Non-Bloch band theory of non-Hermitian systems

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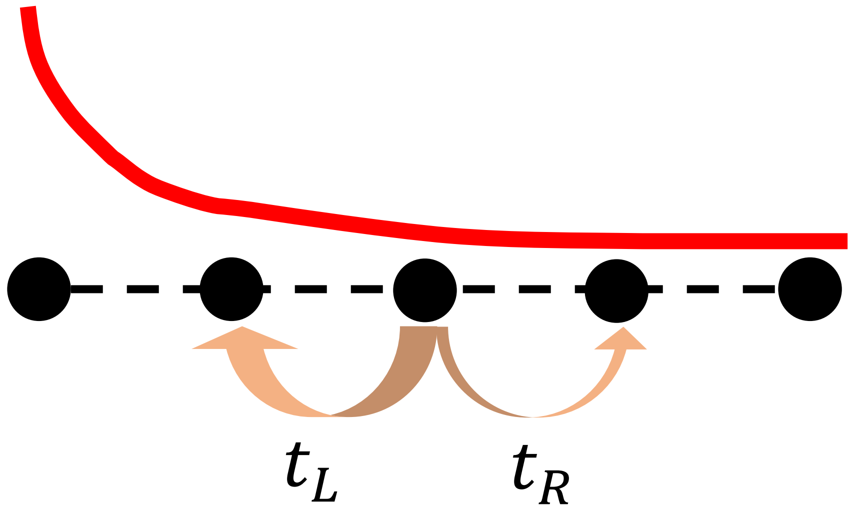
Introduction

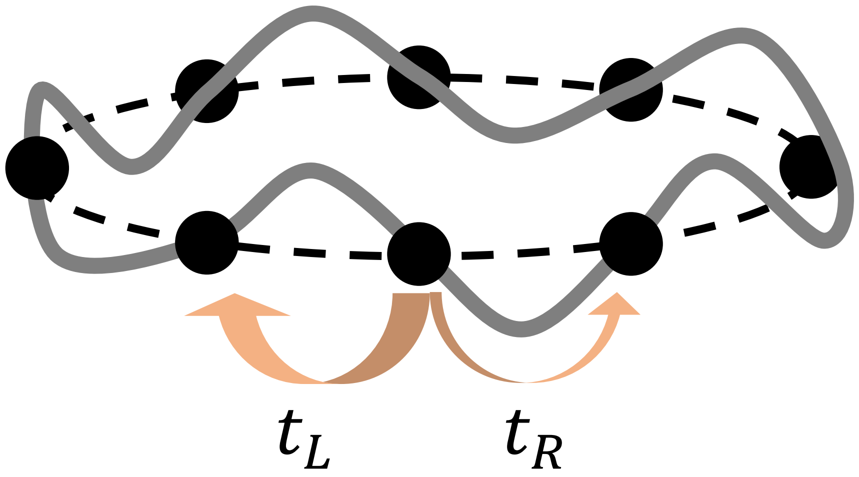
✓ Non-Hermitian skin effect S. Yao et al., Phys. Rev. Lett. **121**, 086803 (2018)

➢ The eigenstates in the bulk are localized at either end of an open chain.



Open boundary condition

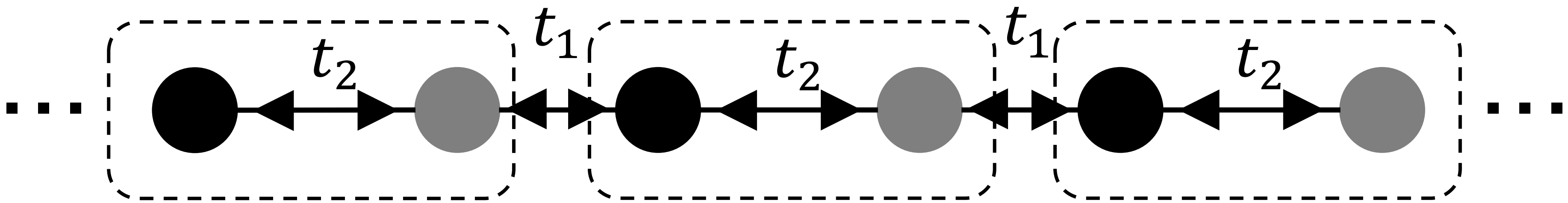
➢ The eigenstates in the bulk extend over a whole system in a periodic chain.



Periodic boundary condition

Introduction

✓ Su-Schrieffer-Heeger model (Hermitian system) W. P. Su et al., Phys. Rev. B **22**, 2099 (1980)

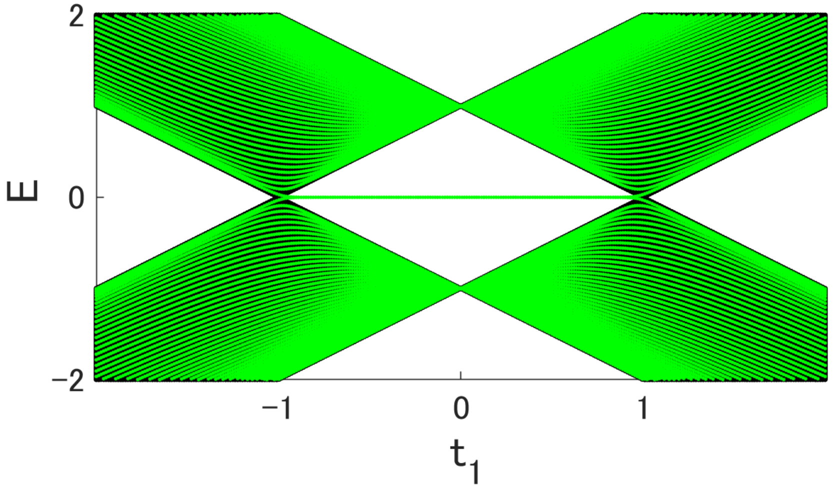


➢ Bloch Hamiltonian:

$$H(k) = \begin{pmatrix} 0 & t_1 + t_2 e^{-ik} \\ t_1 + t_2 e^{ik} & 0 \end{pmatrix}$$

$k \in \mathbb{R}$: Black

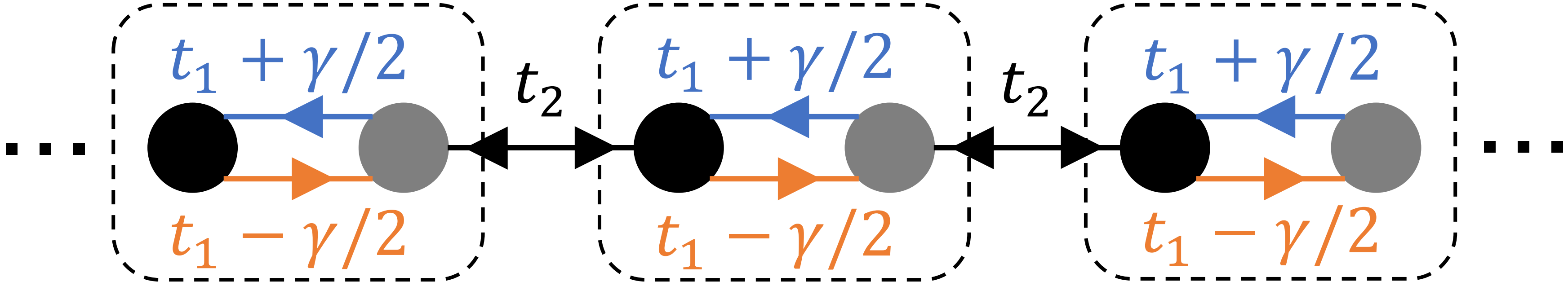
Finite open chain: Green



➡ The Bloch band theory reproduces the energy levels under an open boundary condition.

Introduction

✓ Non-Hermitian Su-Shrieffer-Heeger model S. Yao et al., Phys. Rev. Lett. **121**, 086803 (2018)



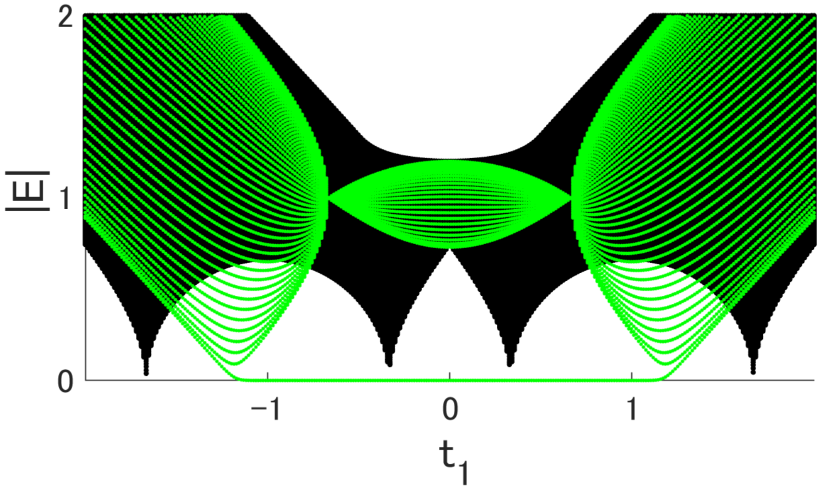
➤ Bloch Hamiltonian:

$$H(k)=\begin{pmatrix} 0 & t_1+\gamma/2+t_2e^{-ik} \\ t_1-\gamma/2+t_2e^{ik} & 0 \end{pmatrix}$$

The conventional Bloch band theory fails in a non-Hermitian system.

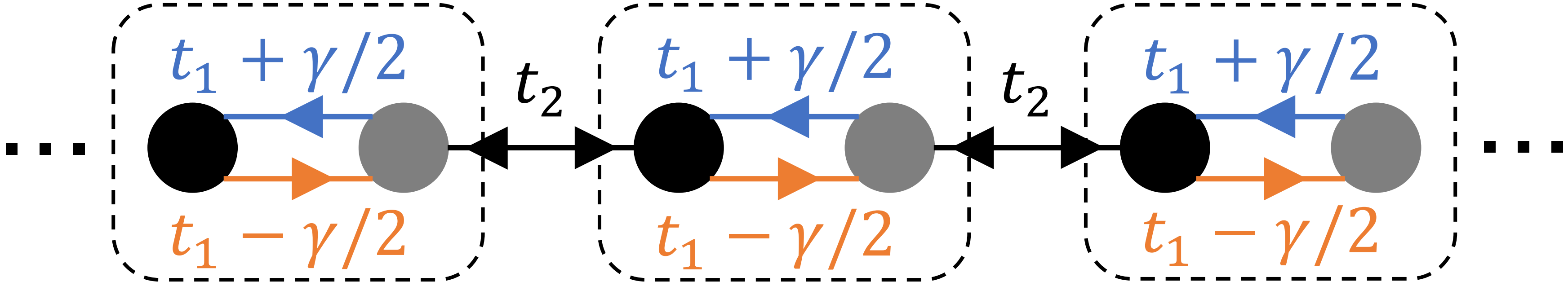
$k\in\mathbb{R}$: Black

Finite open chain: Green



Introduction

✓ Non-Hermitian Su-Shrieffer-Heeger model S. Yao et al., Phys. Rev. Lett. **121**, 086803 (2018)

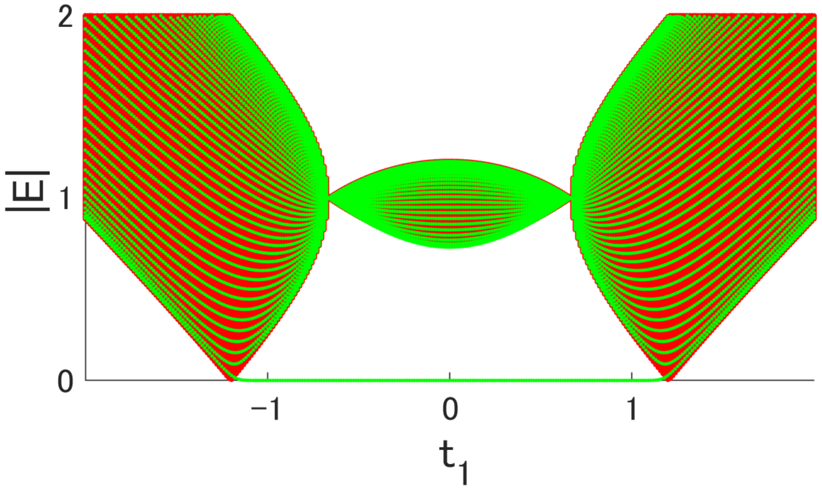


➢ Bloch Hamiltonian:

$$H(k) = \begin{pmatrix} 0 & t_1 + \gamma/2 + t_2 e^{-ik} \\ t_1 - \gamma/2 + t_2 e^{ik} & 0 \end{pmatrix}$$

$k \in \mathbb{C}$: Red

Finite open chain: Green



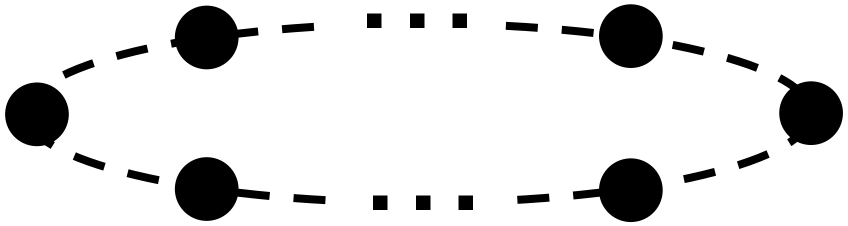
➡ The energy bands match the energy levels in the complex Bloch wave number k.

↑ Non-Bloch band theory

Introduction

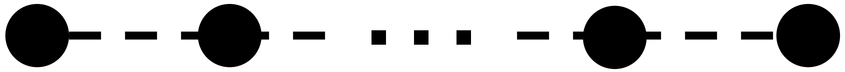
✓ 1D Hermitian system

- Periodic boundary condition:



Bloch wave number k: real

- Open boundary condition:



The Bloch wave number is ill-defined because of no translational symmetry

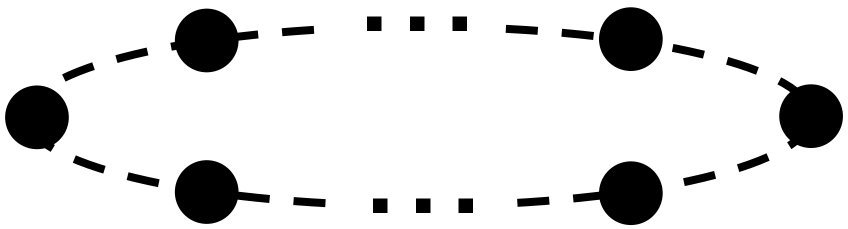
The energy bands in a periodic chain are asymptotically same as the energy levels in an open chain in the limit of a large system size.

➡ We can use the real Bloch wave number for analyzing an open chain.

Introduction

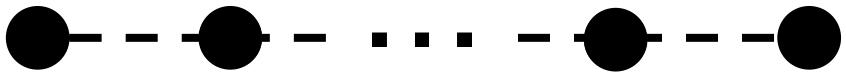
✓ 1D non-Hermitian system

➢ Periodic boundary condition:



Bloch wave number k: real

➢ Open boundary condition:



The Bloch wave number is ill-defined because of no translational symmetry

The energy bands in a periodic chain and an open chain are different even in the limit of a large system size because of the non-Hermitian skin effect.

➡ We cannot adopt the real Bloch wave number for studying an open chain.

Non-Bloch band theory

The Bloch wave number should become complex in a non-Hermitian system.

How can we determine the complex Bloch wave number giving the energy bands in a non-Hermitian system?

Main result

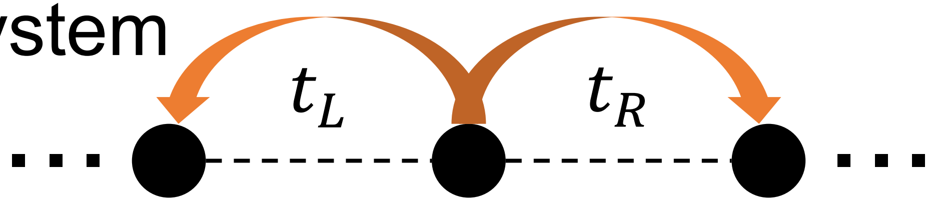
We determine the Brillouin zone $\beta \equiv e^{ik}$ for the complex Bloch wave number $k \in \mathbb{C}$.

Non-Bloch band theory

KY and S. Murakami, Phys. Rev. Lett. **123**, 066404 (2019)

KY, T. Yoda, and S. Murakami, Phys. Rev. Research **4**, 023089 (2022)

Non-Bloch band theory in a tight-binding system



✓ <u>Simple model</u>

- <u>Schrödinger equation:</u> $t_R\psi_{n-1} + t_L\psi_{n+1} = E\psi_n$

- <u>Ansatz:</u> $\psi_n = (\beta_1)^n\phi^{(1)} + (\beta_2)^n\phi^{(2)}$ with $t_R\beta^{-1} + t_L\beta = E$

Eigenvalue equation

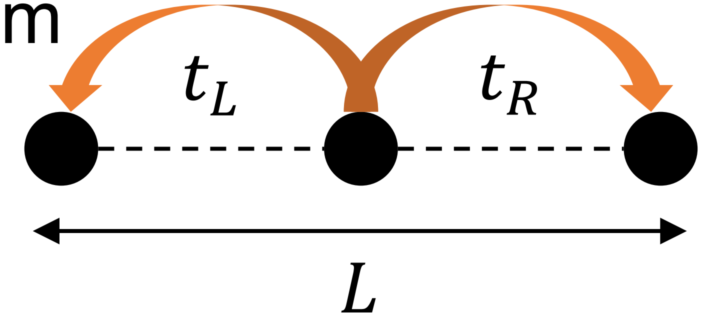
$\beta \sim e^{ik}$

k: Bloch wave number

cf. Hermitian case ($t_L = t_R \equiv t$):

$$E = t\left(e^{ik} + e^{-ik}\right) = 2t\cos k$$

Non-Bloch band theory in a tight-binding system



✓ Simple model

➢ Schrödinger equation: $t_R\psi_{n-1} + t_L\psi_{n+1} = E\psi_n$

➢ Ansatz: $\psi_n = (\beta_1)^n\phi^{(1)} + (\beta_2)^n\phi^{(2)}$ with $t_R\beta^{-1} + t_L\beta = E$

✓ Open boundary condition ($\psi_0 = \psi_{L+1} = 0$)

$$\begin{pmatrix} 1 & 1 \\ (\beta_1)^{L+1} & (\beta_2)^{L+1} \end{pmatrix}\begin{pmatrix} \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

➡ $\left(\frac{\beta_1}{\beta_2}\right)^{L+1} = 1$: Boundary equation

➡ $|\beta_1| = |\beta_2|$

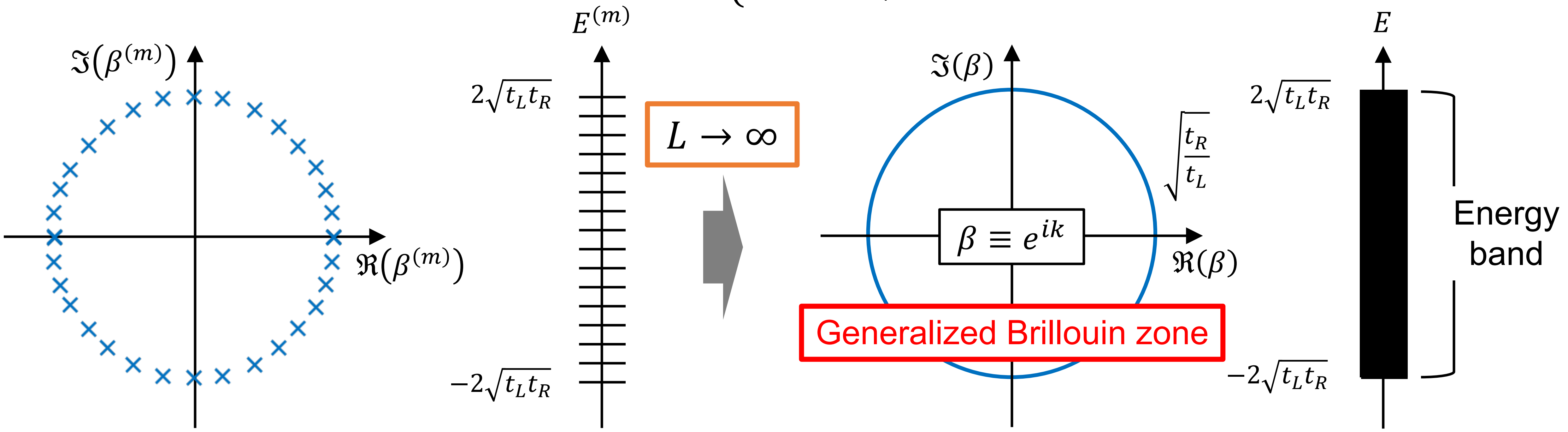
➡ $|\beta_1| = |\beta_2| = \sqrt{\frac{t_R}{t_L}}$

Non-Bloch band theory in a tight-binding system

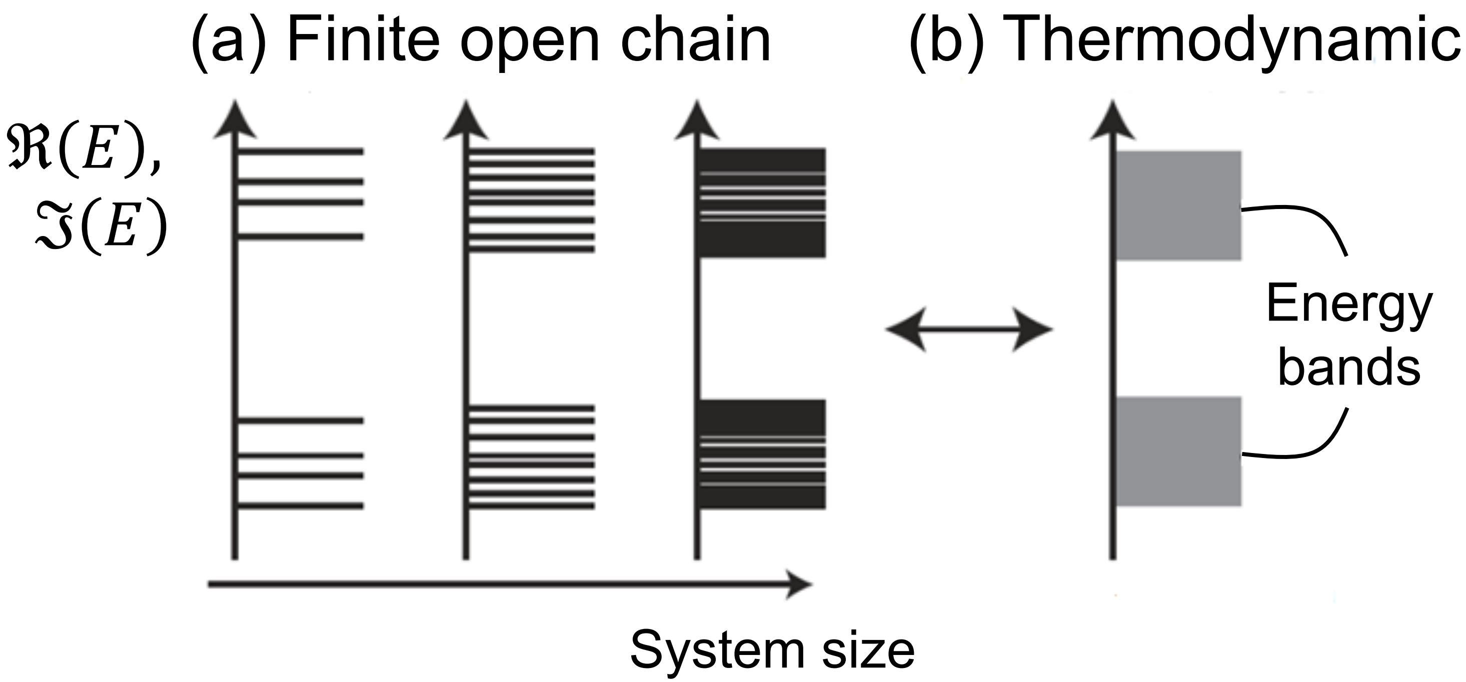
✓ Simple model

➢ "Plane wave" and energy level:

$$\begin{cases} \beta_1^{(m)} = \sqrt{\dfrac{t_R}{t_L}}\, e^{i\theta_m} \\ \beta_2^{(m)} = \sqrt{\dfrac{t_R}{t_L}}\, e^{-i\theta_m} \\ E^{(m)} = 2\sqrt{t_L t_R}\cos\theta_m \end{cases}, \left(\theta_m = \frac{m\pi}{L+1}, m = 1, \ldots, L\right)$$



Non-Bloch band theory in a tight-binding system



Concept

We should define the generalized Brillouin zone to construct the energy bands.

Non-Bloch band theory in a tight-binding system

✓ Non-Hermitian tight-binding system

➢ Hamiltonian:

$$H = \sum_{n} \sum_{i=-N}^{N} \sum_{\mu,\nu=1}^{q} t_{i,\mu\nu} c^{\dagger}_{n+i,\mu} c_{n,\nu}$$

➢ Non-Bloch matrix:

$$[\mathcal{H}(\beta)]_{\mu\nu} = \sum_{i=-N}^{N} t_{i,\mu\nu} \beta^{i}, (\mu, \nu = 1, \dots, q)$$

$$(\beta \sim e^{ik})$$

➢ Eigenvalue equation: $\det[\mathcal{H}(\beta) - E] = 0$

➡ The $2M(= 2qN)$ solutions satisfy $|\beta_1| \leq \cdots \leq |\beta_{2M}|$.

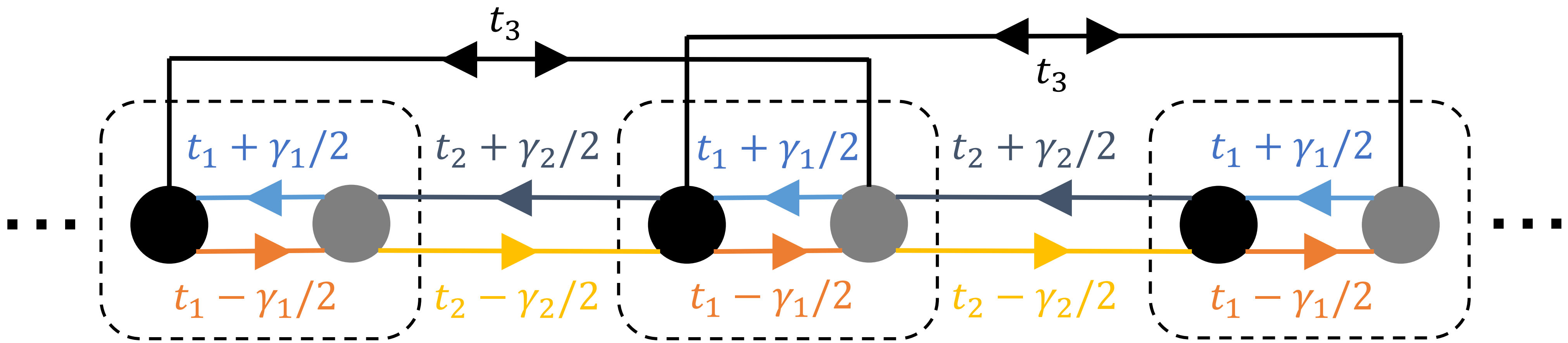
Main result

Condition for the generalized Brillouin zone: $|\beta_M| = |\beta_{M+1}|$

➡ The trajectory of β_M and β_{M+1} forms the generalized Brillouin zone.

Non-Bloch band theory in a tight-binding system

✓ Non-Hermitian Su-Schrieffer-Heeger model



➢ Eigenvalue equation:

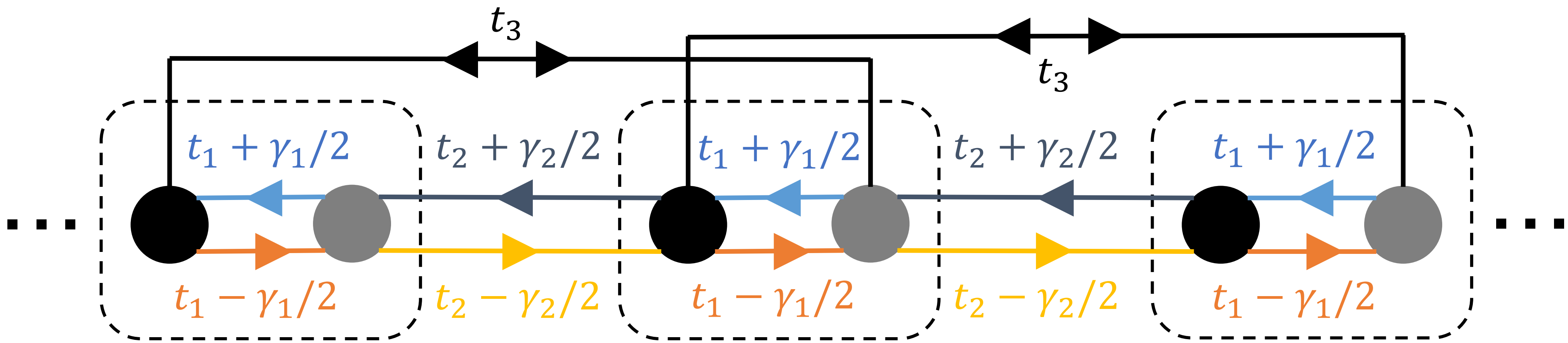
$$\left[\left(t_2-\frac{\gamma_2}{2}\right)\beta^{-1}+\left(t_1+\frac{\gamma_1}{2}\right)+t_3\beta\right]\left[t_3\beta^{-1}+\left(t_1-\frac{\gamma_1}{2}\right)+\left(t_2+\frac{\gamma_2}{2}\right)\beta\right]=E^2$$

Condition for the generalized Brillouin zone: $|\beta_2|=|\beta_3|$

for the solutions of the eigenvalue equation with $|\beta_1|\le|\beta_2|\le|\beta_3|\le|\beta_4|$

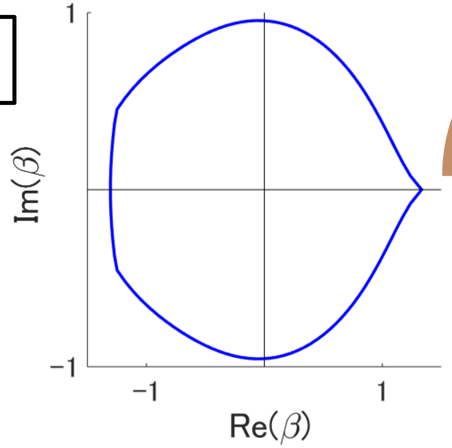
Non-Bloch band theory in a tight-binding system

✓ Non-Hermitian Su-Schrieffer-Heeger model

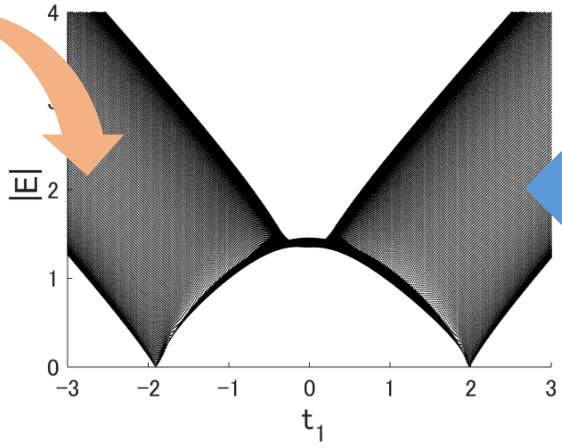


➢ Generalized Brillouin zone:

$\beta \equiv e^{ik}$

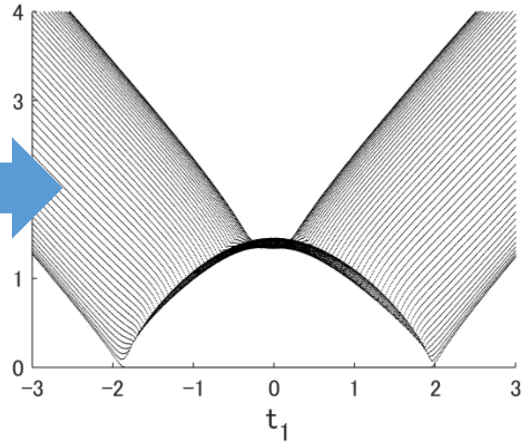


➢ Energy bands :



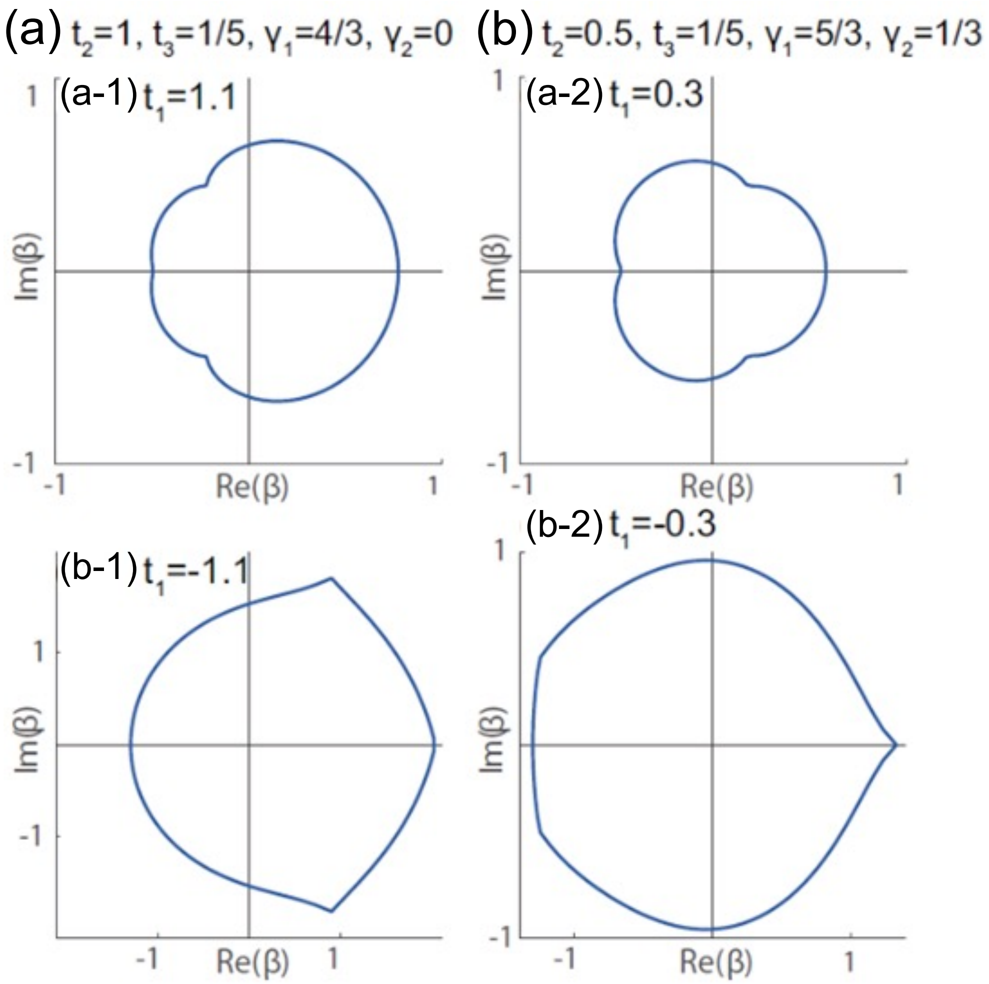
Match

➢ Energy levels ($L = 100$):



Non-Bloch band theory in a tight-binding system

✓ Generalized Brillouin zone (GBZ) $\left(\beta = e^{ik}, k \in \mathbb{C}\right)$



Feature

- GBZ forms loops encircling the origin on the β plane.
- GBZ depends on the system parameters.
- GBZ can have the cusps.
- GBZ becomes a unit circle when the system becomes Hermitian ($k \in \mathbb{R}$).

Non-Bloch band theory in a continuous system

So far,…

Most of the previous works studied some non-Hermitian systems by using a tight-binding model.

- The Non-Bloch band theory has been applied only to a tight-binding system.

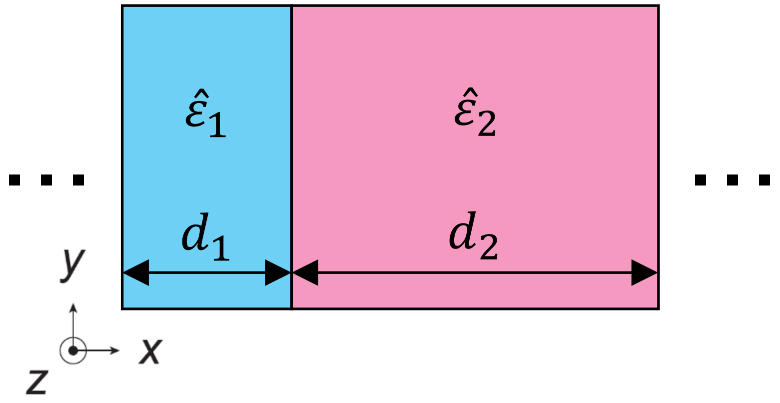
✓ Purpose

We construct the non-Bloch band theory in a continuous system.

KY, T. Yoda, and S. Murakami, Phys. Rev. Research **4**, 023089 (2022)

Non-Bloch band theory in a continuous system

✓ Photonic crystal



Lattice constant: $d_1 + d_2 \equiv a$

- Dielectric tensor: $\hat{\varepsilon}_i = \begin{pmatrix} \varepsilon_{i,xx} & \varepsilon_{i,xy} & 0 \\ \varepsilon_{i,yx} & \varepsilon_{i,yy} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, (i = 1,2)$
- Transverse-electric modes: $H_z(x, y) = H(x)e^{ik_y y}$

✓ Eigenvalue equation:

$$\nabla\times\left(\frac{1}{\hat{\varepsilon}(\boldsymbol{r})}\nabla\times\boldsymbol{H}(\boldsymbol{r})\right) = \left(\frac{\omega}{c}\right)^2 \boldsymbol{H}(\boldsymbol{r})$$

$(\equiv \hat{\eta}(\boldsymbol{r}))$

- Wave equation:

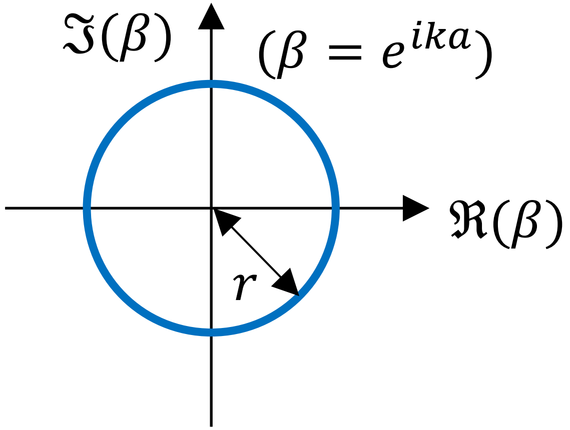
$$\left[-\frac{d}{dx}\eta_{yy}(x)\frac{d}{dx} - \frac{i}{2}\left(-2k_y\eta_{xy}(x)\frac{d}{dx} + \frac{d}{dx}\left(-2k_y\eta_{yx}(x)\right)\right) + k_y^2\eta_{xx}(x)\right]H(x) = \left(\frac{\omega}{c}\right)^2 H(x)$$

Gauge potential term

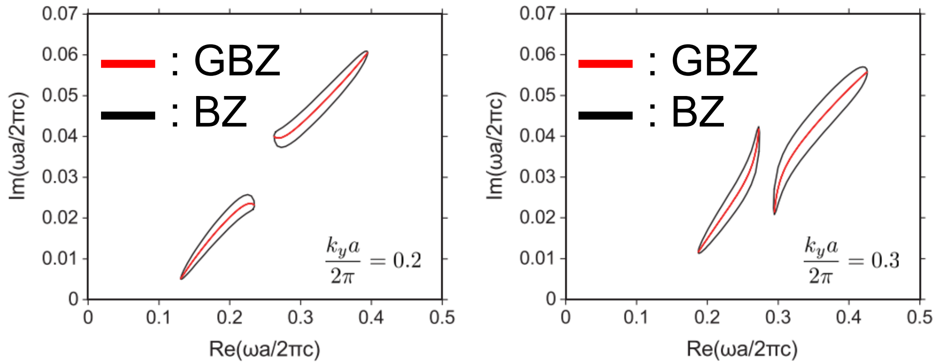
Non-Bloch band theory in a continuous system

✓ Photonic crystal

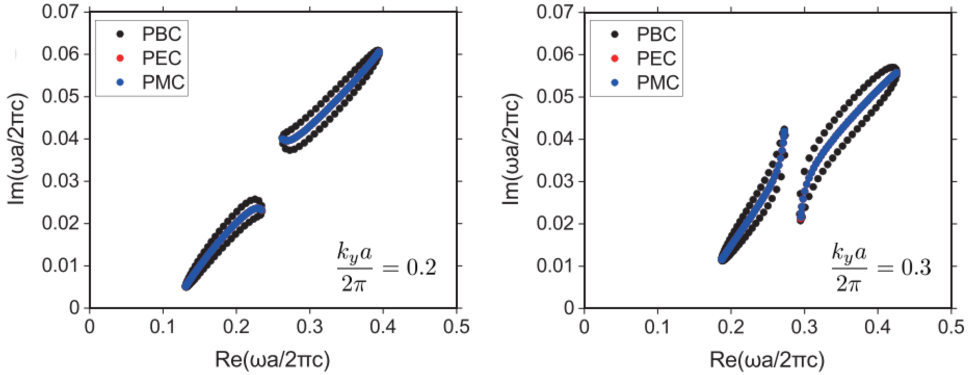
- Generalized Brillouin zone: $r = \exp\left(\frac{k_y}{2}\Im\sum_{i=1}^{2}\frac{\varepsilon_{i,xy}+\varepsilon_{i,yx}}{\varepsilon_{i,xx}}d_1\right)$

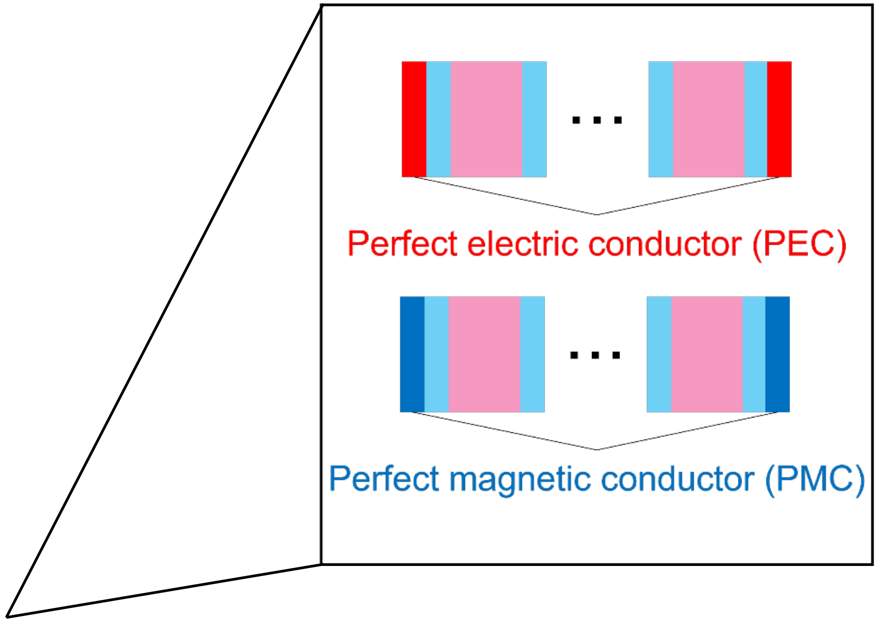


- Band:



- Eigenvalue:





PEC: $E_y(0) = E_y(L) = 0$

PMC: $H_z(0) = H_z(L) = 0$

Summary

- ✓ We construct the non-Bloch band theory in a non-Hermitian tight-binding system and in a non-Hermitian continuous system.

  KY and S. Murakami, Phys. Rev. Lett. **123**, 066404 (2019) KY, T. Yoda, and S. Murakami, Phys. Rev. Research **4**, 023089 (2022)

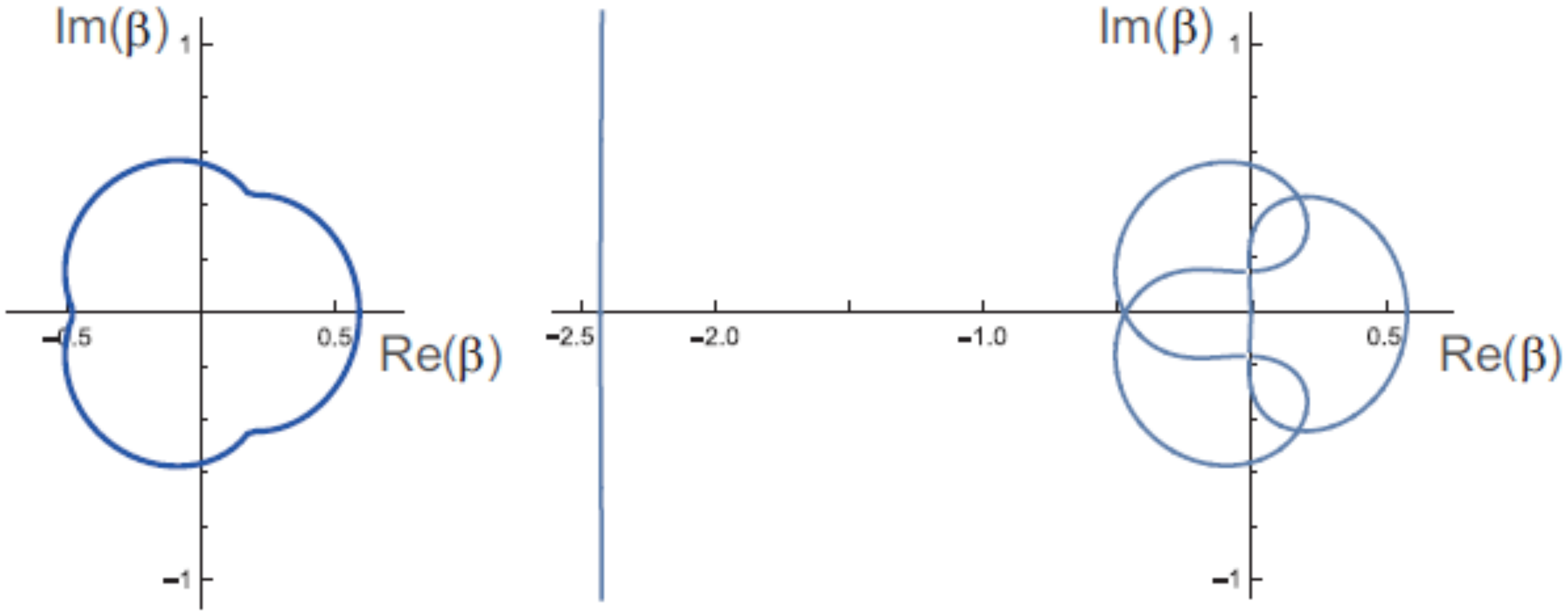
- ✓ We show that energy bands are obtained from the generalized Brillouin zone $\beta = e^{ik}$ $(k \in \mathbb{C})$.

- ✓ In a tight-binding system, the generalized Brillouin zone can have cusps and depends on the system parameters.

- ✓ In a continuous system, the generalized Brillouin zone forms a circle, which means that the localization lengths of all the skin modes are common.

Supplemental Material

(a) Trajectories of $|\beta_2| = |\beta_3|$ (b) Trajectories of $|\beta_i| = |\beta_j|, (i \neq j)$



The cusps appear
when <u>three of the four solutions of the eigenvalue equation</u>
<u>share the same absolute value</u>.

Supplemental Material

✓ <u>Condition for the generalized Brillouin zone</u> $(|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq |\beta_4|)$

➢ <u>Boundary equation:</u>

$$[(\beta_1\beta_2)^{L+1} + (\beta_3\beta_4)^{L+1}](X_1 - X_2)(X_3 - X_4) + [(\beta_1\beta_4)^{L+1} + (\beta_2\beta_3)^{L+1}](X_1 - X_4)(X_2 - X_3)$$
$$-[(\beta_1\beta_3)^{L+1} + (\beta_2\beta_4)^{L+1}](X_1 - X_3)(X_2 - X_4) = 0$$

- $X_j = \frac{1}{(t_2 - \gamma_2/2)\beta_j^{-1} + (t_1 + \gamma_1/2) + t_3\beta_j}, (j = 1, \dots, 4)$
- L: system size

Supplemental Material

✓ Condition for the generalized Brillouin zone $(|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq |\beta_4|)$

➢ Boundary equation:

$$[(\beta_1\beta_2)^{L+1} + (\beta_3\beta_4)^{L+1}](X_1 - X_2)(X_3 - X_4) + [(\beta_1\beta_4)^{L+1} + (\beta_2\beta_3)^{L+1}](X_1 - X_4)(X_2 - X_3)$$
$$-[(\beta_1\beta_3)^{L+1} + (\beta_2\beta_4)^{L+1}](X_1 - X_3)(X_2 - X_4) = 0$$

Leading term

- $X_j = \frac{1}{(t_2 - \gamma_2/2)\beta_j^{-1} + (t_1 + \gamma_1/2) + t_3\beta_j}, (j = 1, \dots, 4)$
- L: system size

If $|\beta_2| \neq |\beta_3|$,

we have $(X_1 - X_2)(X_3 - X_4) = 0$ in the thermodynamic limit $L \to \infty$.

➡ It does not lead to the energy bands.

Supplemental Material

✓ Condition for the generalized Brillouin zone $(|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq |\beta_4|)$

➢ Boundary equation:

$$[(\beta_1\beta_2)^{L+1} + (\beta_3\beta_4)^{L+1}](X_1 - X_2)(X_3 - X_4) + [(\beta_1\beta_4)^{L+1} + (\beta_2\beta_3)^{L+1}](X_1 - X_4)(X_2 - X_3)$$

$$-[(\beta_1\beta_3)^{L+1} + (\beta_2\beta_4)^{L+1}](X_1 - X_3)(X_2 - X_4) = 0$$

Leading term: $(\beta_3\beta_4)^{L+1}$

Leading term: $(\beta_2\beta_4)^{L+1}$

- $X_j = \frac{1}{(t_2-\gamma_2/2)\beta_j^{-1}+(t_1+\gamma_1/2)+t_3\beta_j}, (j = 1, \dots, 4)$
- L: system size

If $|\beta_2| = |\beta_3|$,

we have $\left(\frac{\beta_3}{\beta_2}\right)^{L+1} = \frac{(X_1-X_3)(X_2-X_4)}{(X_1-X_2)(X_3-X_4)}$ in the thermodynamic limit $L \to \infty$.

➡ We can get a dense set of the solutions by changing the relative phase between β_2 and β_3 .

= Energy band