Non-Hermitian quantum mechanics and localization

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N.H. and D.R. Nelson:
PRL 77, 570 (1996)
PRB 56, 8651 (1997)
PRB 58, 8384 (1998)
N.H.:
Physica A 254, 317 (1998)

Why non-Hermitian?

- provides a new viewpoint of Hermitian physics
- expresses physical phenomena effectively

(1) Anderson localization
Localization length
(2) Resonant states
Resonance lifetime
(3) Flux-line pinning in type-II superconductors
Vortex depinning transition

Outline

1. Non-Hermitian Anderson model and delocalization
2. Motivation I — Localization length
3. How does the delocalization happen?
4. Numerical examples
5. Motivation II — Resonant states
6. Motivation III — Vortex depinning in type-II superconductors

Non-Hermitian Anderson model

Continuum model

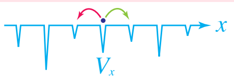
$$H=\frac{(\boldsymbol{p}+i\boldsymbol{g})^2}{2m}+V(\boldsymbol{x})$$

$\boldsymbol{g}$: "imaginary vector potential" (constant)
$V(\boldsymbol{x})$: random scalar potential

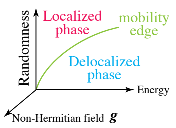
$\boldsymbol{g}=\boldsymbol{0}$: (conventional) 1-electron Anderson model

Lattice model

$$H=-\frac{t}{2}\sum_{\boldsymbol{x}}\sum_{\nu=1}^{d}\left(e^{\boldsymbol{g}\cdot\boldsymbol{e}_\nu}|\boldsymbol{x}+\boldsymbol{e}_\nu\rangle\langle\boldsymbol{x}| + e^{-\boldsymbol{g}\cdot\boldsymbol{e}_\nu}|\boldsymbol{x}-\boldsymbol{e}_\nu\rangle\langle\boldsymbol{x}|\right)+\sum_{\boldsymbol{x}}V_{\boldsymbol{x}}|\boldsymbol{x}\rangle\langle\boldsymbol{x}|$$



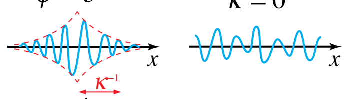
asymmetric hopping *vs.* randomness



Non-Hermitian delocalization

$\boldsymbol{g}=\boldsymbol{0}$: localized states

$\psi\sim e^{-\kappa|\boldsymbol{x}|}$ $\kappa=0$



(i) $\boldsymbol{g}\uparrow$: delocalized at $g=g_c$

(ii) delocalization ~ complex eigenvalue

(iii) inverse localization length: $\kappa=g_c$

How does it happen?
Imaginary gauge transformation

$$H_g = \frac{(\boldsymbol{p}+i\boldsymbol{g})^2}{2m} + V(\boldsymbol{x})$$

$$e\boldsymbol{A} \leftrightarrow i\boldsymbol{g}$$

$\boldsymbol{g}=\boldsymbol{0}$: $H_0\psi_0 = \varepsilon_0\psi_0$

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$\boldsymbol{g}\neq\boldsymbol{0}$: $H_g\psi_g = \varepsilon\psi_g$

$\psi_g = \psi_0 e^{\boldsymbol{g}\cdot\boldsymbol{x}}$

$\varepsilon = \varepsilon_0$: real, fixed

$$\psi_0 \sim e^{-\kappa|\boldsymbol{x}|}$$

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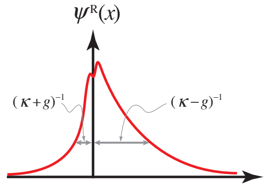
$$\psi_g^{\mathrm{R}} \sim e^{-\kappa|\boldsymbol{x}|+\boldsymbol{g}\cdot\boldsymbol{x}}$$

integrable for $|\boldsymbol{g}|<\kappa$ only



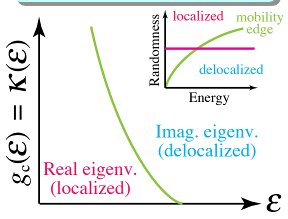
$|\boldsymbol{g}|>\kappa$: $\psi^{\mathrm{R}} \sim e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, $\varepsilon \sim \frac{(\boldsymbol{k}+i\boldsymbol{g})^2}{2m}$

Motivation

(1) Localization length of *Hermitian* Anderson model

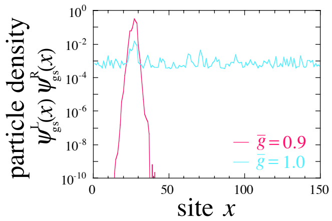
Hermitian case ($\boldsymbol{g}=\boldsymbol{0}$): $\psi_0 \sim e^{-\kappa|\boldsymbol{x}|}$

Complex eigenvalue at $\boldsymbol{g}=\boldsymbol{g}_c \longrightarrow \kappa = \boldsymbol{g}_c$



Delocalization of eigenfunction

1D: L_x = 1000 (Diagonalized 1000 × 1000 matrix)

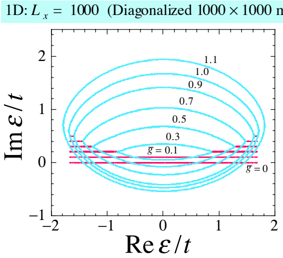


Note: Particle Density $\neq |\psi(x)|^2$

$$\left.\begin{array}{l} H|\psi^{\mathrm{R}}\rangle = \varepsilon|\psi^{\mathrm{R}}\rangle \\ \langle\psi^{\mathrm{L}}|H = \varepsilon\langle\psi^{\mathrm{L}}| \end{array}\right\} \rightarrow \langle\psi^{\mathrm{L}}| \neq |\psi^{\mathrm{R}}\rangle^{\dagger}$$

Complex spectrum and delocalization

1D: L_x = 1000 (Diagonalized 1000 × 1000 matrix)

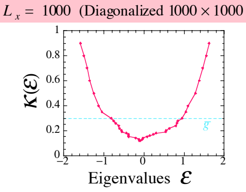


real eigenvalue
localized state
$\kappa > g$

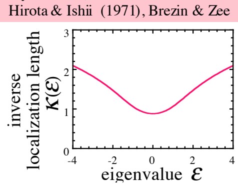
complex eigenv.
delocalized state
$\kappa < g$

Localization length

1D: L_x = 1000 (Diagonalized 1000 × 1000 matrix)

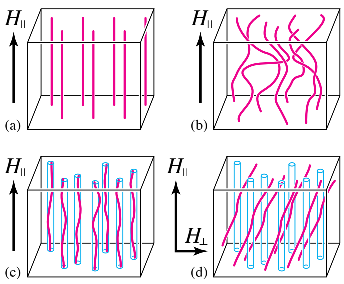


Lloyd model (Lorentzian random distribution):
Hirota & Ishii (1971), Brezin & Zee (1998)



Motivation

(3) Vortex pinning in high-T_c superconductors

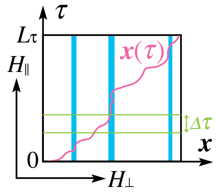


Path-integral mapping

$$E_{cl}[x(\tau)] = \int_0^{L_\tau} d\tau \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 - g\cdot\frac{dx}{d\tau} + V(x)\right]$$

$$g \equiv \Phi_0 H_\perp/(4\pi)$$

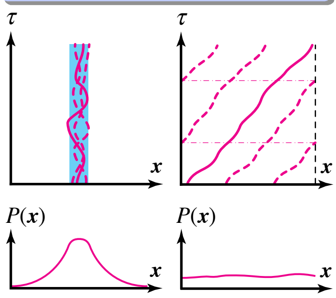
Nelson and Vinokur, PRB **48**, 13060 (1993)



TM : $T(\Delta\tau) \approx \exp(-\Delta\tau H)$

$$Z = \int \mathcal{D}x\, e^{-E_{cl}[x(\tau)]} = \langle \psi^f | e^{-L_\tau H} | \psi^i \rangle$$

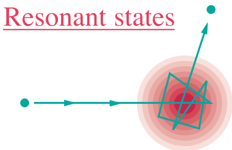
Pinning and localization



pinning	depinning
⬇	⬇
localized fn.	delocalized fn.

Motivation

(2) Resonant states



$$E_{res} = E_r - i\Gamma/2 \approx \frac{(k-i\gamma)^2}{2m}$$

Γ^{-1}: Resonance lifetime

$\psi_{res}(r) \sim e^{i(k-i\gamma)r}$ as $r \to \infty$

Unintegrable eigenfn.

$$H = \frac{(p+ig)^2}{2m} + V(r);\quad g \equiv -g_0 \frac{x}{r}$$

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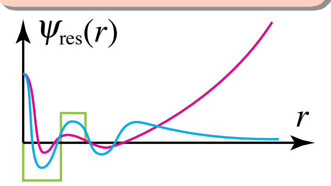
$$\psi_{res}(r;g) \sim e^{ikr-(g_0-\gamma)r}$$



Aguilar, Balslev & Combes, Comm. Math. Phys. **22** (1971)

complex rotation: $\begin{cases} r \to re^{i\theta} \\ p \to pe^{-i\theta} \end{cases}$

Other applications

- CDW pinning
  Chen *et al.*,PRB **54**, 12798 (1996)
- Fokker-Planck equation
  Nelson and Shnerb, PRE **58**, 1383 (1998)
- Dirac Fermion in random gauge field
  Mudry *et al.* PRL **80**, 4257 (1998)
- *PT* symmetry
  Bender and Boettcher, PRL **80**, 5243 (1998)
  complex spectrum ↔ *PT* symmetry breaking
- Interactions
  Lehrer and Nelson, cond-mat/9806016
  Boson crystal and delocalization