Physics of complete positivity in open quantum systems

— Let's use "GKLS master equation" in honor of all the contributors,
Lindblad as well as Gorini, Kossakowski, and Shudarshan! —

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What is the universal property of quantum dynamics? If the quantum system is isolated, the best answer will be the *unitarity* based on the Schrödinger equation. With the (time independent) Hamiltonian H, the time evolution operator is given by $U_t = \exp(-iHt/\hbar)$ that maps an initial quantum state*2 ρ to the quantum state $\rho_t = U_t \rho U_t^\dagger$ at time t. Due to the hermiticity $H = H^\dagger$, it is easy to check the unitary condition $U_t U_t^\dagger = U_t^\dagger U_t = \mathbb{I}$*3. But, what if the quantum system is open [1, 3]? Some of you would say "it's *complete positivity* (CP)*4." Well, it is widely believed, especially in the field of open quantum system and recently in quantum information science [4], that a completely positive map Λ_t describes the most general dynamics for open quantum systems. However, is this really true? What is the physical reason behind this? For an isolated system, the unitarity is basically due to the probability interpretation of quantum mechanics. Then, the CP condition can also be understood similarly? Indeed, the CP is usually interpreted to be the condition to map a density operator, including an entanglement, to a density operator on the enlarged system $\mathcal{H} \otimes \mathbb{C}^n$, hence it is tempted to resort to the probability interpretation. However, surprisingly, it turns out that CP condition cannot be understood in such a simple way. We will see that the CP condition introduces strong constraints on the relaxation properties [5]. The main purpose of this lecture is to share this fact by introducing the foundation of CP dynamics. For this purpose, we focus on the quantum Markovian dynamics in the sense that a time evolution map satisfies the semi-group property: $\Lambda_{t+s} = \Lambda_t \circ \Lambda_s \ (t, s \geq 0)$. The dynamics is then described by the so-called *master equation* $d\rho_t/dt = \mathcal{L}\rho_t$ where $\mathcal{L}$ is the generator of the time evolution map: $\Lambda_t = \exp(\mathcal{L}t)$. In 1976, Lindblad [6], and independently, Gorini, Kossakowski, and Sudarshan [7] discovered the general form of the generator:

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right), \tag{1}$$

The first term, called the Hamiltonian part, reflects the unitary dynamics due to the (effective) Hamiltonian H. The second term, called the dissipative part, is in charge of the non-unitary property that causes decoherence, dissipation, relaxation, etc. Due to its generality, the master equation of this form is widely used in many fields of physics, including quantum information science, quantum thermodynamics, or even particle physics. Although the master equation is often called the Lindblad equation, in honor of all the contributors, it is highly recommended to call it the GKLS master equation [8]. This lecture also gives a brief introduction of GKLS master equation.

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*2 Remind that a quantum state is generally described by a density operator (or matrix) ρ, which is a positive operator with unit trace, denoted by $\rho \geq 0$ and $\mathrm{tr}\,\rho = 1$, on the associated Hilbert space $\mathcal{H}$ to the quantum system. The lecture includes a brief review of the foundations of open quantum theory, including the description of quantum states by density operators. However, those who have never heard about many terminologies used in this abstract are recommended to study beforehand e.g., [1] (in Japanese) or [2] (limited time released) available from qr-codes above.

*3 This is true even for a time-dependent Hamiltonian, where in that case $U_t = T\exp\{\int_0^t H(t')dt'/\hbar\}$ where $T\exp$ is the time-ordered exponential.

*4 A linear map (sometimes called a superoperator) Λ on the set of linear operators $\mathcal{L}(\mathcal{H})$ is called *positive* if it preserves the positivity of operators, i.e., if $A \geq 0$, then $\Lambda(A) \geq 0$. It is called a completely positive map if the extension $\Lambda \otimes Id_n$ on $\mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{C}^n)$ for arbitrary natural number n is positive.