Derivation of the Lieb-Robinson bound

I. DEFINITION AND NOTATION

A. Definition of the system

We consider a finite-volume lattice system with spins (or sites) labeled as $i = 1, 2, \ldots, n$, where the Hilbert space of each spin is d-dimensional. We denote the finite set of sites as X, Y, Z and so on, and the cardinality of X as $|X|$ (e.g. $X = \{i_1, i_2, \ldots, i_{|X|}\}$). We define the distance $d_{X,Y}$ as the shortest path length for which one needs to connect X to Y.

B. Definition of k-local operator

We first define the *k-local operator* $O^{(k)}$ as follows:

$$O^{(k)} = \sum_{|Z| \leq k} o_Z, \tag{1}$$

where o_X is a local operator supported on the finite set Z. The k-local operator includes the interactions of up to k-body coupling with finite k. Explicitly, this operator can be described as

$$O^{(k)} = \sum_{i_1 < i_2 < \cdots < i_k} \sum_{\mu_1, \ldots, \mu_k} o^{\mu_1, \ldots, \mu_k}_{i_1, \ldots, i_k} s^{\mu_1}_{i_1} \otimes \cdots \otimes s^{\mu_k}_{i_k}, \tag{2}$$

where $\{s^\mu_i\}_{\mu,i}$ are operator bases at the site i; for example, when we consider a (1/2)-spin system, $\{s^\mu_i\}_{\mu,i} = \{\sigma^x_i, \sigma^y_i, \sigma^z_i\}$ with $\{\sigma^\mu_i\}_{\mu=x,y,z}$ the Pauli matrices.

C. Definition of extensiveness

We next define the *extensiveness* as follows: the *k-local operator* $O^{(k)}$ satisfies the extensiveness if

$$\max_{i \in [1,n]} \left(\sum_{Z: Z \ni i} \|o_Z\| \right) \leq g, \tag{3}$$

where g is a $\mathcal{O}(1)$ constant and $\sum_{Z:Z \ni i}$ denotes the summation with respect to the support Z which contains the spin i. Note that if the operator $O^{(k)}$ satisfies the extensiveness, it also satisfies

$$\|O^{(k)}\| \leq gn, \tag{4}$$

because of

$$\|O^{(k)}\| \leq \sum_Z \|o_Z\| \leq \sum_{i=1}^n \sum_{Z:Z \ni i} \|o_Z\| \leq gn.$$

D. Definition of the Hamiltonian

In the following, we consider Hamiltonians which are extensive k-local operators with $k = \mathcal{O}(1)$:

$$H = \sum_{|Z| \leq k} h_Z \quad \text{with} \quad \sum_{Z:Z \ni i} \|h_Z\|_2 \leq g \quad \text{for} \quad i = 1, 2, \ldots, n, \tag{5}$$

We also define the finite-range interaction as follows;

$$\sum_{Z:Z \ni \{i,j\}} \|h_Z\| = 0, \quad \text{for} \quad d_{i,j} > \ell_H, \tag{6}$$

where ℓ_H corresponds to the interaction length.

II. LIEB-ROBINSON BOUND

In this section, we show the Lieb-Robinson bound and its proof in the case of the finite-range interacting systems; even if we consider more general systems (e.g. with exponentially decaying interactions), the Lieb-Robinson bound can be also obtained and its proof is essentially the same as in this case.

The Lieb-Robinson bound restricts the velocity of the information transfer in the time evolution of quantum many-body systems. In other words, if we send some information from a subsystem X to a subsystem Y by the use of the time-evolution of a local Hamiltonian, we need finite time which is proportional to the distance between X and Y.

Lieb-Robinson bound. Let O_X and O_Y be arbitrary operators on the subsystems X and Y, respectively. We then bound the norm of the commutator $[O_X(t), O_Y]$ from above by

$$\|[O_X(t), O_Y]\| \le \frac{2}{k}\|O_X\| \cdot \|O_Y\| \cdot |X| \frac{(2kg|t|)^{m_0}}{m_0!}, \tag{7}$$

with

$$m_0 = \left\lfloor \frac{d_{X,Y}}{\ell_H} + 1 \right\rfloor, \tag{8}$$

where $O_X(t) \equiv e^{-iHt} O_X e^{iHt}$ and the Hamiltonian (5) is finite-range as in Eq. (6).

The RHS of the inequality is roughly given by

$$\frac{(C|t|)^{m_0}}{m_0!} \sim \left(\frac{\text{const.} \times |t|}{m_0}\right)^{m_0} \sim e^{-\text{const.} \times d_{X,Y} \log(d_{X,Y}/|t|)}, \tag{9}$$

which implies the super-exponential decay with respect to the distance $d_{X,Y}$.

Proof. We first prove the following inequality:

$$\frac{d}{dt}\|[O_X(t), O_Y]\| \le 2\|O_X\| \cdot \|[H_X(t), O_Y]\|. \tag{10}$$

In order to prove this inequality, we start with the following equation:

$$\begin{aligned}
\|[O_X(t+\delta t), O_Y]\| &= \|[e^{-iH(t+\delta t)} O_X e^{iH(t+\delta t)}, O_Y]\| \\
&= \|[e^{-iHt} e^{-iH\delta t} O_X e^{iH\delta t} e^{iHt}, O_Y]\| \\
&= \|[(O_X - i\delta t[H, O_X] + \mathcal{O}(\delta t^2), O_Y(-t)]\| \\
&= \|[(O_X - i\delta t[H_X, O_X] + \mathcal{O}(\delta t^2), O_Y(-t)]\|,
\end{aligned} \tag{11}$$

where we define H_X as

$$H_X = \sum_{Z_1 : Z_1 \cap X \neq \emptyset} h_{Z_1}. \tag{12}$$

We then obtain

$$\begin{aligned}
\|[e^{-iH_X\delta t} O_X e^{iH_X\delta t}, O_Y(-t)]\| + \mathcal{O}(\delta t^2) =& \|[O_X, e^{iH_X\delta t} O_Y(-t) e^{-iH_X\delta t}]\| + \mathcal{O}(\delta t^2) \\
=& \|[O_X, O_Y(-t) + i\delta t[H_X, O_Y(-t)]\| + \mathcal{O}(\delta t^2) \\
=& \|[O_X(t), O_Y]\| + i\delta t[O_X, [H_X, O_Y(-t)]\| + \mathcal{O}(\delta t^2) \\
\le& \|[O_X(t), O_Y]\| + 2\delta t\|O_X\| \cdot \|[H_X(t), O_Y]\| + \mathcal{O}(\delta t^2).
\end{aligned} \tag{13}$$

From (11) and (13), we prove the inequality (10) by taking $\delta t \to 0$.

By integrating the inequality (10) and utilizing the fact of $\|[O_X(t=0), O_Y]\| = \|[O_X, O_Y]\| = 0$, we obtain the following inequality:

$$\begin{aligned}
\|[O_X(t), O_Y]\| &\le 2\|O_X\| \sum_{Z_1 : Z_1 \cap X \neq \emptyset} \left(\|[h_{Z_1}(0), O_Y]\| + \int_0^t \|[h_{Z_1}(t_1), O_Y]\| dt_1 \right) \\
&= 2\|O_X\| \sum_{Z_1 : Z_1 \cap X \neq \emptyset} \int_0^t \|[h_{Z_1}(t_1), O_Y]\| dt_1,
\end{aligned} \tag{14}$$

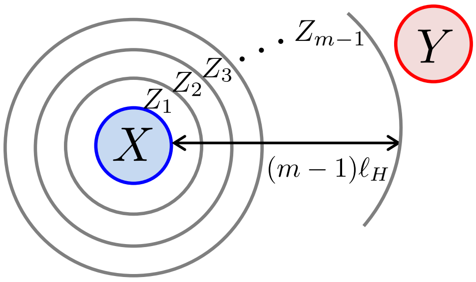

FIG. 1: The regions $\{Z_j\}_{j=1}^m$ in the inequality (16) is defined as $Z_j : Z_j \cap Z_{j-1} \neq \emptyset$. Because the interactions in the Hamiltonian are short-range with interaction length ℓ_H, for each of the terms h_{Z_j}, the subset Z_j satisfies $d_{X,i} < j\ell_H$ for an arbirtrary site $i \in Z_j$.

where we use $\|[h_{Z_1}(0), O_Y]\| = 0$ as long as $Z_1 \cap Y = \emptyset$. Because we assume the finite-range interaction, we here note that arbitrary site i in the set $Z_1 : Z_1 \cap X$ satisfies $d_{i,X} < \ell_H$ for $\forall i \in Z_1$(Figure 1). If all the commutators $\|[h_{Z_1}, O_Y]\|$ vanish, namely $d_{X,Y} > \ell_H$ we apply the same process to $\|[h_{Z_1}(t), O_Y]\|$ and obtain

$$\|[O_X(t), O_Y]\| \leq 4\|O_X\| \sum_{Z_1:Z_1\cap X\neq\emptyset} \|h_{Z_1}\| \sum_{Z_2:Z_2\cap Z_1\neq\emptyset} \int_0^t \int_0^{t_1} \|[h_{Z_2}(t), O_Y]\| dt_2 dt_1 \tag{15}$$

as long as $Z_1 \cap Y = \emptyset$. Note that arbitrary site i in the set Z_2 satisfies $d_{i,X} < 2\ell_H$.

By iteratively applying this process, we have

$$\begin{aligned}\|[O_X(t), O_Y]\| \leq & 2^m \|O_X\| \sum_{Z_1:Z_1\cap O_X\neq\emptyset} \|h_{Z_1}\| \sum_{Z_2:Z_2\cap Z_1\neq\emptyset} \|h_{Z_2}\| \cdots \sum_{Z_{m-1}:Z_{m-1}\cap Z_{m-2}\neq\emptyset} \|h_{Z_{m-1}}\| \\ & \sum_{Z_m:Z_m\cap Z_{m-1}\neq\emptyset} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} \|[h_{Z_m}(t_m), O_Y]\| dt_m dt_{m-1} \cdots dt_1. \\ \leq & 2^m \|O_X\| \sum_{Z_1:Z_1\cap O_X\neq\emptyset} \|h_{Z_1}\| \sum_{Z_2:Z_2\cap Z_1\neq\emptyset} \|h_{Z_2}\| \cdots \sum_{Z_{m-1}:Z_{m-1}\cap Z_{m-2}\neq\emptyset} \|h_{Z_{m-1}}\| \\ & \sum_{Z_m:Z_m\cap Z_{m-1}\neq\emptyset} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} 2\|h_{Z_m}\| \cdot \|O_Y\| dt_m dt_{m-1} \cdots dt_1\end{aligned} \tag{16}$$

under the assumption that all the commutators $\|[h_{Z_{m-1}}, O_Y]\|$ vanish. From $d_{X,i} < (m-1)\ell_H$ for an arbitrary site $i \in Z_{m-1}$, as long as $d_{X,Y} \geq (m-1)\ell_H$, we can ensure $[h_{Z_{m-1}}, O_Y] = 0$. Therefore, the condition $[h_{Z_{m-1}}, O_Y] = 0$ is ensured up to $m = m_{\max}$ with

$$m_{\max} = \left\lfloor \frac{d_{X,Y}}{\ell_H} + 1 \right\rfloor. \tag{17}$$

Because of the extensiveness (5) of the Hamiltonian, we obtain

$$\sum_{Z_1:Z_1\cap X\neq\emptyset} \|h_{Z_1}\| \leq \sum_{i:i\in X} \sum_{Z_1:Z_1\ni i} \|h_{Z_1}\| \leq \sum_{i:i\in X} g = g|X|. \tag{18}$$

Similarly, we have

$$\sum_{Z_1:Z_1\cap Z_2\neq\emptyset} \|h_{Z_2}\| \leq \sum_{i:i\in Z_1} \sum_{Z_2:Z_2\ni i} \|h_{Z_2}\| \leq \sum_{i:i\in Z_1} g = g|Z_1| \leq kg, \tag{19}$$

where we utilize k-locality of the Hamiltonian as in Eq. (5).

By the use of the inequalities (18) and (19) in (16), we obtain

$$\begin{aligned}\|[O_X(t), O_Y]\| \leq & 2^{m+1} \|O_X\| \cdot \|O_Y\| \cdot |X| k^{m-1} g^m \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} dt_m dt_{m-1} \cdots dt_1 \\ = & \frac{2}{k} \|O_X\| \cdot \|O_Y\| \cdot |X| \frac{(2kgt)^m}{m!}.\end{aligned} \tag{20}$$

By taking m as $m_{\max} =: m_0$, we can obtain the main inequality. $\square$