Non-Hermitian Hamiltonian and the Zeros of the Riemann Zeta Function

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The Riemann hypothesis asserts that the nontrivial zeros of the Riemann zeta function should be of the form $\frac{1}{2} + \mathrm{i}\lambda_n$, where the set of numbers $\{\lambda_n\}$ are real. The so-called Hilbert-Pólya conjecture assumes that $\{\lambda_n\}$ should correspond to the eigenvalues of an operator $\hat{H}$ that is Hermitian. The discovery of such an operator $\hat{H}$, if it exists, thus amounts to providing a proof of the Riemann hypothesis. In 1999 Berry and Keating conjectured that such an operator $\hat{H}$ should correspond to a quantisation of the classical Hamiltonian $H = xp$. Since then, the Berry-Keating conjecture has been investigated intensely in the literature, but its validity has remained elusive up to now.

In this talk I will sketch the proof of the validity of the Berry-Keating conjecture. Specifically, I will derive the Hamiltonian, whose classical counterpart is $H = xp$, having the property that with a suitable boundary condition on its eigenstates, its the eigenvalues $\{\lambda_n\}$ corresponding to the nontrivial zeros of the Riemann zeta function. This Hamiltonian is not Hermitian, but is PT symmetric in a special way. A formal argument will be given for the associated metric operator and the formally 'Hermitian' counterpart Hamiltonian. Numerical evidence indicates that the eigenvalues of $\hat{H}$ are real, hinting at the validity of the Riemann hypothesis.

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