Non-exponential decay near the continuum threshold

Part 1: bound states and anti-bound states

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Osaka Metropolitan University

S. G., T. Petrosky, L. Simine, D. Segal,

Fortschr. Phys. **61**, 261 (2013)

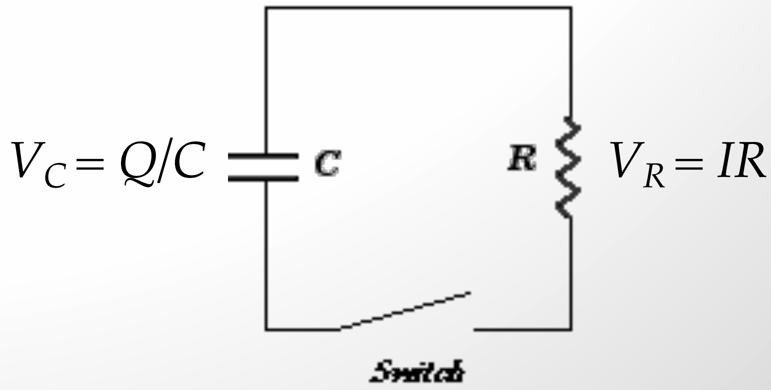
Stat&QuantPhys Winter School 2024

21-22 February

Overview:

Exponential decay is ubiquitous in nature.

➢ Classical physics: ***RC* circuit**



Kirchoff's loop rule:

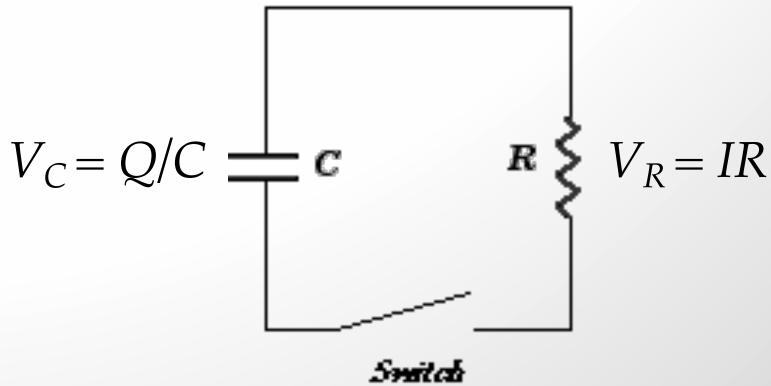
$$IR + \frac{Q}{C} = 0$$

Overview:

Exponential decay is ubiquitous in nature.

- Classical physics: ***RC* circuit**

Kirchoff's loop rule:



$$\frac{dQ}{dt} + \frac{1}{RC}Q = 0$$

$$\Longrightarrow \quad Q = Q_0 e^{-\frac{t}{t_\gamma}}$$

exponential lifetime:

$$t_\gamma = RC$$

Overview:

Exponential decay is ubiquitous in nature.

- Classical physics: **beer foam**

$$h(t) = h_0 e^{-\frac{t}{\tau}}$$

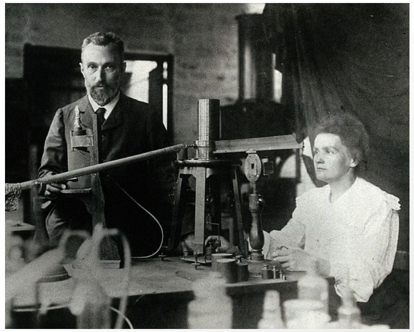
τ - beer-dependent parameter

A Leike, Eur. J. Phys. **23**, 21 (2002).

Overview:

Exponential decay is ubiquitous in nature.

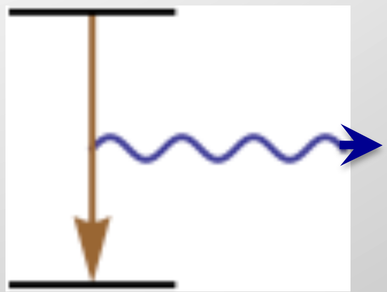
➢ <u>Quantum physics</u>: **nuclear decay**



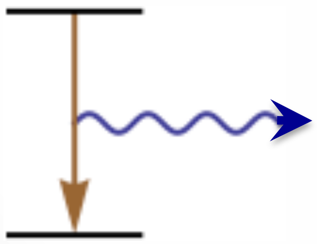
$$h(t) = h_0 e^{-\Gamma t}$$

Γ - decay width (inverse lifetime)

➢ **atomic relaxation**

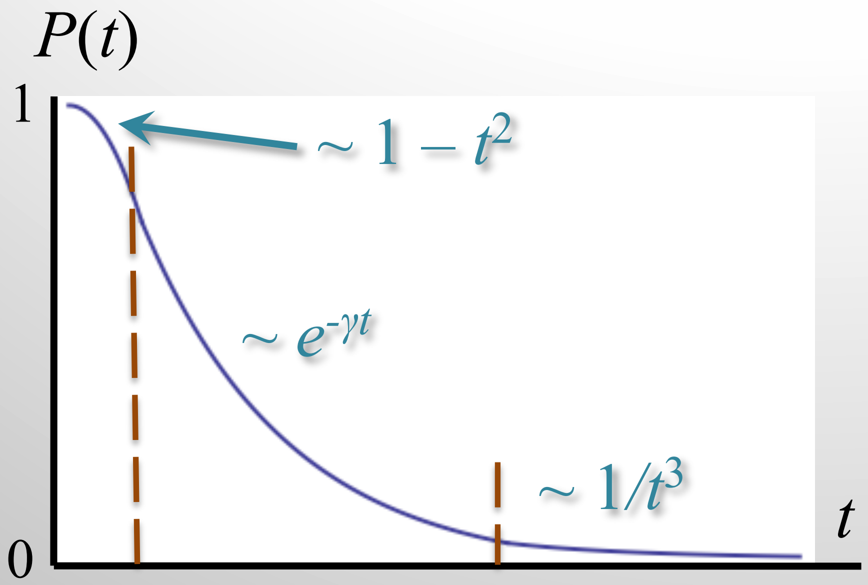


Overview:



Quantum physics: deviations from exponential decay always exist in QM

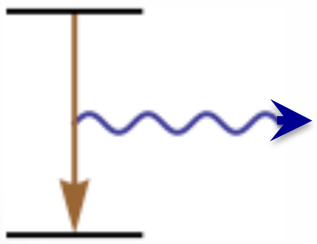
- Deviations from exponential decay exist at least on **extremely short** and **extremely long** time scales



Survival Probability:

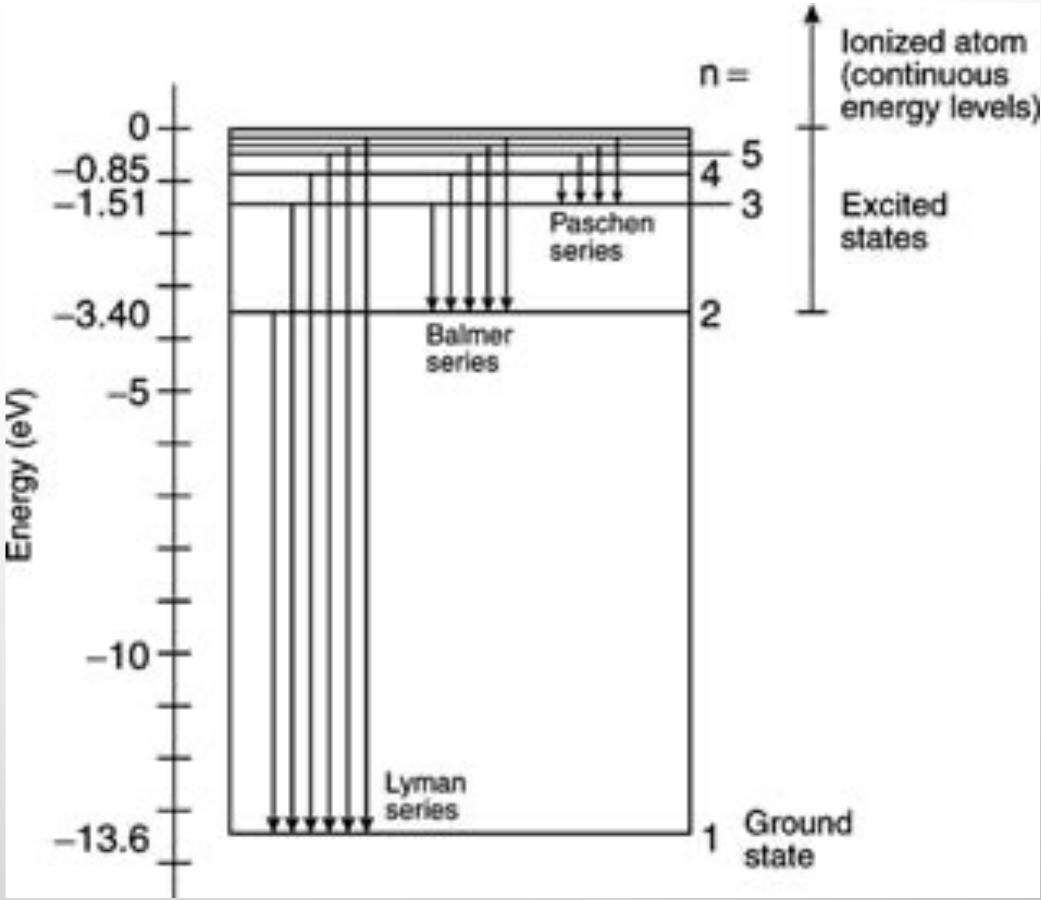
$$P(t) = \left|\langle\psi_0|e^{-iHt}|\psi_0\rangle\right|^2$$

Overview:

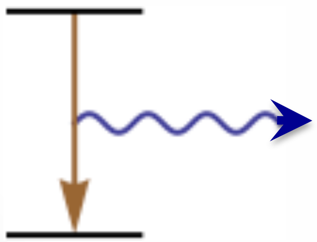


Deviations from exponential decay result from influence of the **continuum threshold**

Hydrogen spectrum:

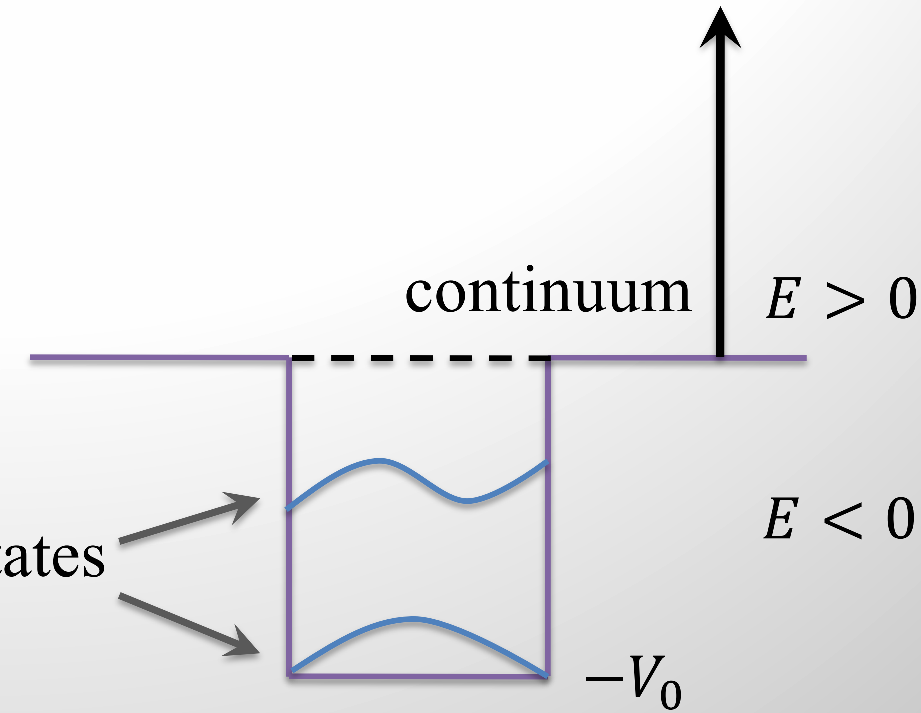


Overview:

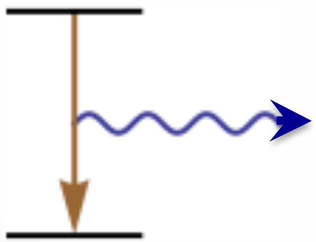


Bound state influence on the long-time deviations

Good ole′ finite square well:



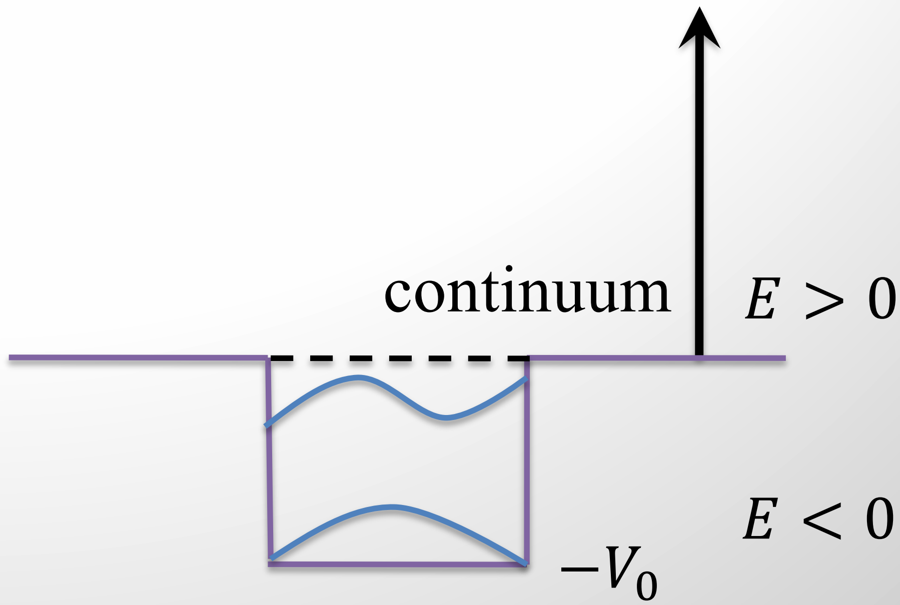
Overview:



<u>Bound state</u> influence on the long-time deviations

Good ole′ finite square well:

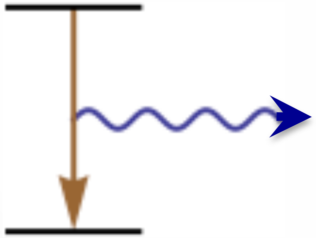
As well becomes more shallow bound state eventually vanishes at continuum threshold (**delocalization**)



<u>Bound state becomes an **anti-bound state**</u>

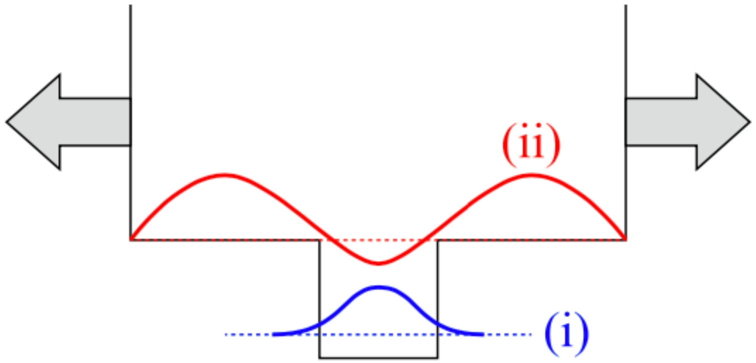
(virtual bound state)

Overview:



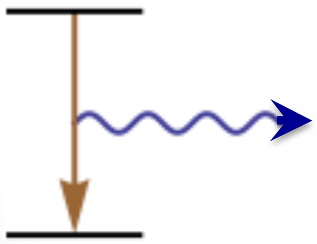
What is an **anti-bound state?**

(ii) bound state
(i) bound state



N. Hatano and G. Ordonez,
J. Math. Phys. **55**, 122106 (2014).

Overview:



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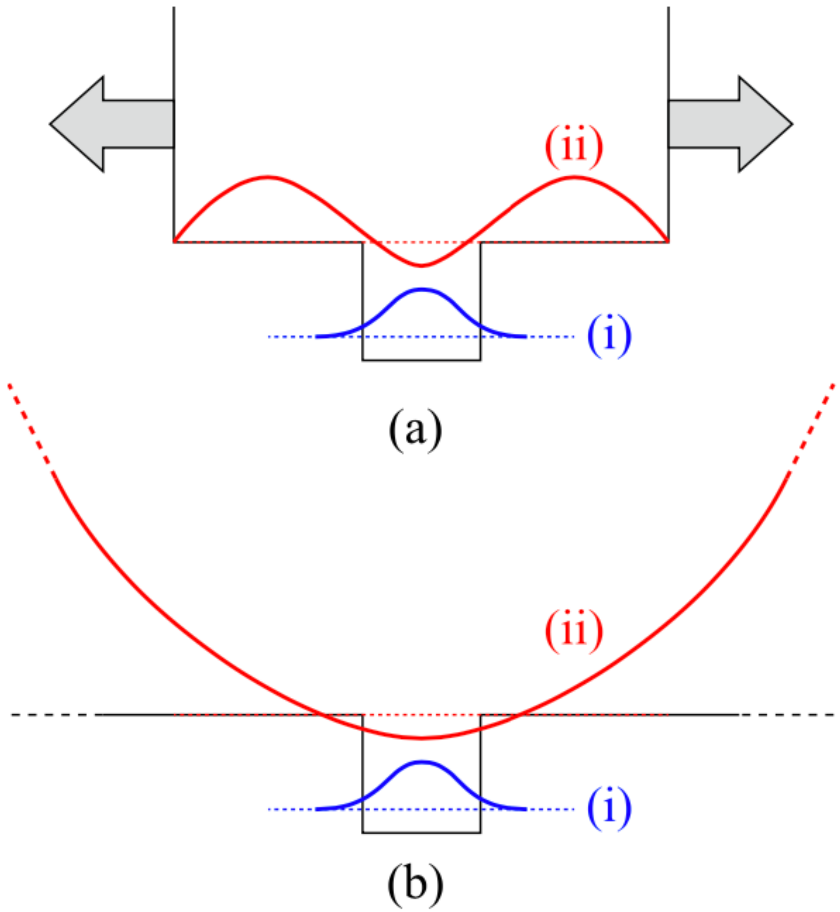
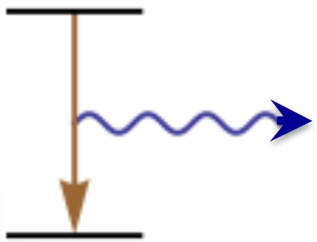


(ii) anti-bound state

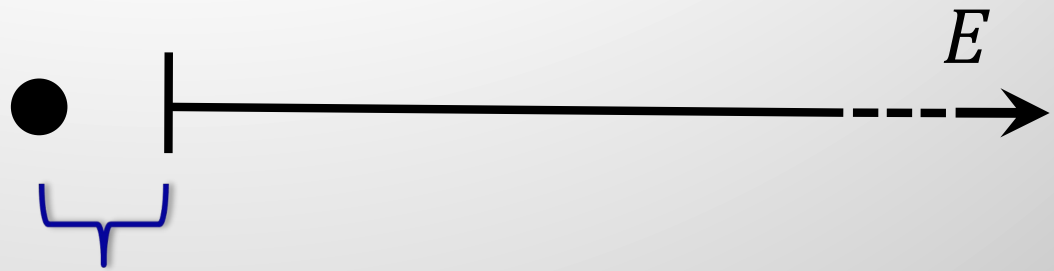
(i) bound state

Overview:



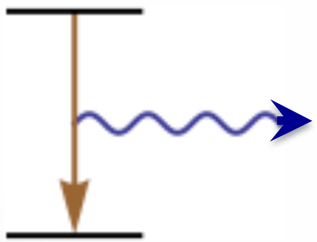
Deviations from exponential decay in quantum mechanics

- Long time deviations result from the **continuum threshold**
- **Question:** What happens when **bound state** appears near the continuum threshold?



threshold gap: Δ_Q

Overview:



Deviations from exponential decay in quantum mechanics

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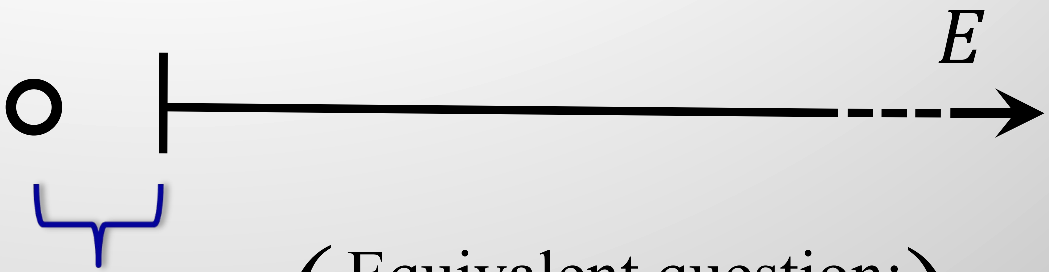
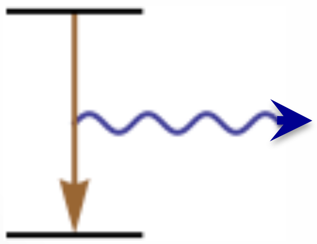


threshold gap: Δ_Q

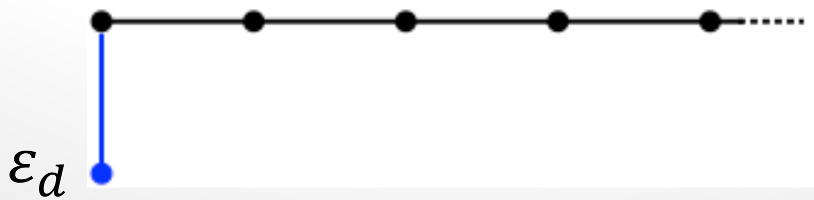
(Equivalent question: **anti-bound state**)

Overview:



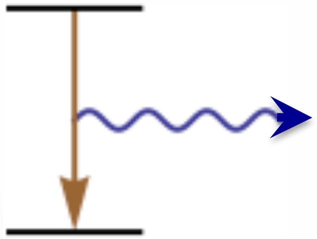
Deviations from exponential decay in quantum mechanics

Nanowire model: semi-infinite chain with end-point impurity



- Certain system parameters: **exponential decay vanishes completely**

Overview:



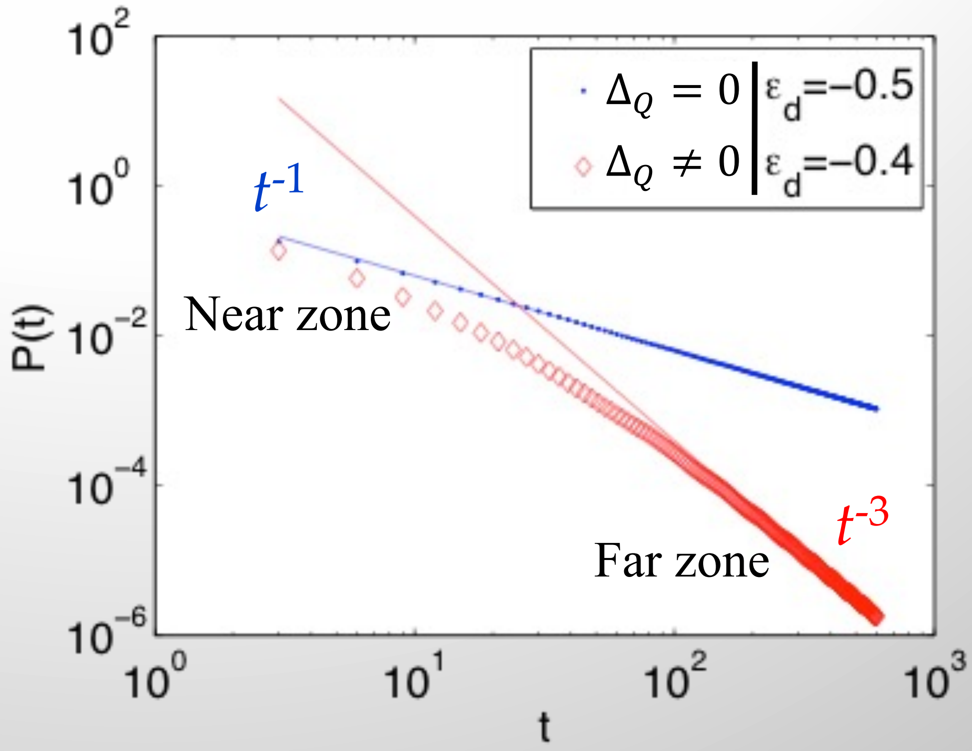
Nanowire model: system dynamics are fixed according to timescale $(\Delta_Q)^{-1}$

Long time 'near zone'

$$1 \ll t \ll \frac{1}{\Delta_Q}$$

Long time 'far zone'

$$\frac{1}{\Delta_Q} \ll t$$



Outline:

- Nanowire fabrication methods
  - Prototype model
- Open quantum systems - definition
  - Non-exponential decay - details
- Long-time deviations
  - Prototype model: t^{-1} --> t^{-3}
  - Comparison model

Nanowire fabrication: top-down methods

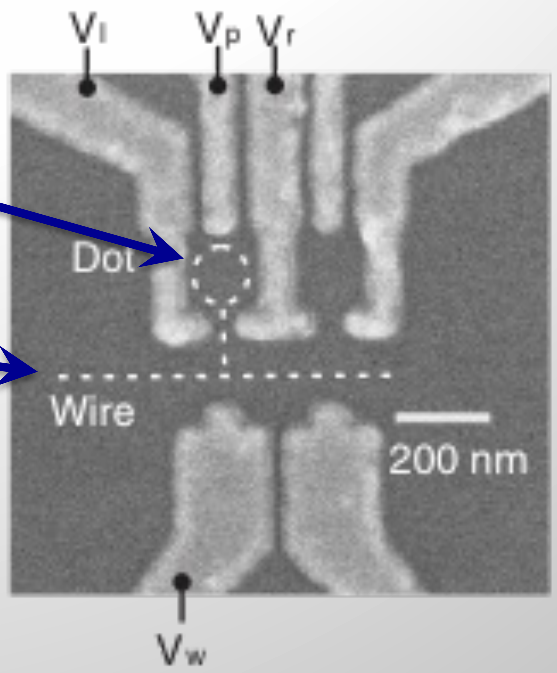
Effective 1-D systems with impurities can be constructed at the nanoscale by various methods:

- ‘Top-down’ vs. ‘bottom-up’ methods

Top-down approach: quantum dot-quantum wire etched into AlGaAs/GaAs heterointerface:

Localized quantum dot

Effective 1-D quantum wire



T. Otsuka, E. Abe, S. Katsumoto, Y. Iye, G. L. Khym, and K. King, J. Phys. Soc. Japan **76**, 084706 (2007).

Nanowire fabrication: bottom-up methods

Bottom-up approach: structures are created rather than simply modified at the nanoscale.

Laser-assisted catalytic growth of nanowire superlattices:

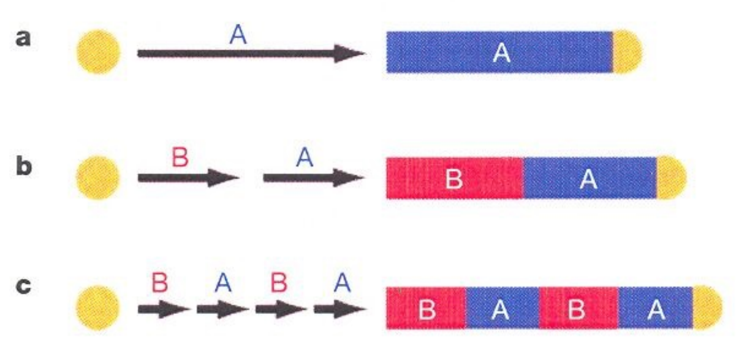
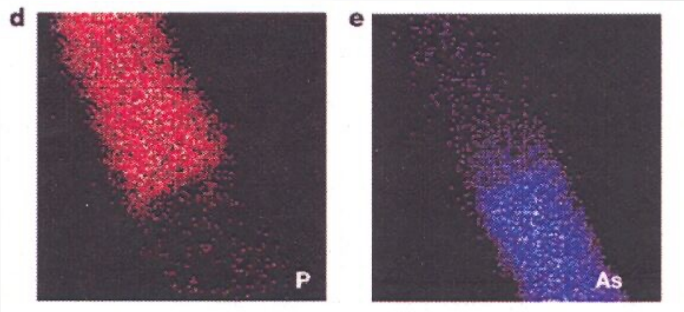


Figure 1 Synthesis of nanowire superlattices. **a**, A nanocluster catalyst (shown gold) nucleates and directs one-dimensional semiconductor nanowire (blue) growth with the catalyst remaining at the terminus of the nanowire. **b**, Upon completion of the first growth step, a different material (red) can be grown from the end of the nanowire. **c**, Repetition of steps **a** and **b** leads to a compositional superlattice within a single nanowire.

TEM images show clear localization in layers:



Gallium Phosphide layer

Gallium Arsenide layer

M. S. Gudiksen, L. J. Lauhon, J. Wang, D. C. Smith, and C. M. Lieber, Nature **415**, 617 (2002)

Prototype Model

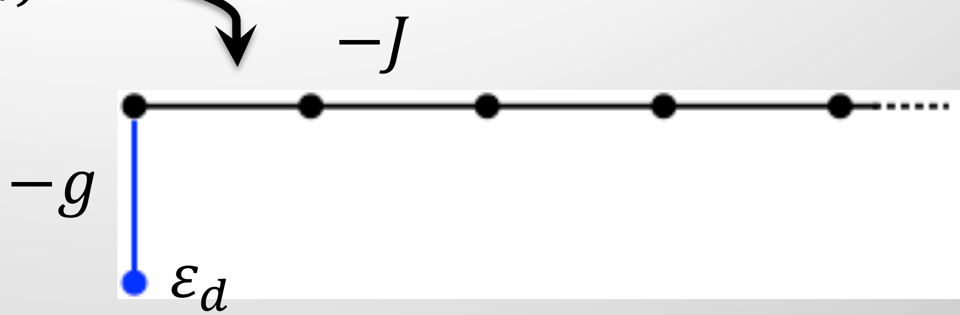
<u>Prototype Model</u>: simplified chain model with end-point impurity

tight-binding chain:

$$H = -J \sum_{i=1}^{\infty} \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right)$$

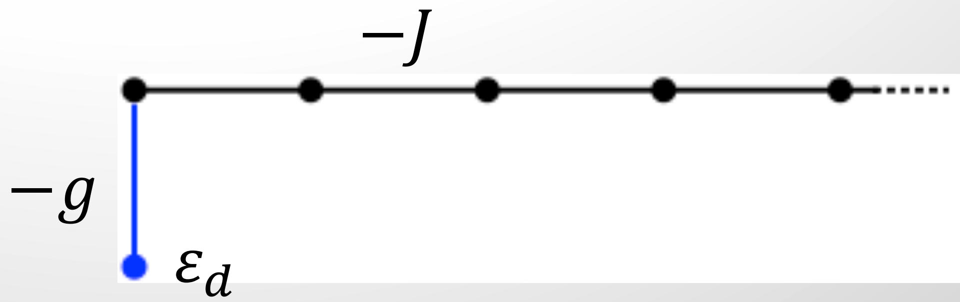
we will use (in this talk):

$$J = g = 1/2$$



Open Quantum Systems

Our prototype model is an example of an **open quantum system** (**OQS**).

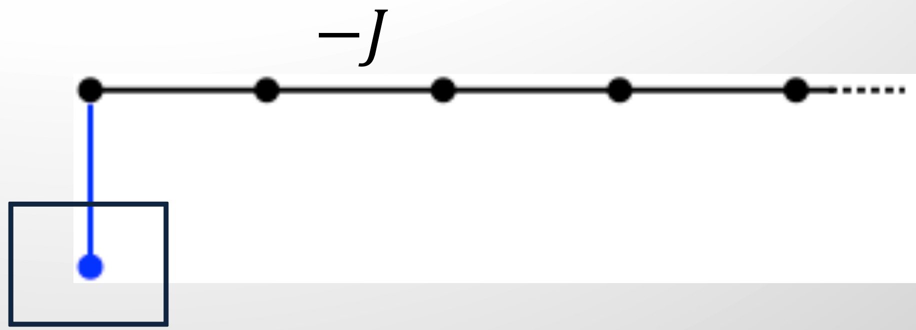


Open Quantum Systems

Our prototype model is an example of an **open quantum system** (**OQS**). Open quantum system consists of:

- Discrete system H_{D}

discrete component: H_{D} →

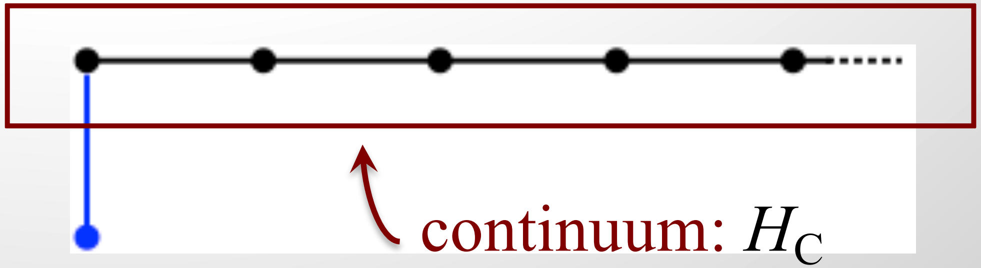


$$H_D = \varepsilon_d d^\dagger d$$

Open Quantum Systems

Our prototype model is an example of an **open quantum system** (**OQS**). Open quantum system consists of:

- Discrete system H_D
- Embedded in a larger system (continuum) H_C

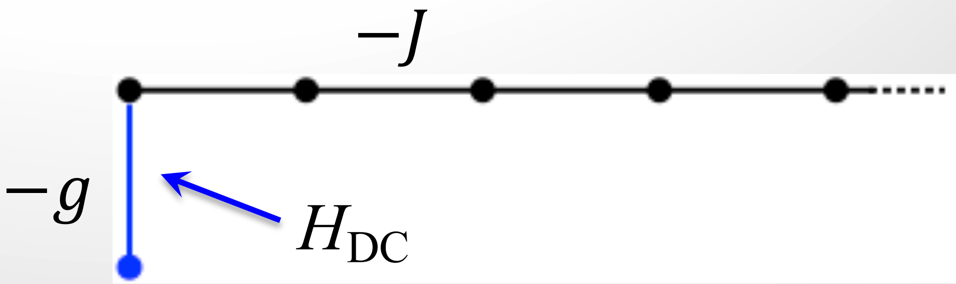


$$H_C = -\frac{1}{2}\sum_{n=1}^{\infty}\left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right)$$

Open Quantum Systems

Our prototype model is an example of an **open quantum system** (**OQS**). Open quantum system consists of:

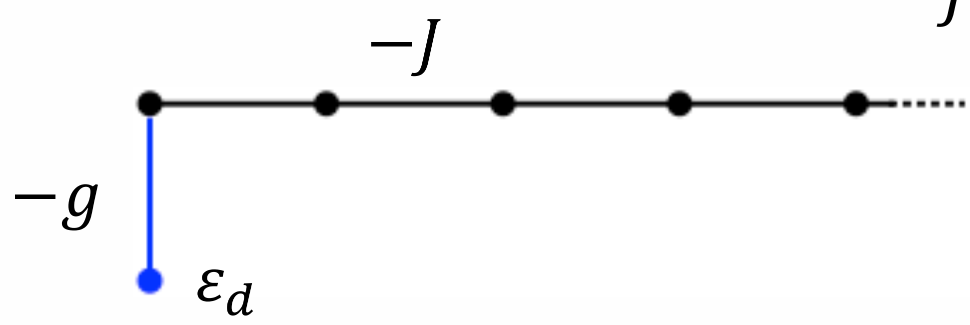
- Discrete system H_D
- Embedded in a larger system (continuum) H_C
- Coupled via H_{DC}



$$H_{DC} = \frac{1}{2}\left(c_1^\dagger d + d^\dagger c_1\right)$$

$J = g = 1/2$

Prototype model: Full Hamiltonian



$J = g = 1/2$

$$H = H_D + H_C + H_{DC}$$

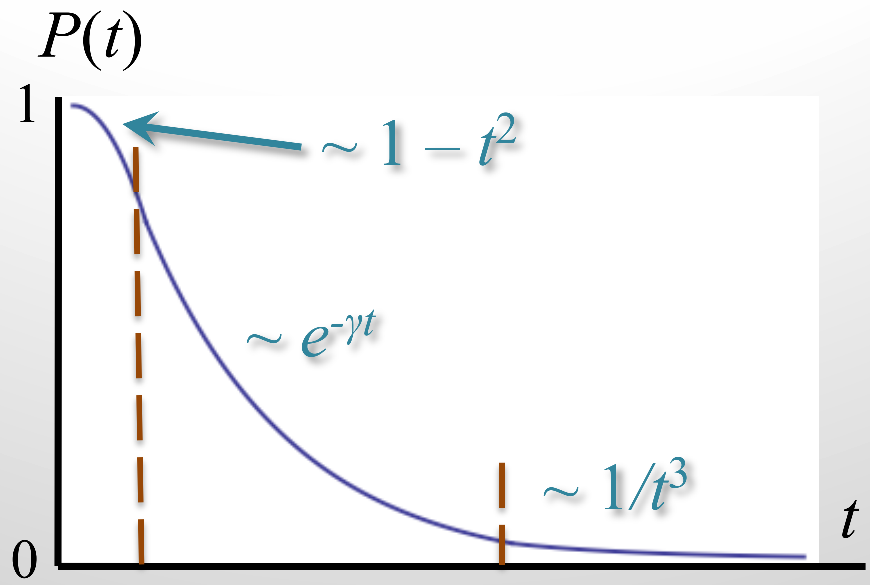
$$H = \varepsilon_d d^\dagger d - \frac{1}{2}\sum_{n=1}^{\infty}\left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right) - \frac{1}{2}\left(c_1^\dagger d + d^\dagger c_1\right)$$

Deviations from exponential decay

Quantum systems yield deviations from exponential decay on *at least* very short and very long time scales:

C. B. Chiu, B. Misra, and E. C. G. Sudarshan, Phys. Rev. D **16**, 520 (1977).

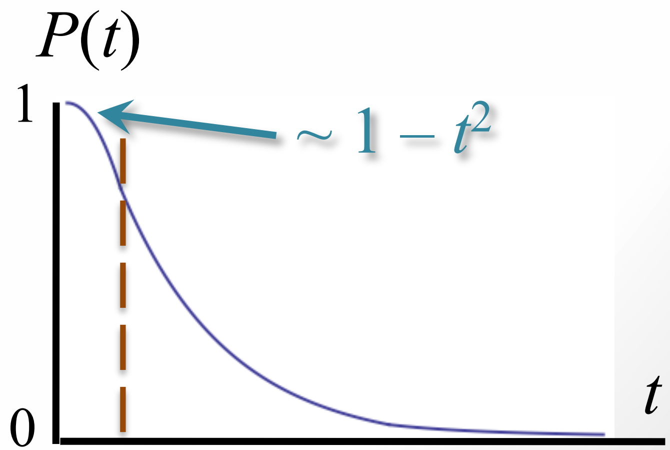
J. Martorell, J. G. Muga, and D. W. L. Spring, Lect. Notes. Phys. **789**, 239 (2009).



Short-time deviations from exponential decay

Short time scales typically give rise to parabolic decay:

$$P(t) \sim 1 - t^2$$



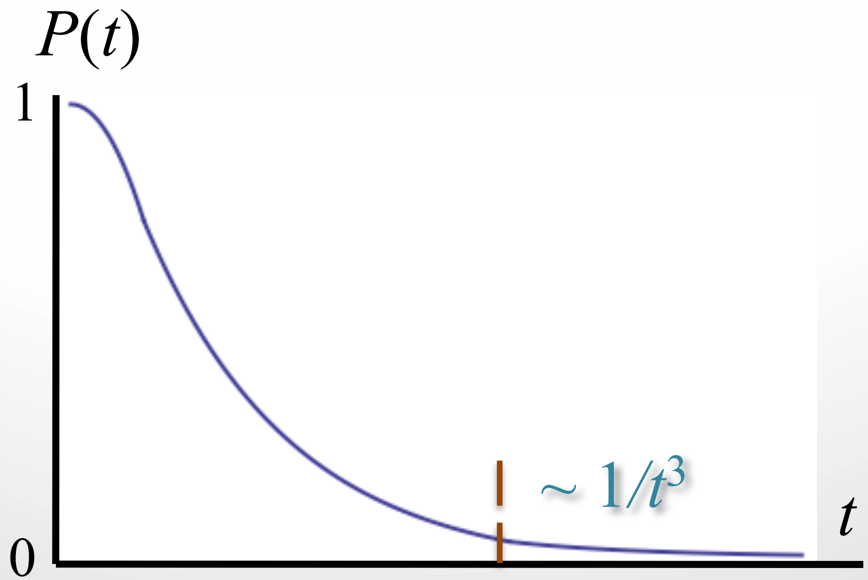
- Quantum Zeno effect ➔ repeated measurements result in decelerated decay
- quantum anti-Zeno effect ➔ accelerated decay
- Experimental confirmation – ultra-cold Na atoms initially trapped in accelerating optical potential:

S. R. Wilkinson, *et al*, Nature (London) **387**, 575 (1997).

M. C. Fischer, *et al*, Phys. Rev. Lett. **87**, 040402 (2001).

Long-time deviations from exponential decay

Long time deviations intimately connected with the continuum threshold.



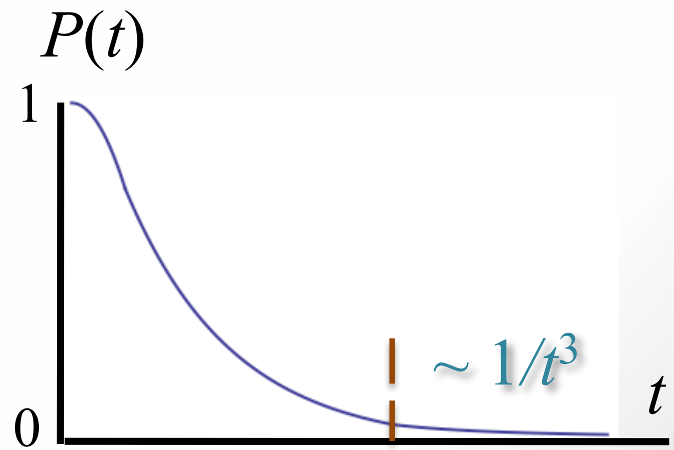
- Mathematically proven for quantum systems:

L. A. Khalfin, Soc. Phys. JETP **6**, 1053 (1958).

M. N. Hack, Phys. Lett. A **90**, 220 (1982).

Long-time deviations from exponential decay

- Typically gives rise to inverse power law decay
- Typical asymptotic decay law: $P(t) \sim t^{-3}$



- Experimental verification: luminescence decay properties of dissolved organic materials following laser excitation:

C. Rothe, et al, Phys. Rev. Lett. **96**, 163601 (2006).

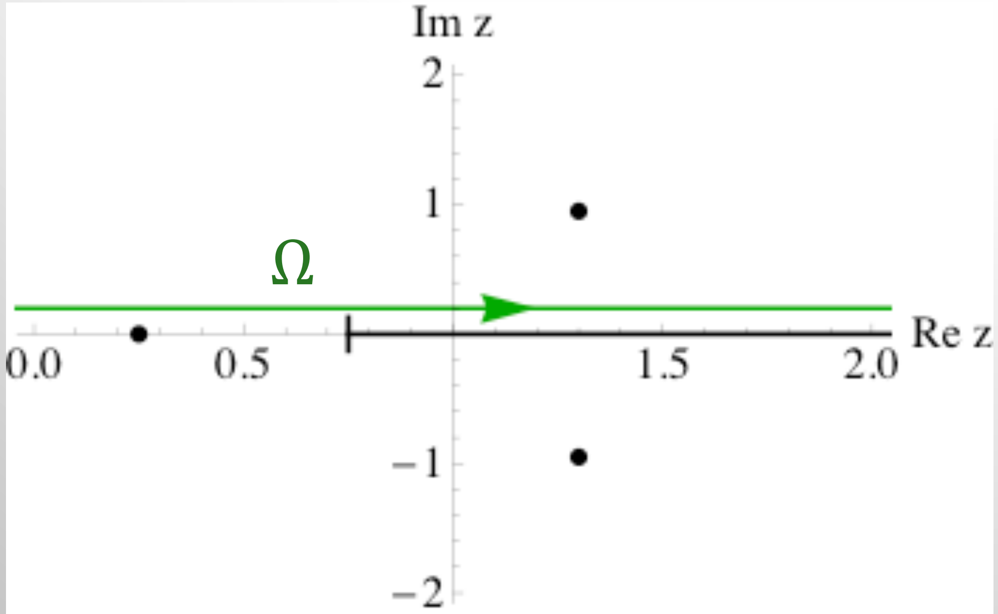
- Array of single-mode optical waveguides:

A. Crespi, et al, Phys. Rev. Lett. **122**, 130401 (2019).

Formalism: survival probability for an initially prepared state

Survival probability: $P(t) = |A(t)|^2$

$$A(t) = \langle \psi_0 | e^{-iHt} | \psi_0 \rangle = \frac{1}{2\pi i} \int_{\Omega} e^{-izt} \left\langle \psi_0 \middle| \frac{1}{z - H} \middle| \psi_0 \right\rangle$$



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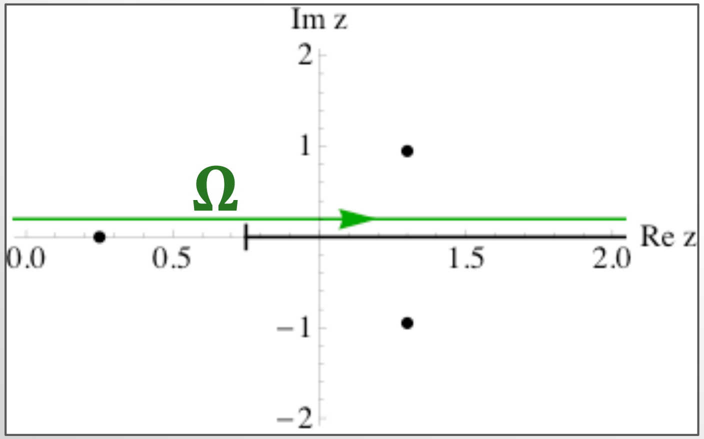
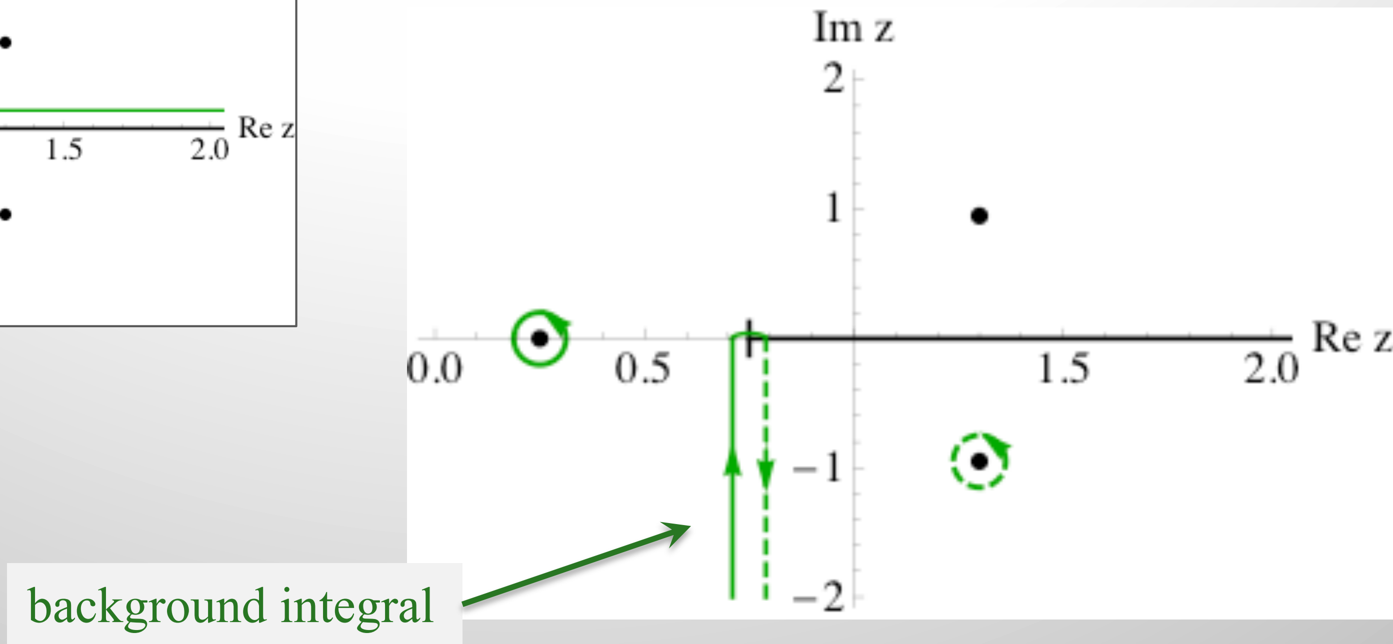
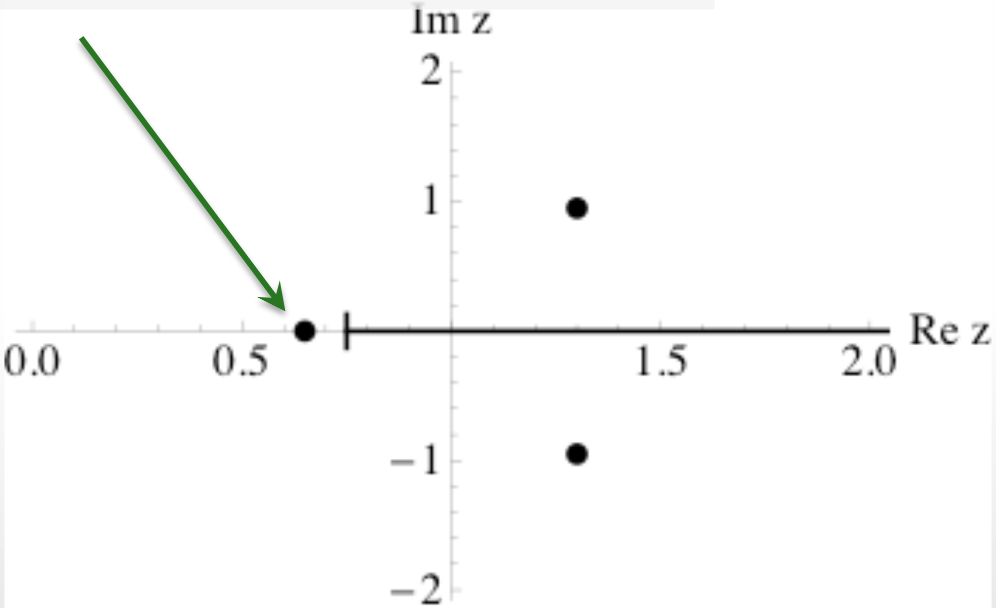




Physical motivations: bound state at threshold

question: what happens as bound state approaches continuum threshold?



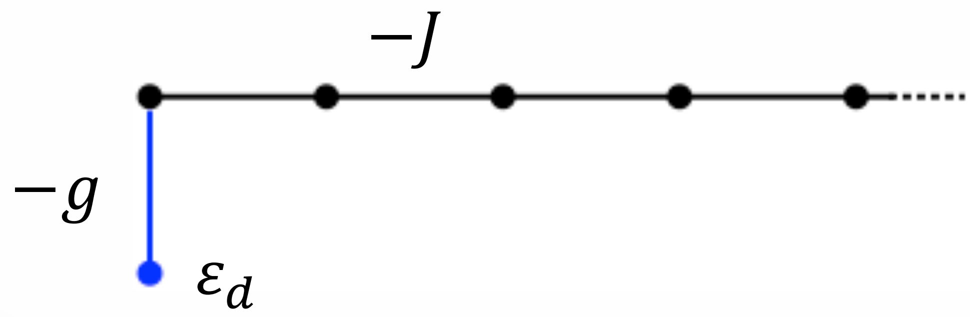
Answer: long-time non-exponential decay effects enhanced as bound state approaches threshold.

Note that bound state transitions to virtual bound state (2^{nd} sheet) after reaching threshold

Prototype model: continuum and discrete spectra

Return to prototype:

$J = g = 1/2$



$$H = H_D + H_C + H_{DC}$$

$$H = \varepsilon_d d^\dagger d - \frac{1}{2}\sum_{n=1}^{\infty}\left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right) - \frac{1}{2}\left(c_1^\dagger d + d^\dagger c_1\right)$$

Prototype model: continuum spectrum

take continuum limit, introduce half-chain Fourier series:

$$H = \varepsilon_d d^\dagger d - \frac{1}{2}\sum_{n=1}^{\infty}\left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right) - \frac{1}{2}\left(c_1^\dagger d + d^\dagger c_1\right)$$

$(N \to \infty)$

Prototype model: continuum spectrum

take continuum limit, introduce half-chain Fourier series:

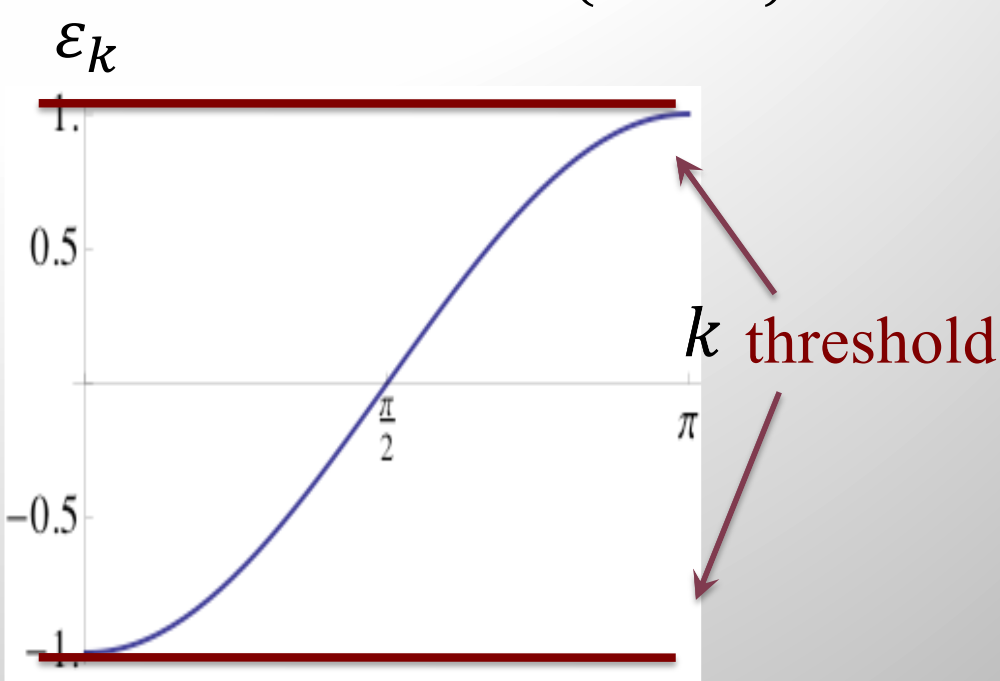
$$H = \varepsilon_d d^\dagger d + \int_0^\pi dk\, \varepsilon_k c_k^\dagger c_k + \int_0^\pi dk\, V_k \left(c_k^\dagger d + d^\dagger c_k \right)$$

$(N \to \infty)$

Continuum:

$k \in [0, \pi]$ on $\varepsilon_k = -\cos k$

$$V_k = -\frac{\cos k}{\sqrt{\pi}}$$



Prototype model: discrete spectrum

Obtain the discrete spectrum from:

$$\left\langle d \left| \frac{1}{z-H} \right| d \right\rangle = \frac{1}{z - \varepsilon_d - \Sigma(z)}$$

yields linear polynomial

Discrete dispersion relation:

$$z - \varepsilon_d - \Sigma(z) = z - \varepsilon_d - \frac{1}{2}\left(z - \sqrt{z^2 - 1}\right) = 0$$

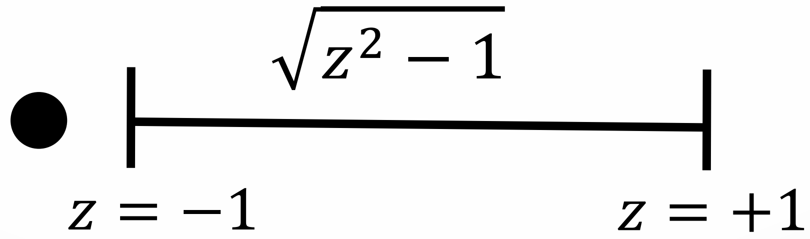
yields solution:

$$z_L = \varepsilon_d + \frac{1}{4\varepsilon_d}$$

however, let's examine the dispersion more carefully…

Prototype model:
Bound state absorption into continuum

We re-write the dispersion slightly: $z - 2\varepsilon_d + \sqrt{z^2 - 1} = 0$



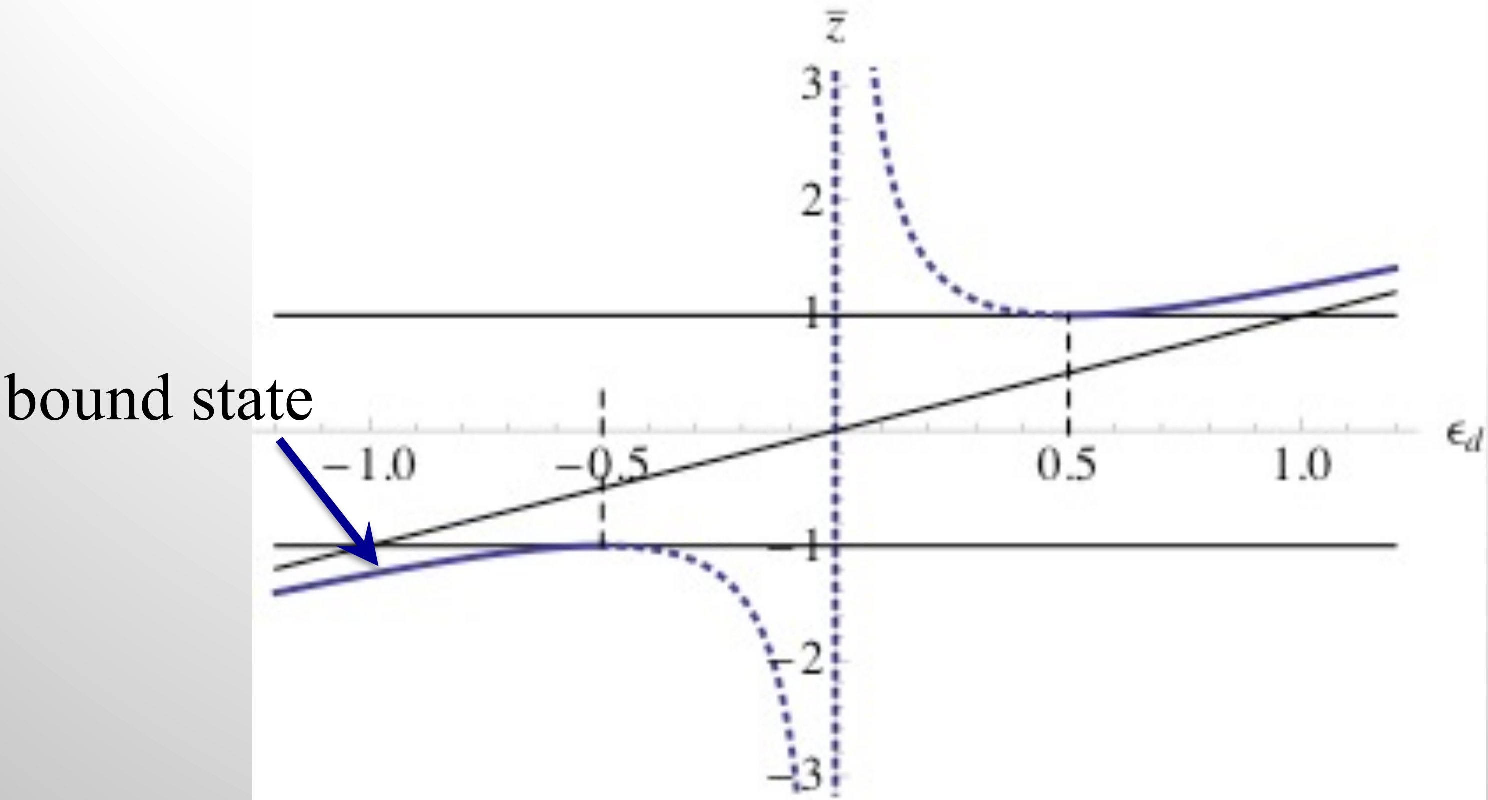
We see the root vanishes for $z = \pm 1$, which occurs at:

$$\varepsilon_d = \pm \frac{1}{2}$$

These are the points where solution crosses from one Riemann sheet into the other (localization/delocalization).

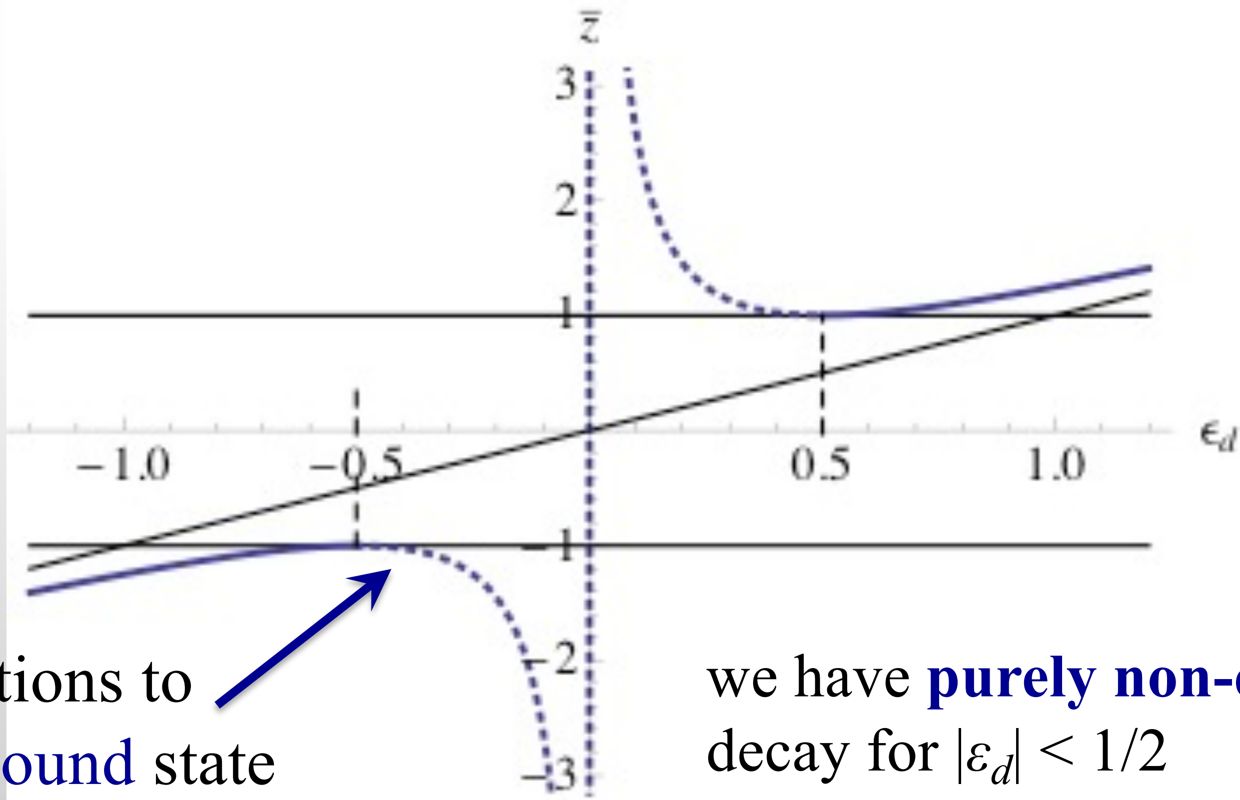
Prototype model: linear dispersion plot

$$z_L = \varepsilon_d + \frac{1}{4\epsilon_d}$$



Prototype model: linear dispersion plot

$$z_L = \varepsilon_d + \frac{1}{4\epsilon_d}$$

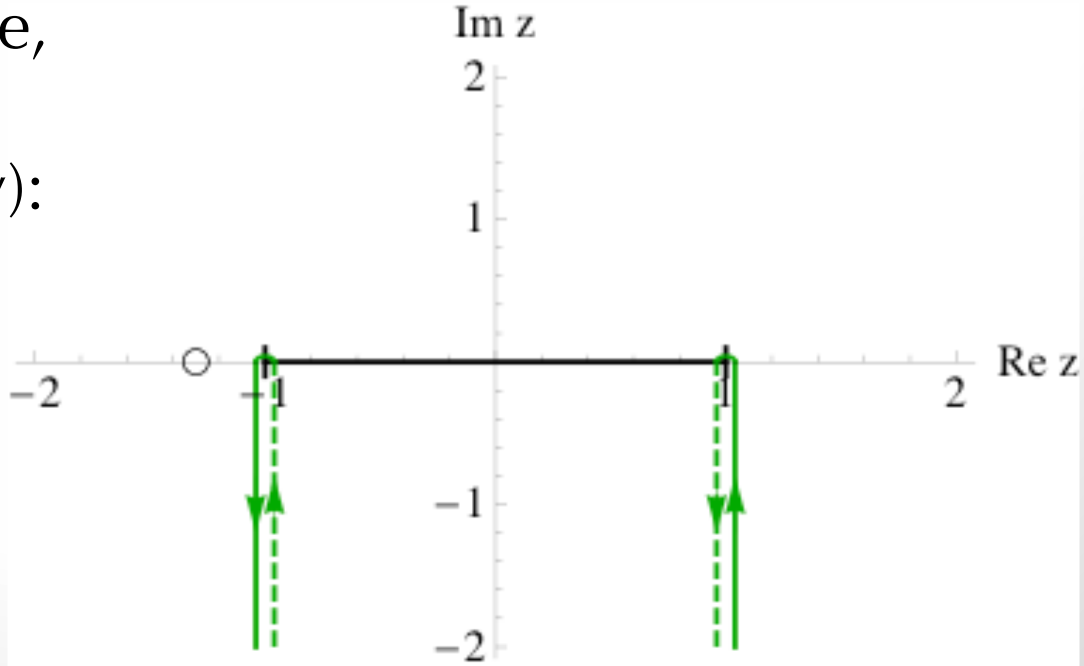


transitions to anti-bound state

we have **purely non-exponential** decay for $|\varepsilon_d| < 1/2$

Long-time dynamics for prototype model

Focus on the complete, non-exponential case (anti-bound state only):

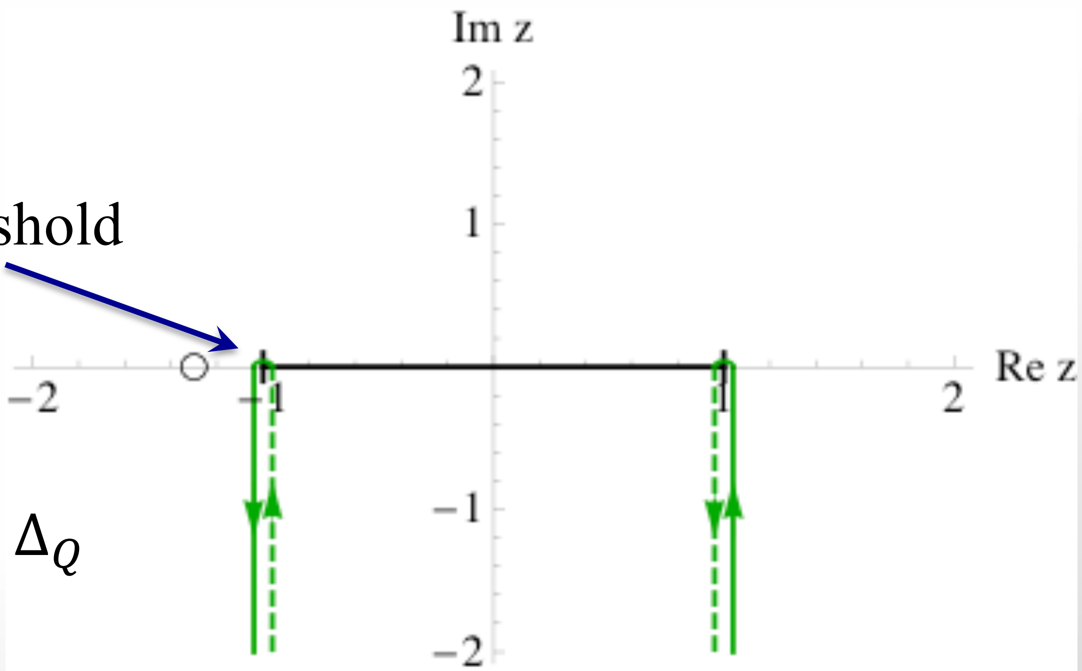


$$A_{th}^{-}(t) = \frac{1}{2\pi i}\left[\int_{-1}^{-1-i\infty} \frac{e^{-izt}}{\eta^{I}(z)}dz - \int_{-1-i\infty}^{-1} \frac{e^{-izt}}{\eta^{II}(z)}dz\right]$$

$$\eta^{I}(z) = z - \varepsilon_d - \Sigma^{I}(z)$$

Long-time dynamics for prototype model

vicinity of lower threshold



define $z_L = -1 + \Delta_Q$

$$A_{th}^{-}(t) = \frac{1}{2\pi i}\left[\int_{-1}^{-1-i\infty} \frac{e^{-izt}}{\eta^{I}(z)} dz - \int_{-1-i\infty}^{-1} \frac{e^{-izt}}{\eta^{II}(z)} dz\right]$$

$$= \frac{1}{2\pi i \varepsilon_d} \int_{-1}^{-1-i\infty} e^{-izt} \frac{\sqrt{z^2-1}}{z - z_L(\Delta_Q)} dz$$

Long-time dynamics: near zone and far zone

$$A_{th}^{-}(t) = \frac{e^{-it}}{2\pi i \varepsilon_d t^2} \int_0^{\infty} e^{-s} \frac{\sqrt{s^2 - 2its}}{\Delta_Q + i\frac{s}{t}} dz$$

Consider the timescale: $1 \ll t \ll \dfrac{1}{\Delta_Q}$ (Long-time 'near zone')

amplitude: $A_{th}^{-}(t) \sim t^{-1/2}$ $|A_{th}^{-}(t)|^2 \sim t^{-1}$

Long-time dynamics: near zone and far zone

$$A_{th}^{-}(t) = \frac{e^{-it}}{2\pi i \varepsilon_d t^2} \int_0^{\infty} e^{-s} \frac{\sqrt{s^2 - 2its}}{\Delta_Q + i\frac{s}{t}} dz$$

Consider the timescale: $1 \ll t \ll \frac{1}{\Delta_Q}$ (Long-time ‘near zone’)

amplitude: $A_{th}^{-}(t) \sim t^{-1/2}$ $|A_{th}^{-}(t)|^2 \sim t^{-1}$

Asymptotic limit: $\frac{1}{\Delta_Q} \ll t$ (Long-time ‘far zone’)

$A_{th}^{-}(t) \sim t^{-3/2}$ $|A_{th}^{-}(t)|^2 \sim t^{-3}$

Note that for $\Delta_Q = 0$, the near zone becomes fully asymptotic

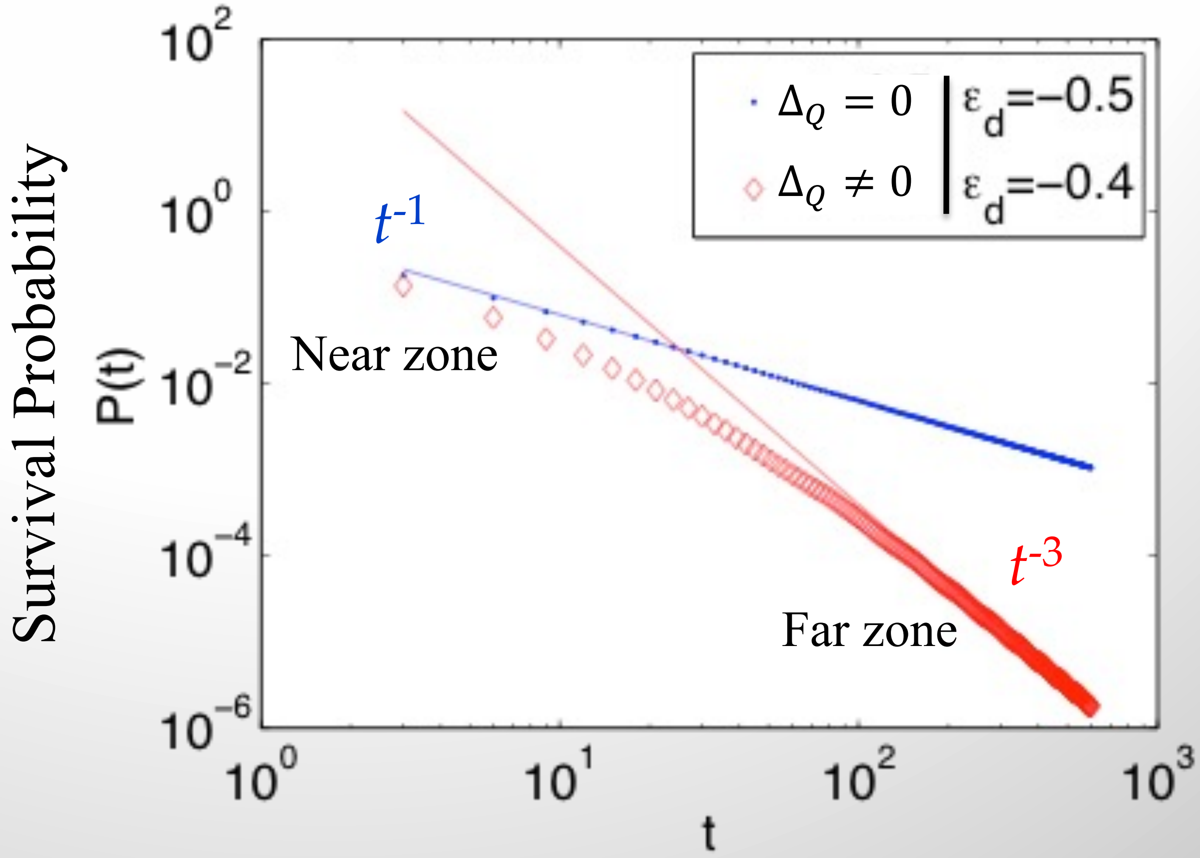
Long-time dynamics: numerical results for prototype model

Near zone

$|A_{th}^{-}(t)|^2 \sim t^{-1}$

Far zone

$|A_{th}^{-}(t)|^2 \sim t^{-3}$



S. Garmon, T. Petrosky, L. Simine, and D. Segal, Fortschr. Phys. **61**, 261 (2013).

Long-time dynamics for general open quantum systems

Similar effect observed in the following works:

S. Longhi, Phys. Rev. Lett. **97**, 110402 (2006).

Axel D. Dente, Raúl A. Bustos-Marún, and Horacio M. Pastawski,
Phys. Rev. A **78**, 062116 (2008).

One can demonstrate the time scale separating near and far zones should be inversely related to Δ_Q fairly generally:

S. Garmon, T. Petrosky, L. Simine, and D. Segal, Fort. Physik, **61**, 261 (2013).

Another approach: using a bound state in continuum as a method to eliminate the effect of the resonance from the decay

S. Garmon, K. Noba, G. Ordonez, and D. Segal,
Phys. Rev. A, **99**, 010102(R) (2019).

Summary

Bound state influence on long time dynamics in OQS:

- Bound state transition to anti-bound state (virtual bound state) at continuum threshold
- Purely non-exponential dynamics when only anti-bound states are present
- Long time dynamics for prototype model:
  - Long-time near zone: $P(t) \sim t^{-1}$
  - Long-time far zone: $P(t) \sim t^{-3}$
- Amplification of non-exponential decay as bound state transitions to anti-bound state; near zone becomes asymptotic dynamics