Workshop on Non-Hermitian Quantum Mechanics
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Exact Bethe-ansatz solutions for open quantum many-body systems

Masaya Nakagawa
(Univ. of Tokyo)

in collaboration with
Masahito Ueda (Tokyo), Kazuki Yamamoto, Masaki Tezuka, and Norio Kawakami (Kyoto)

MN, N. Kawakami, and M. Ueda, PRL 126, 110404 (2021)

K. Yamamoto, **MN**, M. Tezuka, M. Ueda, and N. Kawakami, PRB 105, 205125 (2022)

Quantum integrability

- **Integrable model**: interacting, many-body, but still exactly solvable!

$$H = J\sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

e.g. 1D Heisenberg model

First solved by Hans Bethe: **Bethe ansatz**

[H. Bethe, Z. Phys. 71, 205 (1931)]

- Triumphs in **equilibrium** systems & dynamics in **isolated** systems

$$Z = \mathrm{Tr}[e^{-\beta H}], \qquad i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

- **Open quantum system**: coupled to environment, non-unitary dynamics

$$\frac{d\rho(t)}{dt} = \mathcal{L}\rho(t)$$

$\rho(t)$: density matrix of the system

$\mathcal{L}$: **Liouvillian superoperator (non-Hermitian!)**

This talk:

Exact solutions for open quantum many-body systems

Non-Hermitian Bethe ansatz

Outline

1. Introduction

   - Quantum master equation & Liouvillian superoperator

2. Exact solution of a 1D dissipative Hubbard model

   [MN, N. Kawakami, and M. Ueda, PRL 126, 110404 (2021)]

3. Exact solution of a 1D non-Hermitian XXZ model

   [K. Yamamoto, MN, M. Tezuka, M. Ueda, and N. Kawakami,
   PRB 105, 205125 (2022)]

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4. Summary

Quantum master equation

■ Time-evolution eq. for Markovian open quantum systems

[Gorini, Kossakowski, and Sudarshan, J. Math. Phys. 17, 821 (1976)]

[Lindblad, Commun. Math. Phys. 48, 119 (1976)]

Quantum master equation in the GKSL form

(ρ: density matrix of the system)

$$\frac{d\rho}{dt} = \mathcal{L}\rho = -i[H,\rho] + \sum_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\}\right)$$

Hamiltonian part

Dissipation (L_α: Lindblad operator)

Question:

Exact solution of the quantum master equation?

Liouvillian spectrum

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_\alpha (L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\}) \equiv \mathcal{L}\rho$$

Liouvillian superoperator (non-Hermitian)

■ Diagonalization of Liouvillian

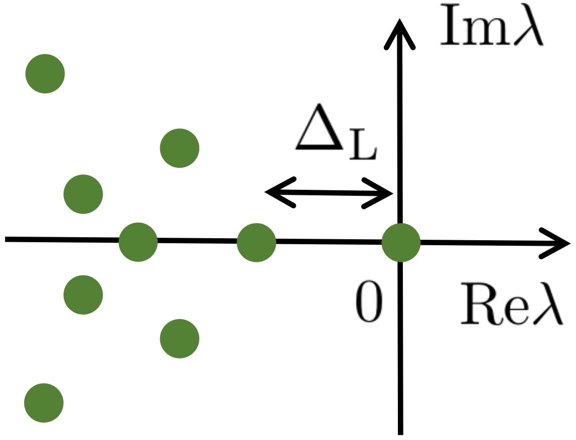
$$\mathcal{L}\rho_n = \lambda_n \rho_n \quad \Rightarrow \quad \rho(t) = \sum_n c_n e^{\lambda_n t} \rho_n$$

$\{\lambda_n\}_n$ **: Liouvillian spectrum**

■ General properties

- $\lambda_n \in \mathbb{C}$ **(non-Hermiticity)**, $\mathrm{Re}\lambda_n \leq 0$
- $\lambda_0 = 0$ **: steady state**
- $\Delta_\mathrm{L} \equiv \min_{\lambda_n \neq 0}[-\mathrm{Re}\lambda_n]$ **: Liouvillian gap**

  → inverse of the relaxation time



[cf. nontrivial cases: Haga, MN *et al*., PRL 127, 070402; Mori and Shirai, PRL 125, 230604]

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Setup

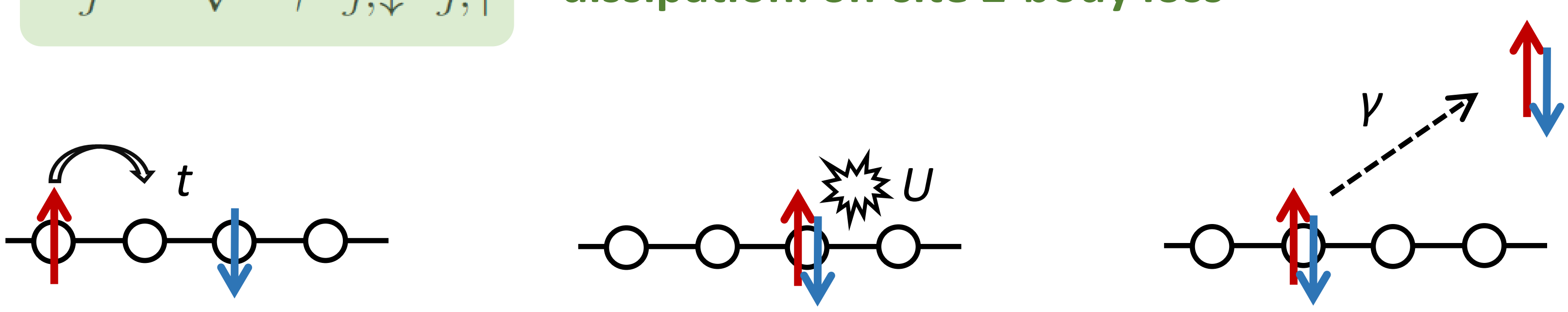
■ Model: 1D dissipative Fermi-Hubbard model

$$\frac{d\rho}{d\tau} = -i[H,\rho] + \sum_{j=1}^{L}\left(L_j \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho\}\right)$$ quantum master eq.

$$H = -t\sum_{j=1}^{L}\sum_{\sigma=\uparrow,\downarrow}(c_{j,\sigma}^\dagger c_{j+1,\sigma} + \mathrm{H.c.}) + U\sum_{j=1}^{L} n_{j,\uparrow} n_{j,\downarrow},$$ Hubbard Hamiltonian

hopping **interaction** $(n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma})$

$$L_j = \sqrt{2\gamma}\, c_{j,\downarrow} c_{j,\uparrow}$$ **dissipation: on-site 2-body loss**



hopping **interaction** **dissipation**

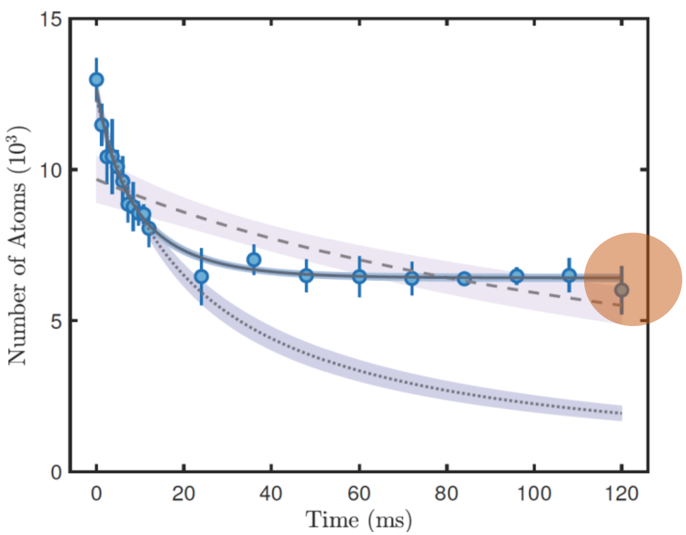
Experimental advance: dissipative Hubbard model

Dissipative Fermi-Hubbard model: experiment

[Sponselee *et al.*, Quant. Sci. Tech. 4, 014002 (2018)]

Ultracold atoms in an optical lattice → Fermi-Hubbard model

Atoms in an excited state → inelastic collisions (2-body loss)



<u>Non-equilibrium steady state</u>:

Nonzero number of particles remain even in the presence of particle loss!

Dissipative Hubbard model:

Strongly interacting dissipative system!

Diagonalization of the Liouvillian

■ Step 1: Decomposition of the Liouvillian

$$\mathcal{L}\rho = -i[H,\rho] + \sum_j (L_j \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho\})$$

$$= -i(H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger) + \sum_j L_j \rho L_j^\dagger$$

"Quantum jump" term

→ decreases particle #

Non-Hermitian Hubbard model

$$H_{\text{eff}} = -t\sum_{j,\sigma}(c_{j,\sigma}^\dagger c_{j+1,\sigma} + \text{H.c.}) + (U - i\gamma)\sum_j n_{j,\uparrow} n_{j,\downarrow}$$

■ Step 2: Diagonalize the non-Hermitian Hamiltonian

$$H_{\text{eff}} |N,a\rangle = E_{N,a} |N,a\rangle \ ,$$ (N: particle number → conserved quantity of H_{eff})

⇨

$$-i(H_{\text{eff}}\varrho_{ab}^{(N,n)} - \varrho_{ab}^{(N,n)} H_{\text{eff}}^\dagger) = \lambda_{ab}^{(N,n)} \varrho_{ab}^{(N,n)},$$

$$\lambda_{ab}^{(N,n)} = -i(E_{N,a} - E_{N+n,b}^*), \ \ \varrho_{ab}^{(N,n)} = |N,a\rangle\langle N+n,b|$$

Diagonalization of the Liouvillian

■ Step 3: Consider matrix elements in the particle-number basis

$$\sum_j L_j \varrho_{ab}^{(N,n)} L_j^\dagger = \sum_{c,d} A_{ab,cd}^{(N,n)} \varrho_{cd}^{(N-2,n)}$$

2-body loss
***N* → *N* - 2**

$$\Rightarrow \quad \mathcal{L} = \begin{pmatrix} \lambda^{(N,n)} & & & \\ A^{(N,n)} & \lambda^{(N-2,n)} & & \\ & A^{(N-2,n)} & & \\ & & & \ddots \end{pmatrix}$$

Triangular structure!

[Torres, PRA 89, 052133 (2014)]

⇒ Liouvillian eigenvalue $\lambda_{ab}^{(N,n)} = -i(E_{N,a} - E_{N+n,b}^*),$

Liouvillian eigenoperator $\rho_{ab}^{(N,n)} = C_{ab}^{(N,n)} \varrho_{ab}^{(N,n)} + \sum_{N'=0}^{N-2} \sum_{a',b'} C_{a',b'}^{(N',n)} \varrho_{a'b'}^{(N',n)}$

Integrable non-Hermitian Hamil. + Triangular structure → Solvable Liouvillian

[MN, N. Kawakami, and M. Ueda, PRL 126, 110404 (2021)]

Non-Hermitian Bethe ansatz

■ Step 4: Solve the non-Hermitian Hubbard model

$$H_{\mathrm{eff}} = -t\sum_{j,\sigma}(c_{j,\sigma}^{\dagger}c_{j+1,\sigma} + \mathrm{H.c.}) + (U - i\gamma)\sum_{j} n_{j,\uparrow}n_{j,\downarrow}$$

complex-valued Hubbard interaction strength!

■ 1D Hubbard model → solvable by Bethe ansatz [Lieb and Wu, PRL 20, 1445 (1968)]

→ Generalized solution of the non-Hermitian case

[MN, N. Kawakami, and M. Ueda, PRL 126, 110404 (2021)]

Bethe eqs.

$$e^{ik_jL} = \prod_{\beta=1}^{M}\frac{\sin k_j - \lambda_\beta + iu}{\sin k_j - \lambda_\beta - iu}, \quad (j = 1, \cdots, N)$$

$$\prod_{j=1}^{N}\frac{\lambda_\alpha - \sin k_j + iu}{\lambda_\alpha - \sin k_j - iu} = -\prod_{\beta=1}^{M}\frac{\lambda_\alpha - \lambda_\beta + 2iu}{\lambda_\alpha - \lambda_\beta - 2iu}, \quad (\alpha = 1, \cdots, M)$$

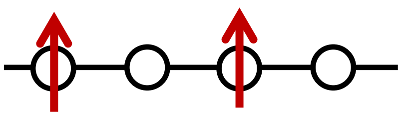
N : # of particles, M : # of down spins, $u = (U - i\gamma)/(4t)$

k_j : quasi-momentum, λ_α : spin rapidity

Ferromagnetic steady state

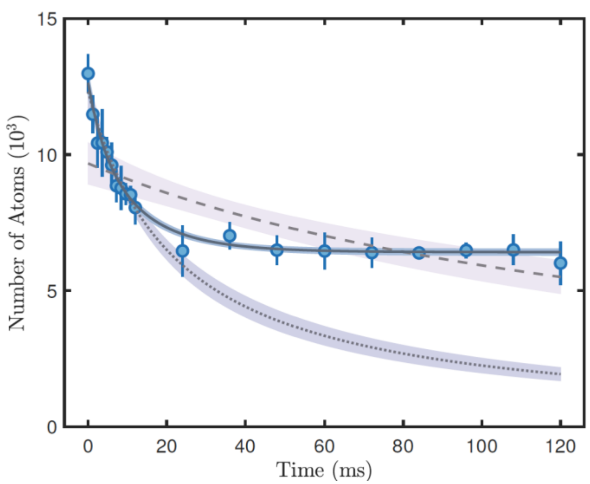
- Steady state: $\mathcal{L}(|\Psi\rangle\langle\Psi|) = 0$, if $H|\Psi\rangle = E|\Psi\rangle$, $L_j|\Psi\rangle = 0$
- Vacuum $|0\rangle\langle 0|$ is a trivial steady state: $H|0\rangle = L_j|0\rangle = 0$
- Other steady states?

  Spin-polarized eigenstate $|\Psi\rangle\langle\Psi|$, $H|\Psi\rangle = E|\Psi\rangle$



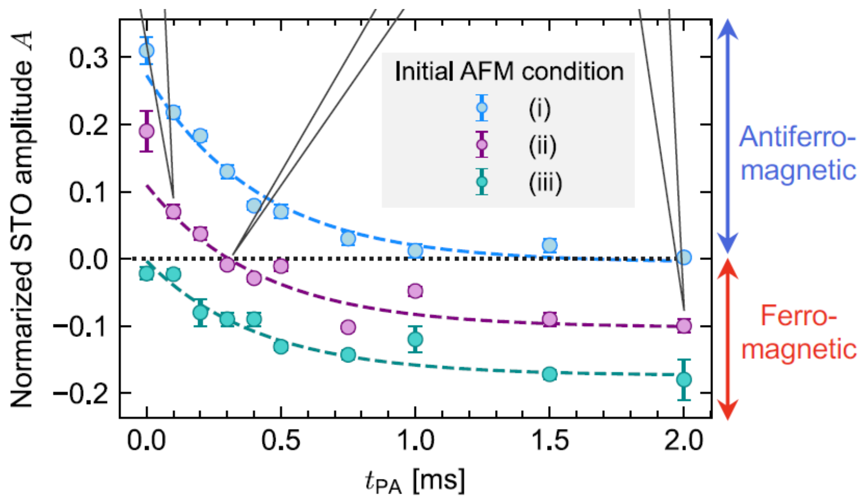
  → No loss: **ferromagnetic steady state!**

  Spin SU(2) symmetry → Ferromagnetic steady states are degenerate



Non-vacuum steady state

[Sponselee *et al*., Quant. Sci. Tech. 4, 014002 (2018)]



Ferromagnetic spin correlation

[Honda, MN *et al*., arXiv: 2205.13162]

Dissipative spin-exchange interaction

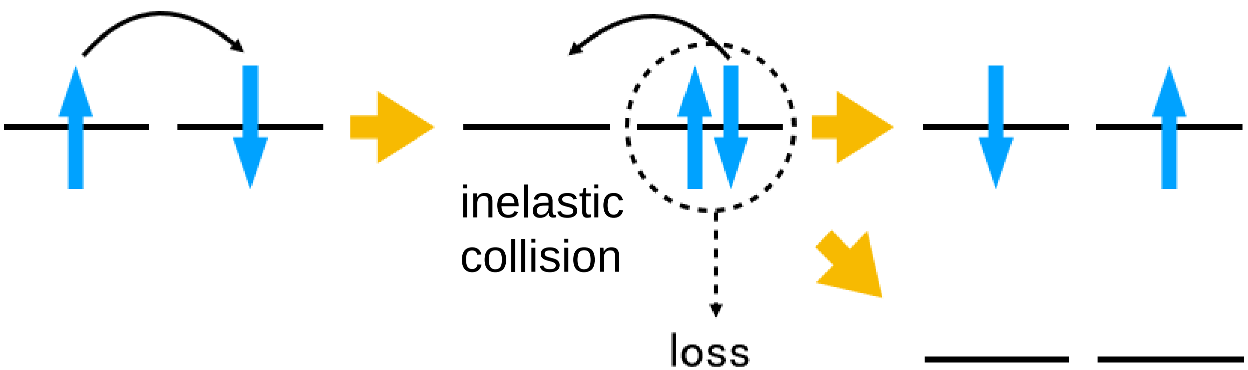
■ Physical origin of the ferromagnetic steady state:

Spin-exchange interaction in the presence of **inelastic collisions**

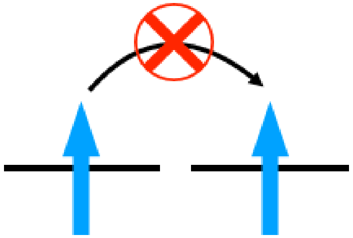
[MN, N. Tsuji, N. Kawakami, and M. Ueda, PRL 124, 147203 (2020)]

◇ **Antiparallel spin configuration → lower the energy,**

but have a finite lifetime



◇ **Parallel spin configuration → higher energy, but free from loss!**



Ferromagnetism is stabilized by dissipation!

Liouvillian gap

- Exact solution of Liouvillian eigenmodes near the steady state

  → **ferromagnetic spin-wave-like excitations**

Dispersion relation from Bethe ansatz ($\Delta P \simeq 0$)

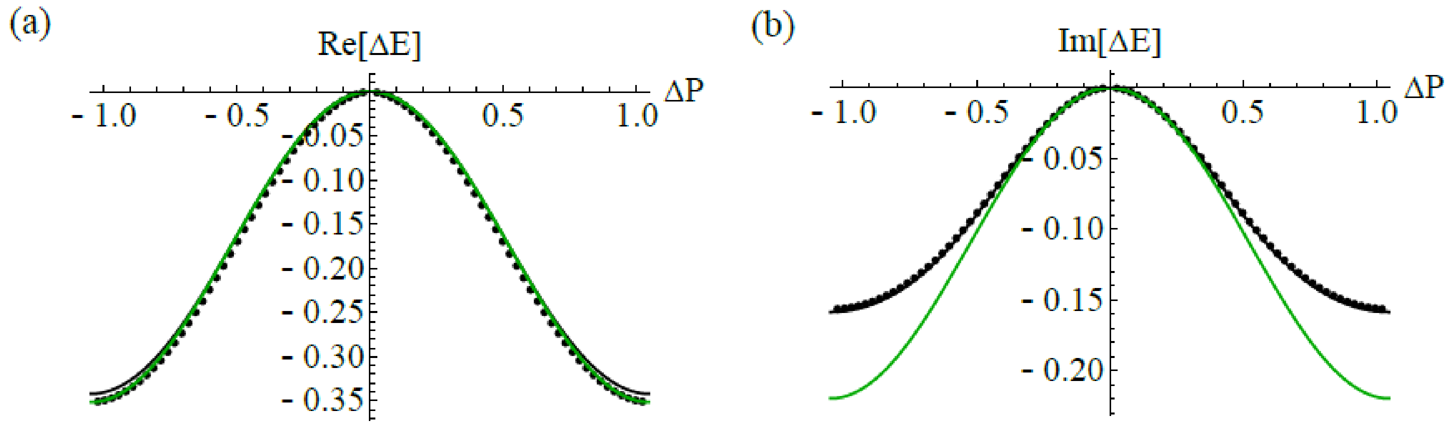
$$\Delta E \simeq -\frac{t}{\pi u}\left(Q_0 - \frac{1}{2}\sin 2Q_0\right)\left(1 - \cos\frac{\pi \Delta P}{Q_0}\right) \quad (Q_0 = \pi N/L)$$

$$\propto (\Delta P)^2$$ **gapless excitation!**

Liouvillian gap

$$\Delta_{\rm L} \propto (\Delta P)^2 \sim \frac{1}{L^2}$$

Relaxation time τ_R diverges as $\tau_R \sim L^2$ in $L \to \infty$ limit



(Dot: numerics, L = 240, N = 80. Black line: numerics in $L \to \infty$. Green line: analytic, applicable to $\Delta P \simeq 0$)

Correlation length

- Another eigenmode (w/ lifetime): Mott insulator (ground state if $\gamma = 0$)

- **Correlation length** ξ defined from charge stiffness D

[Stafford and Millis, PRB 48, 1409 (1993)]

$$D = \frac{d^2 E_0}{d\Phi^2}\Big|_{\Phi=0}$$

Change of the eigenenergy E_0
due to a twisted boundary condition
with angle Φ

$$D \sim \exp[-L/\xi] \quad (L \to \infty)$$

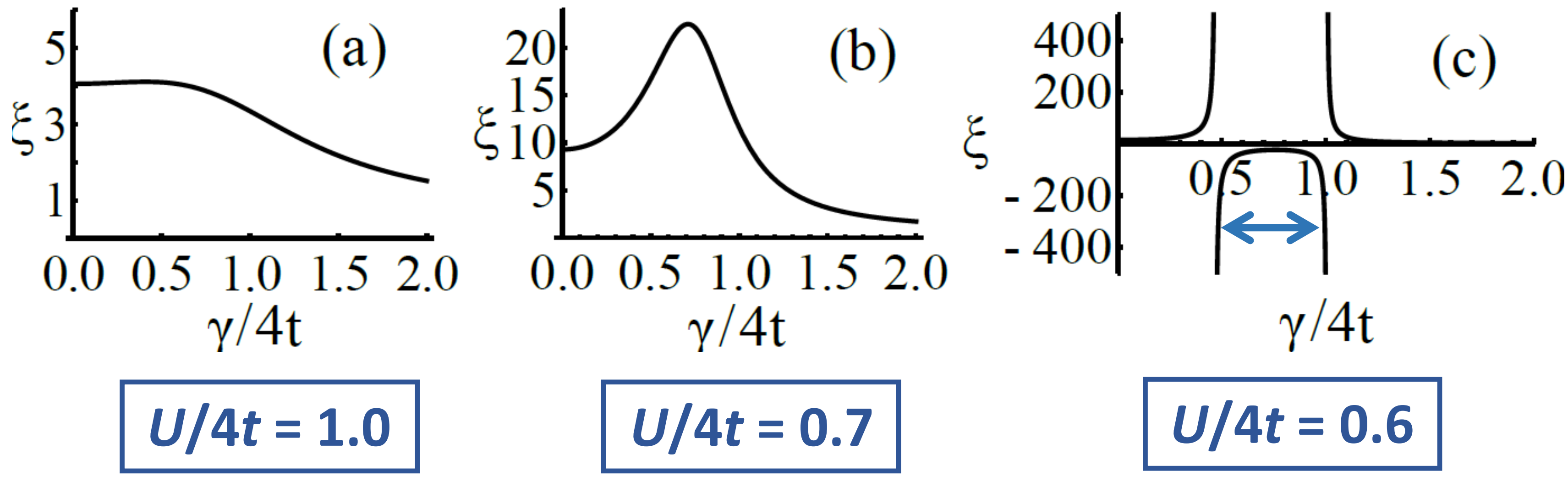
Localized state cannot "feel" the change of
the boundary condition

Exact result from Bethe ansatz

$$1/\xi = \mathrm{Re}\left[\frac{4t}{U - i\gamma}\int_1^\infty dy \frac{\ln(y + \sqrt{y^2 - 1})}{\cosh(2t\pi y/(U - i\gamma))}\right]$$

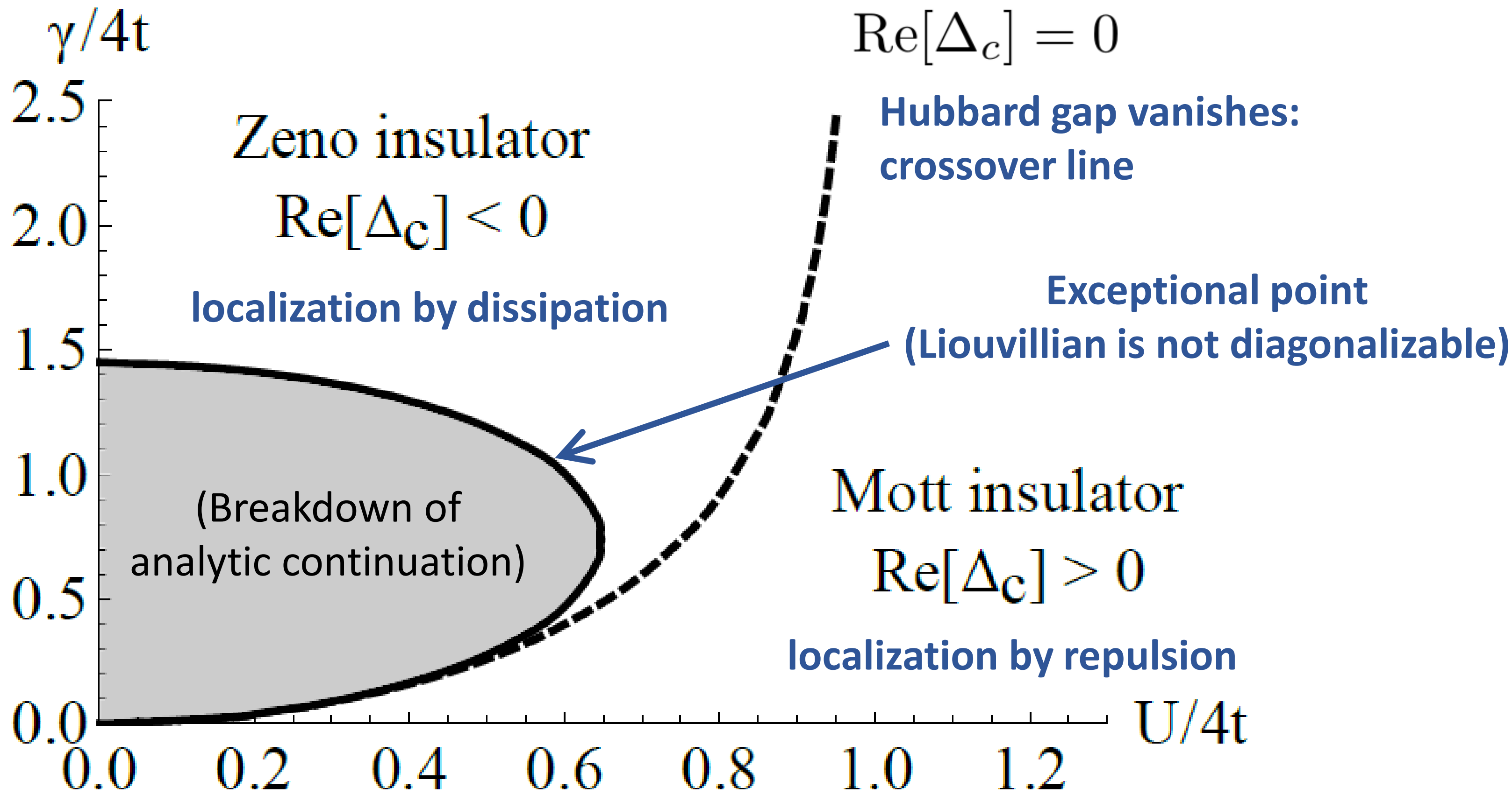
Dissipation-induced divergence of correlation length

■ Correlation length as a function of dissipation strength



- ✓ **Localization due to strong dissipation: <u>quantum Zeno effect</u>**
- ✓ **Dissipation-induced divergence of the correlation length!**
- ✓ **Negative correlation length: breakdown of analytic continuation**

"Phase diagram" of a dissipative Mott insulator



Many-body physics from interplay between interaction and dissipation

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   PRB 105, 205125 (2022)]

4. Summary

Setup

- 1D two-component Bose-Hubbard model + 2-body loss

Non-Hermitian Bose-Hubbard model

$$H_{\text{eff}} = -t_h \sum_j \sum_{\sigma=\uparrow,\downarrow} (b^\dagger_{j,\sigma} b_{j+1,\sigma} + \text{H.c.}) + \sum_j (U_{\uparrow\downarrow} - i\gamma_{\uparrow\downarrow}) n_{j,\uparrow} n_{j,\downarrow} + \frac{1}{2}\sum_{j,\sigma} (U_{\sigma\sigma} - i\gamma_{\sigma\sigma}) n_{j,\sigma}(n_{j,\sigma} - 1)$$

- Unit-filling sector, strong interaction limit ($U_{\sigma\sigma'} >> t$)

→ non-Hermitian XXZ spin chain: exactly solvable!

$$H_{\text{eff}} = \tilde{J} \sum_j (S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \Delta_\gamma S^z_j S^z_{j+1})$$

complex-valued Ising interaction parameter!

Non-Hermitian Tomonaga-Luttinger liquid

- Longest-lived state in the unit-filling sector

→ **Non-Hermitian Tomonaga-Luttinger liquid**

$$H_{\rm eff}^{\rm TL} = \frac{1}{2\pi}\int dx\left[\tilde{u}\tilde{K}(\nabla\theta(x))^2 + \frac{\tilde{u}}{\tilde{K}}(\nabla\phi(x))^2\right]$$ effective field theory

$\tilde{K}$ **: complex-valued Tomonaga-Luttinger parameter**

- Non-unitary quantum critical phenomena

[Ashida, Furukawa, and Ueda, PRA 94, 053615 (2016)]

Hermitian $\langle e^{i2\phi(x)}e^{-2i\phi(0)}\rangle = \left(\frac{\alpha}{x}\right)^{2K}, \quad \langle e^{i2\theta(x)}e^{-2i\theta(0)}\rangle = \left(\frac{\alpha}{x}\right)^{\frac{2}{K}}$ Single critical exponent

Non-Hermitian ${}_R\langle e^{i2\phi(x)}e^{-2i\phi(0)}\rangle_R = \left(\frac{\alpha}{x}\right)^{2K_\phi}, \quad {}_R\langle e^{i2\theta(x)}e^{-2i\theta(0)}\rangle_R = \left(\frac{\alpha}{x}\right)^{\frac{2}{K_\theta}}$ **Two critical exponents!**

$(1/K_\phi = \mathrm{Re}[1/\tilde{K}], K_\theta = \mathrm{Re}[\tilde{K}])$

New universality class: complex extension of conformal field theory

Bethe ansatz solution

- Exact Bethe-ansatz solution of the 1D non-Hermitian XXZ model

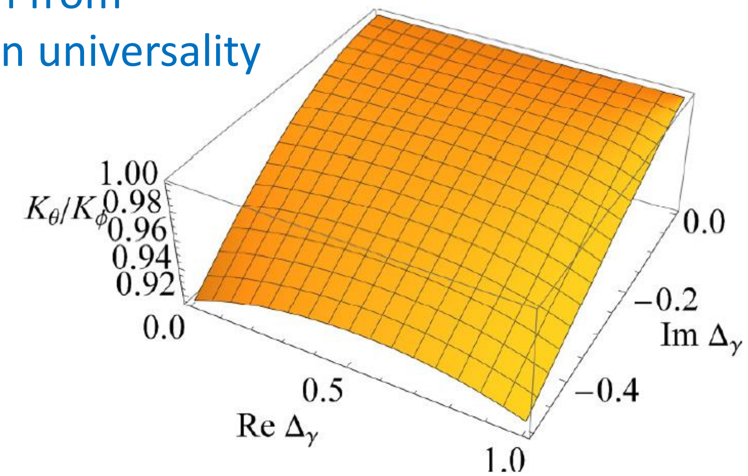
  → **Exact result on the complex-valued Tomonaga-Luttinger parameter**

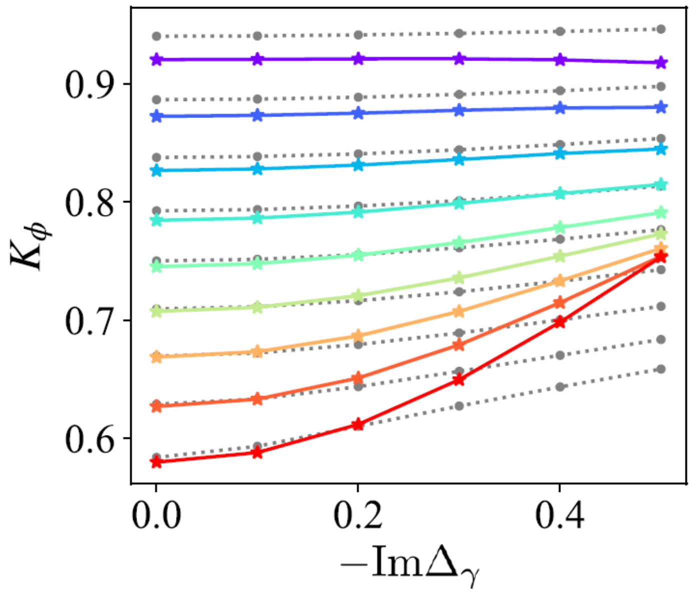
  (extracted from a finite-size correction to the ground-state energy)

  [K. Yamamoto, MN, M. Tezuka, M. Ueda, and N. Kawakami, PRB 105, 205125 (2022)]

$$\tilde{K} = \frac{\pi}{2(\pi - \arccos \Delta_\gamma)}$$

Deviation from Hermitian universality class





Comparison with DMRG calculation

Non-unitary universality class of non-Hermitian many-body systems!

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Summary

- Exact solution for open quantum many-body systems
  - **Triangular structure of the Liouvillian**
  - **Non-Hermitian Bethe ansatz**
- Exact solution of the 1D dissipative Hubbard model
  - **Dissipation-induced ferromagnetism**
  - **Dissipation-induced divergence of the correlation length**
- Exact solution of the 1D non-Hermitian XXZ model
  - **Non-Hermitian Tomonaga-Luttinger liquid & its criticality**

MN, N. Kawakami, and M. Ueda, PRL 126, 110404 (2021)

K. Yamamoto, MN, M. Tezuka, M. Ueda, and N. Kawakami, PRB 105, 205125 (2022)