

Assuming the previous two conditions are satisfied

$$E - \varepsilon_d = \Sigma(z) \sim g^2 \frac{\lambda(E)}{\sqrt{E_{th} - E}}$$

at $\varepsilon_d = E_{th}$

we can write

$$(E - E_{th})^3 \propto -g^4 \lambda^2(E)$$

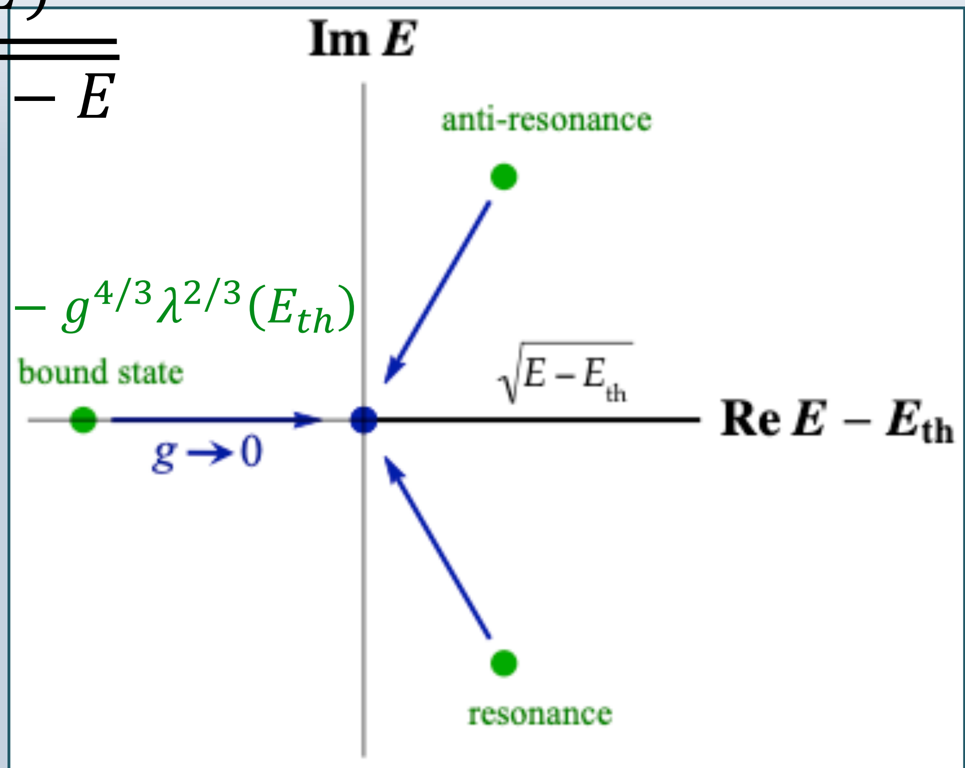
S. Garmon, G. Ordonez, and N. Hatano,
Phys. Rev. Research **3**, 033029 (2021)

Triple-level convergence at threshold

Assuming the previous two conditions are satisfied

$$E - \varepsilon_d = \Sigma(z) \sim g^2 \frac{\lambda(E)}{\sqrt{E_{th} - E}}$$

$$E_B \approx E_{th} - g^{4/3}\lambda^{2/3}(E_{th})$$



at $\varepsilon_d = E_{th}$

$$(E - E_{th})^3 \propto -g^4\lambda^2(E)$$

$$E_R \approx E_{th} - e^{-\pi i/3} g^{4/3}\lambda^{2/3}(E_{th})$$

S. Garmon, G. Ordonez, and N. Hatano,
Phys. Rev. Research **3**, 033029 (2021)

Survival probability near the anomalous EP

Resonance decay width is enhanced near the EP

$$\gamma \approx g^{4/3} \gg g^2$$

Influence on LDOS and absorption:

S. Tanaka, S. Garmon, and T. Petrosky, Phys. Rev. B **73**, 115340 (2006).

S. Garmon, H. Nakamura N. Hatano, and T. Petrosky, Phys. Rev. B **80**, 115318 (2009).

Instinctively, one might expect that the enhanced decay rate results in enhanced exponential decay.

However, near threshold one must also consider the non-Markovian dynamics from the branch-point effect.

L. A. Khalfin, Sov. Phys.-JETP, **6**, 1053 (1958).

Survival probability near the anomalous EP

During timescale $1 < t \ll 1/g^{4/3}$ we can show

$$A(t) \approx e^{2it}\left(1 - \frac{2g^2 t^{3/2}}{3\sqrt{\pi}} e^{-i\pi/4}\right)$$

The $t^{3/2}$ term results from two contributions:

$$\frac{1}{\sqrt{t}} \quad \times \quad t^2 e^{2it}$$

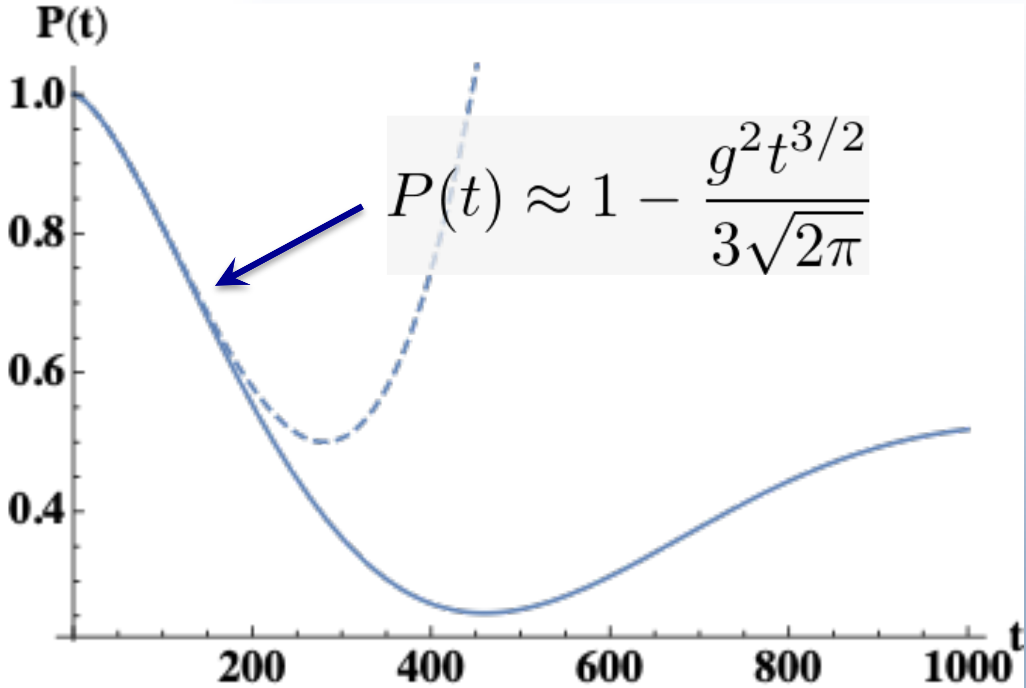
Exponential-power law factor from triple pole (effective EP3)

Inverse root factor results from the influence of the threshold

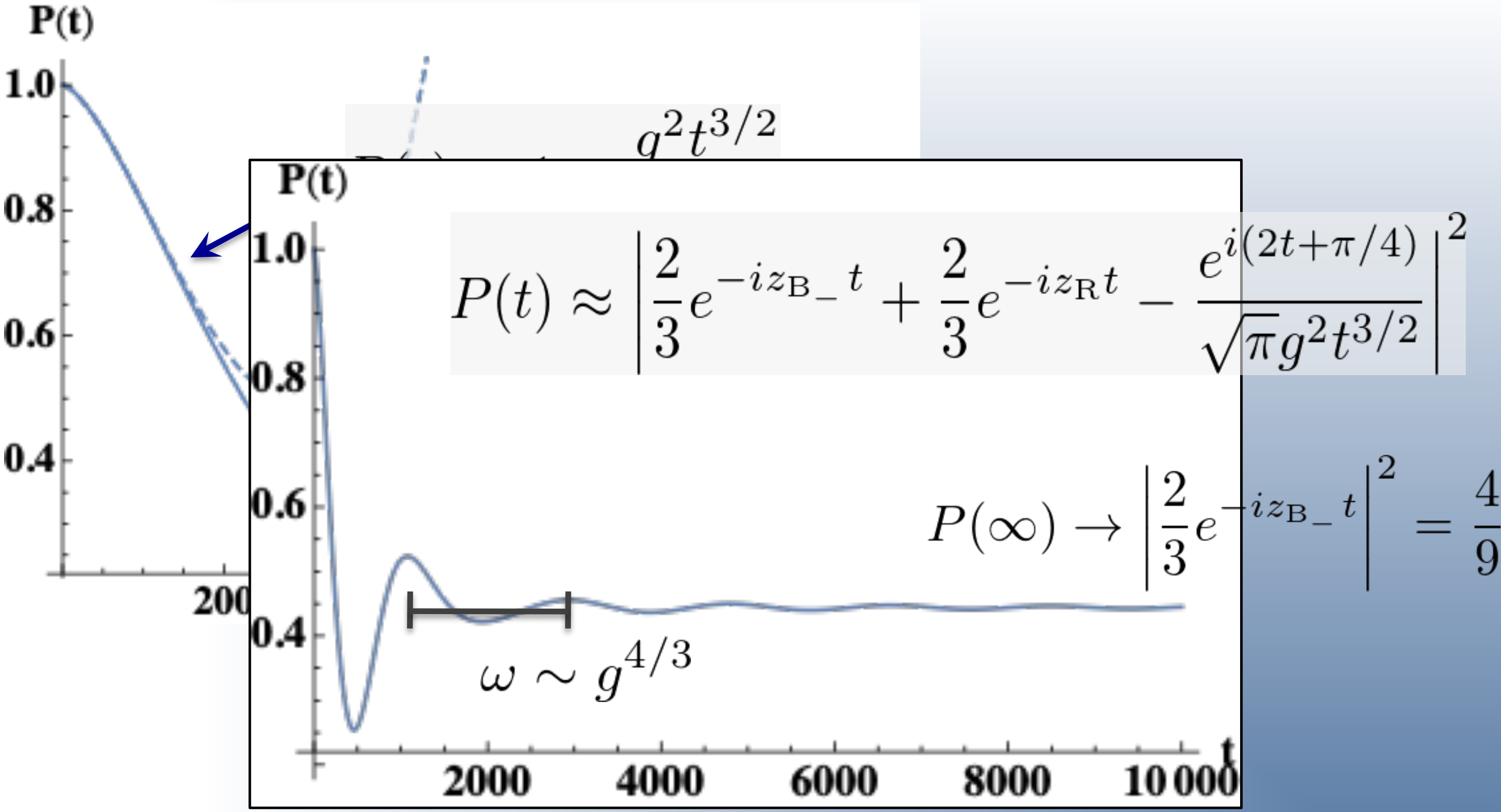
S. Garmon, T. Petrosky, L. Simine, and D. Segal, Fortschr. Phys. **61**, 261 (2013).

S. Garmon and G. Ordonez, J. Math. Phys. **58**, 062101 (2017).

Survival probability: non-Markovian Purcell effect



Survival probability: non-Markovian Purcell effect



While this picture has previously appeared in the literature, the EP analysis is missing

A. G. Kofman, G. Kurizki and B. Sherman, A. J. Mod. Opt. 41, 353 (1994)

Conclusions

- Exceptional points in open quantum systems

- Two coalescing resonances (EP2B) gave rise to a power law-exponential decay:

$$P(t) \sim (1 + C_1 t + C_2 t^2) e^{-\gamma t}$$

- Resonance/anti-resonance pair (EP2A) dynamics were fully non-Markovian
  - Near the threshold we saw an evolution of the form

$$P(t) \approx 1 - C\, t^{1/2}$$

S. Garmon, and G. Ordonez, J. Math. Phys. **58**, 062101 (2017)

S. Garmon, T. Sawada, K. Noba, and G. Ordonez,
J. Phys.: Conf. Ser. **2038**, 012011 (2021).