Complexity Transitions of Boson Sampling in Non-unitary Dynamics

Ken Mochizuki,
Ryusuke Hamazaki
@RIKEN

Ken Mochizuki and R.Hamazaki, in preparation.

Outline

- PT symmetry breaking
- photonic experiments
- boson sampling problem
- motivation

} introduction

- model and PT symmetry breaking
- short-time dynamical complexity transition
- long-time dynamical complexity transition

} results

PT symmetry breaking

Parity-Time symmetry (PT symmetry) $H \neq H^{\dagger}$

PT symmetry of a Hamiltonian

$$(\mathcal{PT})H(\mathcal{PT})^{-1} = H$$

$$H\left|\phi_l\right\rangle = \varepsilon_l\left|\phi_l\right\rangle$$

C. M. Bender and S. Boettcher, Phys. Rev. Lett, **80**, 5243 (1998).

PT symmetry of eigenstates

preserved (oscillation) :

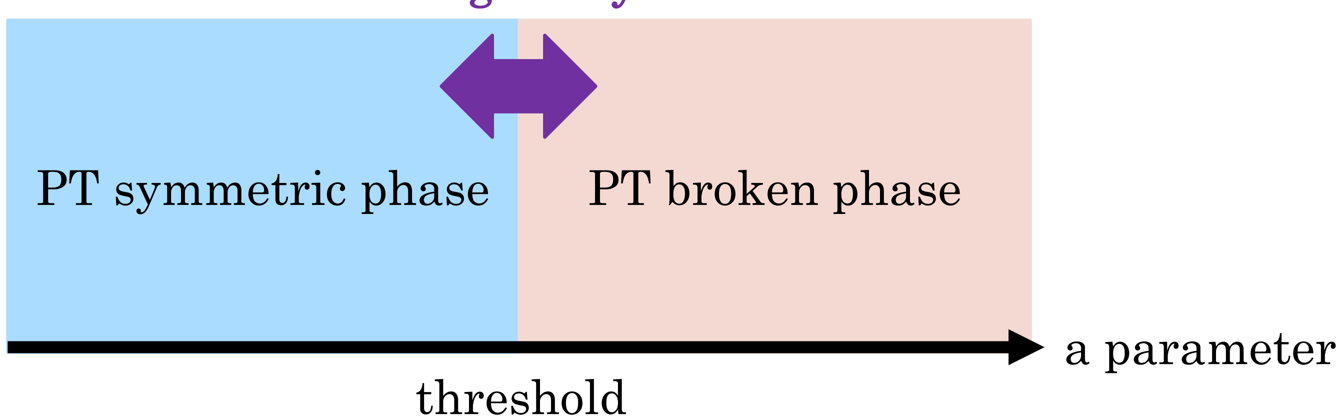
$$\mathcal{PT}\left|\phi_l\right\rangle = \left|\phi_l\right\rangle \to \varepsilon_l : \text{real}$$

broken (attenuation/divergence) :

$$\mathcal{PT}\left|\phi_l\right\rangle \neq \left|\phi_l\right\rangle \to \varepsilon_l : \text{complex}$$

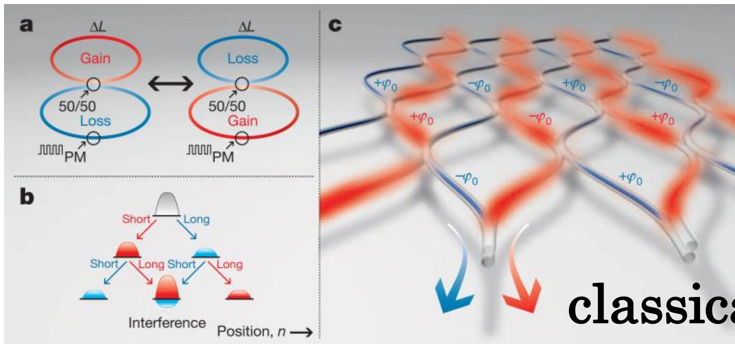
typical behavior in PT symmetric systems:

drastic change of dynamics!

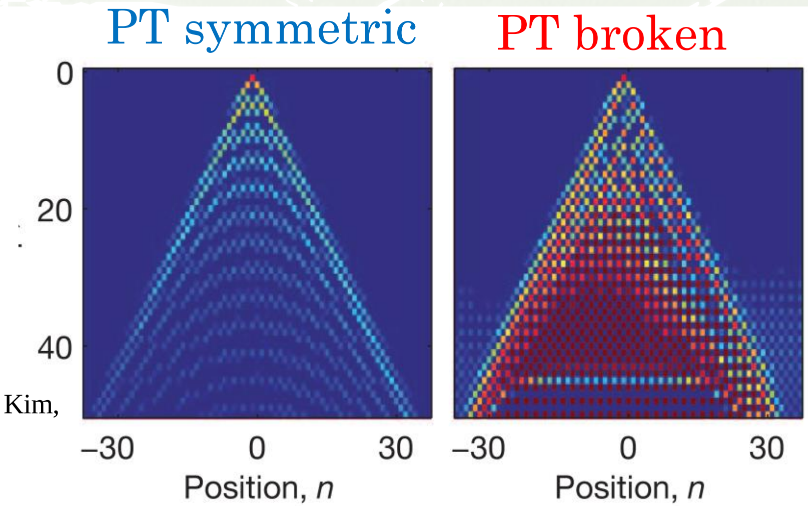


Optical experiments in both classical and quantum regimes

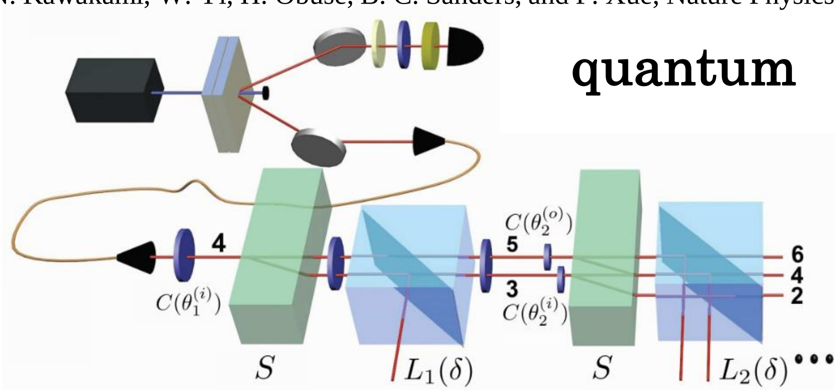
A. Regensburger, C. Bersch, M. A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, Nature **488**, 167 (2012)



classical

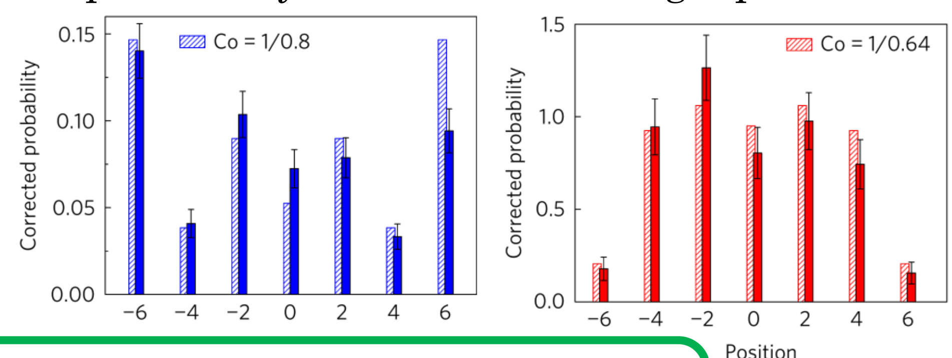


L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, **Ken Mochizuki**, D. Kim, N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nature Physics **13**, 1117 (2017).



quantum

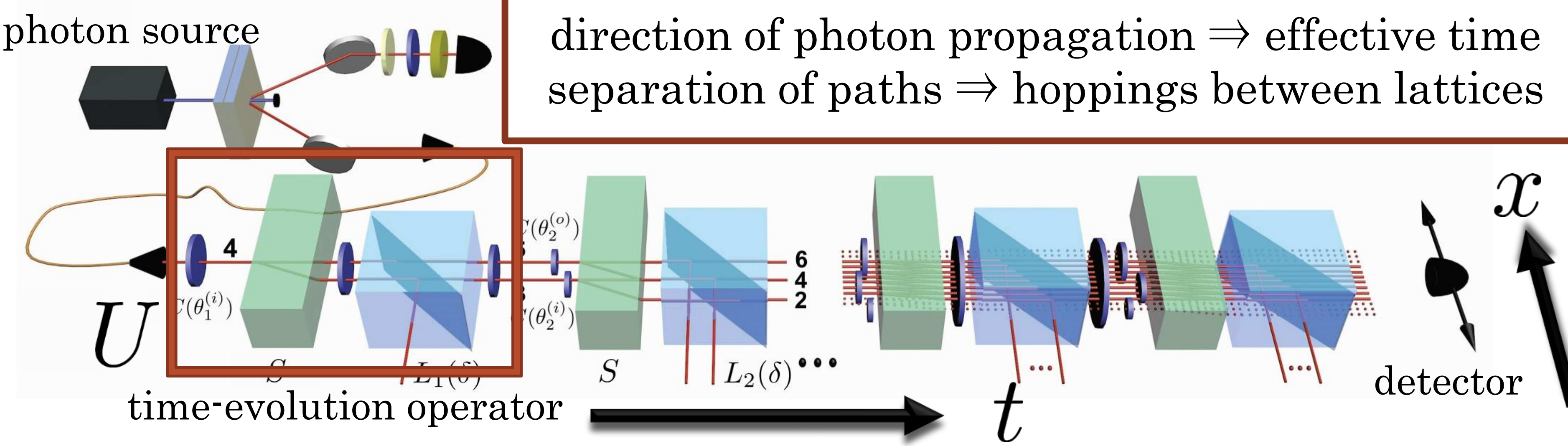
intensity configuration of coherent light
⇔probability distribution of single photons



Results are similar in both systems
⇒ Is there a distinct feature?

A quantum optical experiment (quantum walk)

direction of photon propagation ⇒ effective time
separation of paths ⇒ hoppings between lattices



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$$|\psi(t+1)\rangle = U\,|\psi(t)\rangle\,, \quad |\psi(t)\rangle = U^t\,|\psi(0)\rangle = e^{-iHt}\,|\psi(0)\rangle \qquad U = e^{-iH}$$

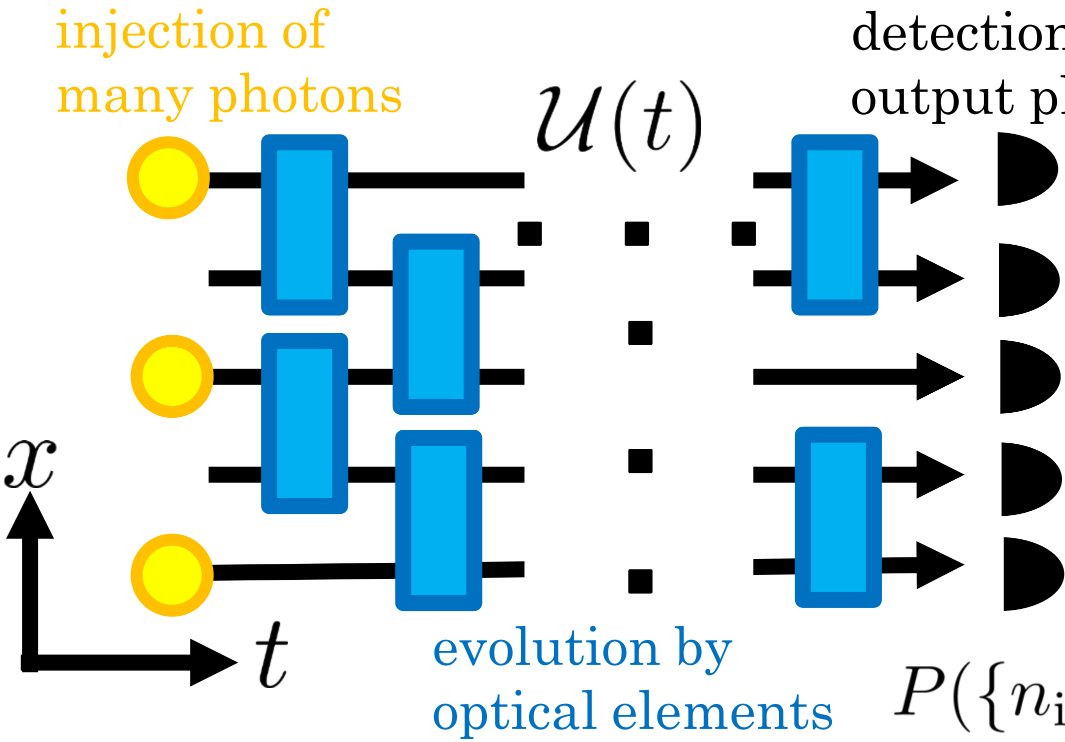
photon loss effect & post selection ⇒ $U^\dagger \neq U^{-1} \leftrightarrow H \neq H^\dagger$

$$|\psi(t)\rangle = \sum_{x=1}^{X} \begin{pmatrix} \psi_h(x,t) \\ \psi_v(x,t) \end{pmatrix} |x\rangle$$

conventional: single photons
many photons ⇒ non-classical behavior?

Extension to cases with many photons： boson sampling problem

single photons ⇒ many photons: boson sampling problem



evolution of
creation operators：

$$\hat{b}_i^\dagger \to \sum_{j=1}^{Y} \mathcal{U}_{ij}(t)\hat{b}_j^\dagger$$

$$\hat{b}_i\hat{b}_j^\dagger - \hat{b}_j^\dagger\hat{b}_i = \delta_{ij}$$

$$P(\{n_{\rm in}\},\{n_{\rm out}\},t)??$$

quantum walks： $\mathcal{U}^{\rm T}(t) = U^t, \quad j=(x,\sigma),\ \sigma=h,v,\ Y=2X$

Probability distribution of photons can be hard to compute
⇒ computational complexity/quantum supremacy

Computational complexity/Quantum supremacy

probability distribution of many photons：

$$P(\{n_{\rm in}\},\{n_{\rm out}\},t) = |{\rm Per}[W(t)]|^2/N(t)\prod_i n_i^{\rm in}!n_i^{\rm out}!$$

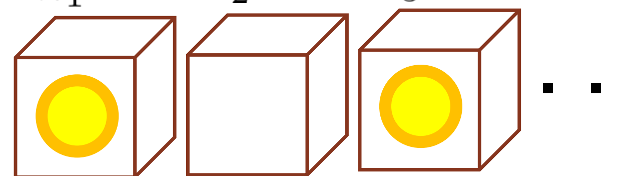
n ：# photons
Y ：# states

$$W_{pq}(t) = [\mathcal{U}(t)]_{{\rm in}_p {\rm out}_q}$$

${\rm in}_p/{\rm out}_p$：p-th input/output state

$$\{n_{\rm in/out}\} = (n_1^{\rm in/out},\cdots,n_Y^{\rm in/out}),\ \sum_{i=1}^{Y} n_i^{\rm in/out} = n$$

$n_1 = 1$ $n_2 = 0$ $n_3 = 1$



$i = 1$ $i = 2$ $i = 3$

$\mathcal{U}(t)$ ：random matrix ⇒ hard to compute

$\mathcal{U}(t)$ ：positive matrix ⇒ easy

U ：local and t：small ⇒ easy

S. Aaronson, A. Arkhipov, STOC’11, 333.
A. Deshpande, *et al*, PRL. **121**, 030501 (2018).

∗hard/easy：within polynomial time of n

physical situation ⇔ hardness for computing the distribution

⬇ may be possible $\qquad \mathcal{U}^{\rm T}(t) = U^t$

characterization of dynamics/phase by computational complexity

Objective

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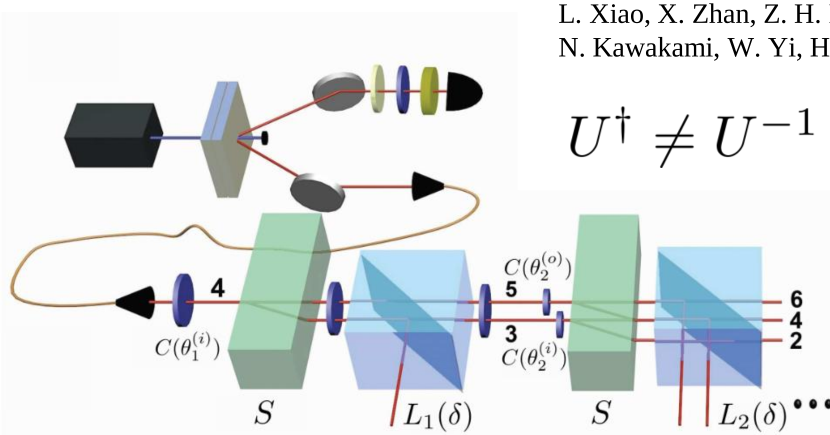
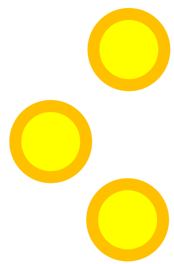


$$U^\dagger \neq U^{-1} \leftrightarrow H \neq H^\dagger$$

$$P(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t) = \frac{|\text{Per}[W(t)]|^2}{N(t)\prod_{j=1}^{Y} n_j^{\text{in}}! n_j^{\text{out}}!}$$

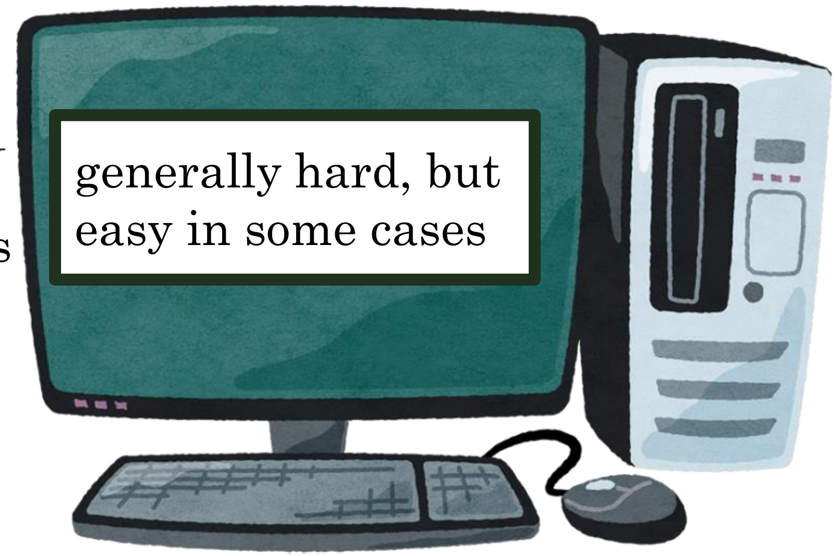


$$(\mathcal{PT})U(\mathcal{PT})^{-1} = U^{-1} \leftrightarrow (\mathcal{PT})H(\mathcal{PT})^{-1} = H$$

extension from single photons to many photons

boson sampling problem

$$\mathcal{U}^{\mathrm{T}}(t) = U^t$$



generally hard, but easy in some cases

computational complexity in PT symmetric non-Hermitian systems?

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PT symmetric model

$$|\psi(t+1)\rangle = U_{\rm gl}\,|\psi(t)\rangle$$

$$U_{\rm gl} = C(\theta_1/2)SG(+\gamma)C(\theta_2)G(-\gamma)SC(\theta_1/2)$$

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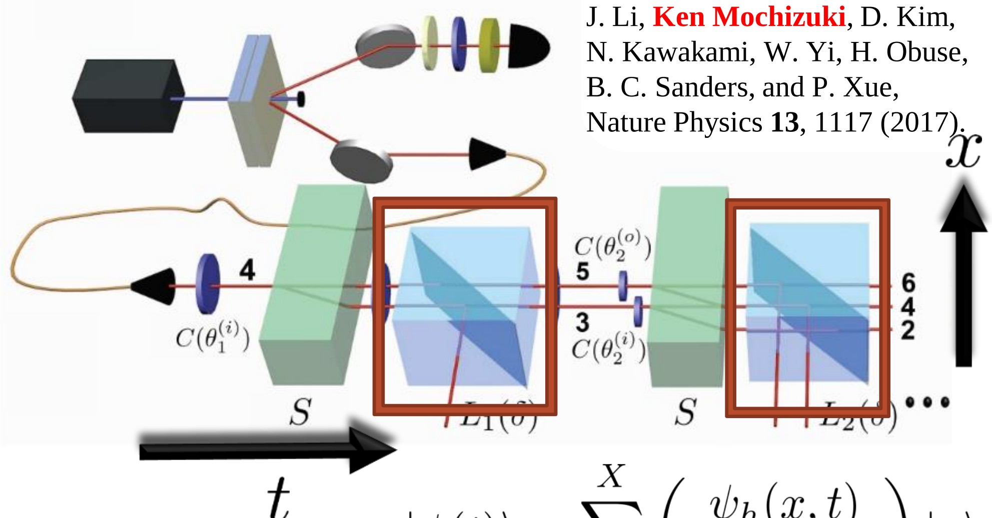
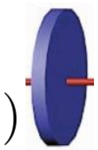


$$|\psi(t)\rangle = \sum_{x=1}^{X} \begin{pmatrix} \psi_h(x,t) \\ \psi_v(x,t) \end{pmatrix} |x\rangle$$

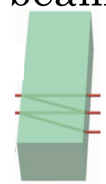
$$\mathcal{U}^{\rm T}(t) = U_{\rm gl}^{t}$$

wave plates：changes of polarizations



$$C(\theta) = \sum_{x=1}^{X} |x\rangle\langle x| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

beam displacers：position shift of photons



$$S = \sum_{x=1}^{X} \begin{pmatrix} |x-1\rangle\langle x| & 0 \\ 0 & |x+1\rangle\langle x| \end{pmatrix}$$

partially polarizing beam splitters：
polarization dependent photon losses
⇒nonunitary dynamics $U_{\rm gl}^{\dagger} \neq U_{\rm gl}^{-1}$

$$G(\gamma) = \sum_{x=1}^{X} |x\rangle\langle x| \begin{pmatrix} e^{+\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}$$

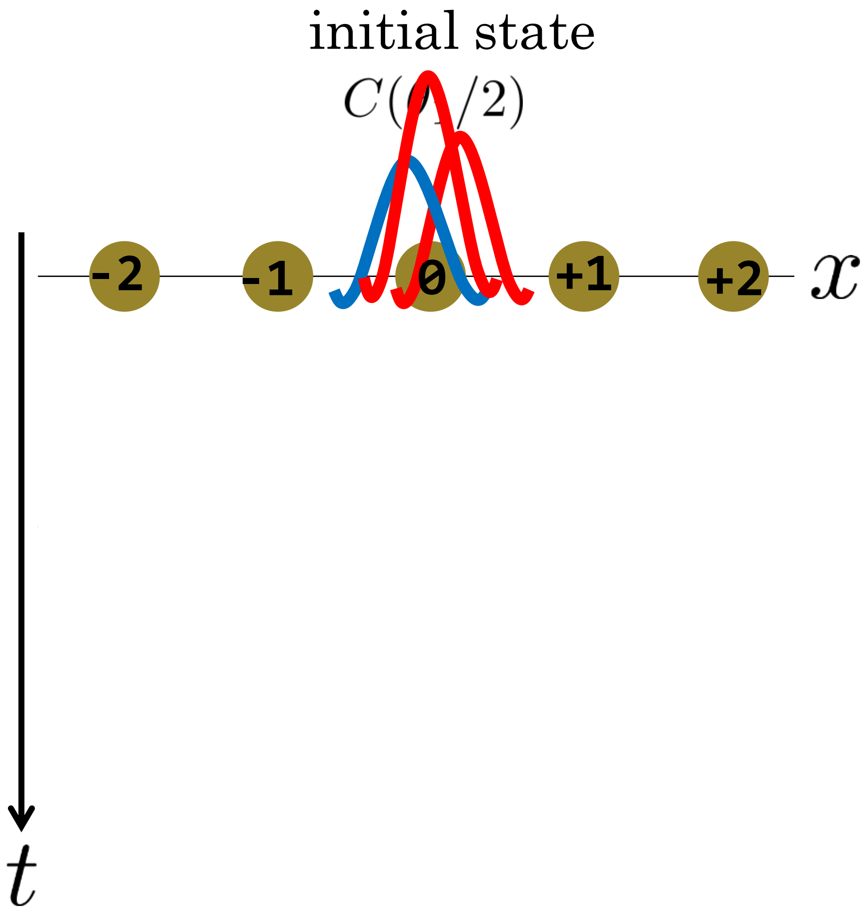
Gain effect of photons is also considered for theoretical simplicity, while only loss exists.

PT symmetry：$(\mathcal{PT})U_{\rm gl}(\mathcal{PT})^{-1} = U_{\rm gl}^{-1}, \quad \mathcal{PT} = \sum_{x} |-x\rangle\langle x| \otimes \sigma_3 \mathcal{K}$

Single-particle dynamics

$$|\psi(t+1)\rangle = U_{\rm gl}\,|\psi(t)\rangle$$

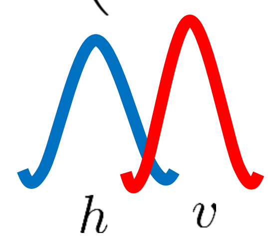
$$U_{\rm gl} = C(\theta_1/2)SG(+\gamma)C(\theta_2)G(-\gamma)SC(\theta_1/2)$$



$$C(\theta) = \sum_{x=1}^{X} |x\rangle\langle x| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$S = \sum_{x=1}^{X} \begin{pmatrix} |x-1\rangle\langle x| & 0 \\ 0 & |x+1\rangle\langle x| \end{pmatrix}$$

$$G(\gamma) = \sum_{x=1}^{X} |x\rangle\langle x| \begin{pmatrix} e^{+\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}$$

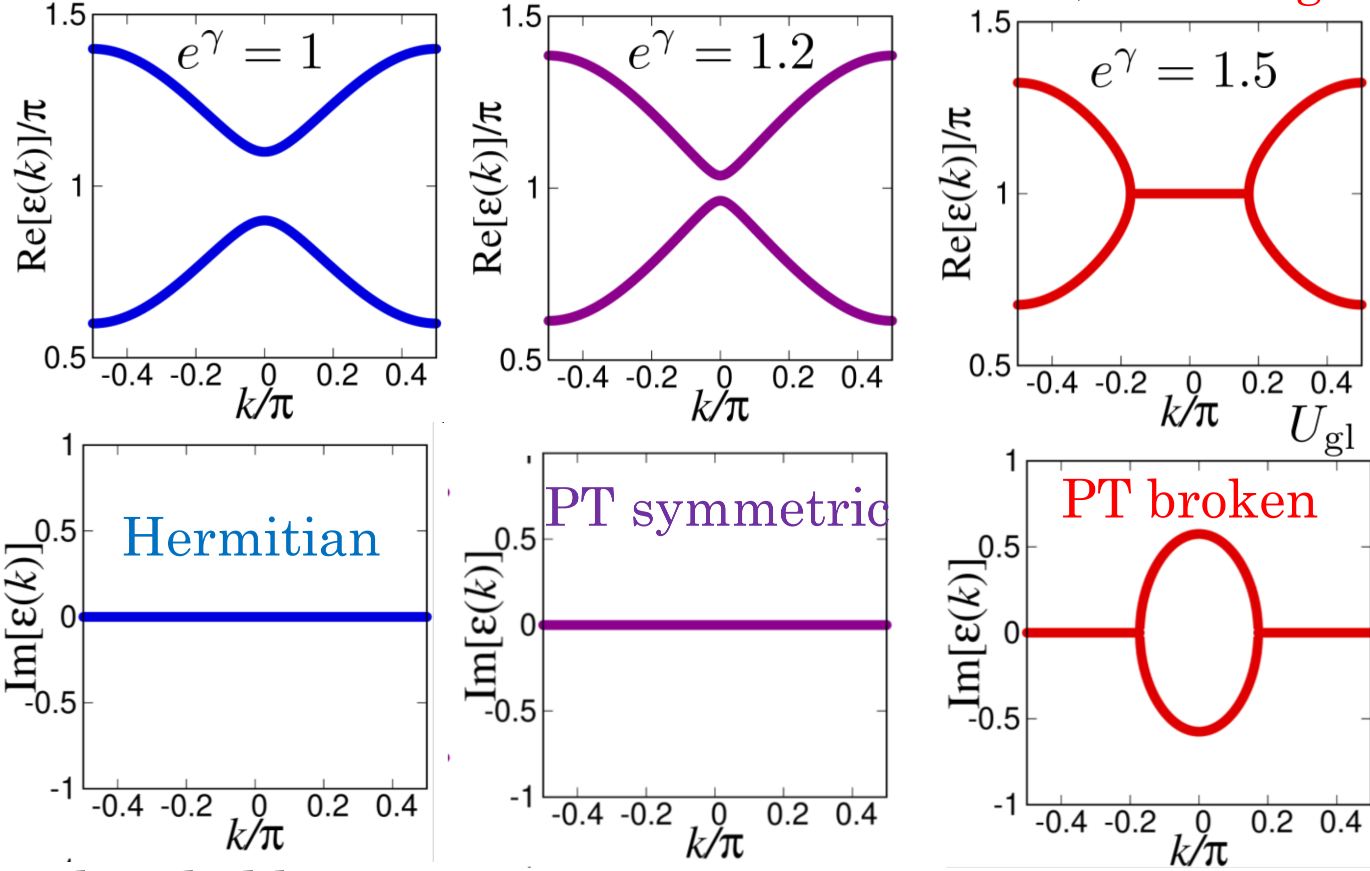


$$|\psi(t)\rangle = \sum_{x=1}^{X} \begin{pmatrix} \psi_h(x,t) \\ \psi_v(x,t) \end{pmatrix} |x\rangle$$

PT symmetry breaking

Ken Mochizuki, D. Kim, and H. Obuse, PRA **93**, 062116 (2016). Ken Mochizuki, D. Kim, N. Kawakami, and H. Obuse, PRA **102**, 062202 (2020).

weak loss → strong loss



$$H(k)\,|\phi_\pm(k)\rangle = \varepsilon_\pm(k)\,|\phi_\pm(k)\rangle$$

$$H(k) = i\log[U_{\rm gl}(k)]$$

$$U_{\rm gl} = \sum_k |k\rangle\langle k| \otimes U_{\rm gl}(k)$$

$$(\mathsf{PT})H(k)(\mathsf{PT})^{-1} = H(k)$$

$$\mathsf{PT} = \sigma_3\mathcal{K}$$

threshold：

$$d(k=0) = -1$$

$$\rightarrow e^{\gamma_{\mathcal{PT}}} \simeq 1.22$$

$$\mathsf{PT}\,|\phi_\pm(k)\rangle = |\phi_\pm(k)\rangle \qquad \mathsf{PT}\,|\phi_\pm(k)\rangle \neq |\phi_\pm(k)\rangle$$

$$e^{-i\varepsilon_\pm(k)} = d(k) \pm \sqrt{d^2(k) - 1} \qquad \theta_1 = 0.65\pi,\ \theta_2 = 0.25\pi$$

$$d(k) = \cos(\theta_1)\cos(\theta_2)\cos(2k) - \sin(\theta_1)\sin(\theta_2)\cosh(2\gamma)$$

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Short-time dynamical complexity transition

hard to compute

actual distribution of many photons

probability distribution of distinguishable particles

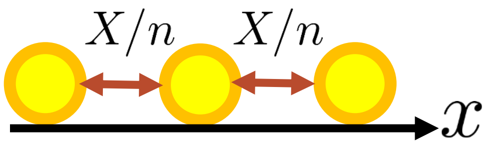
easy to compute

$$|P(t) - P_{\rm d}(t)| = \sum_{\{n_{\rm out}\}} |P(\{n_{\rm in}\}, \{n_{\rm out}\}, t) - P_{\rm d}(\{n_{\rm in}\}, \{n_{\rm out}\}, t)|$$

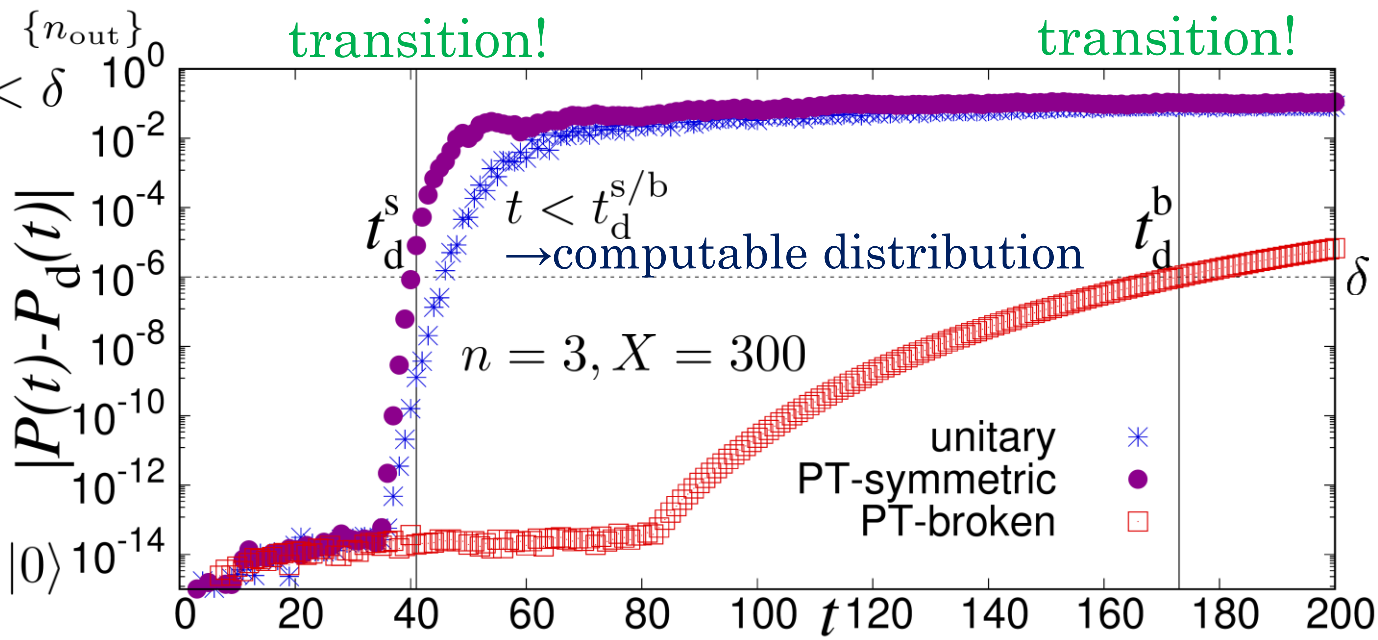
$|P(t) - P_{\rm d}(t)| < \delta$

computable within δ

initial state :



$\hat{b}^\dagger_{x=\frac{X}{6},h}\hat{b}^\dagger_{x=\frac{X}{2},v}\hat{b}^\dagger_{x=\frac{5}{6}X,v}|0\rangle$



$t < t_{\rm d}^{\rm s/b}$: neglectable overlap of single-particle distributions

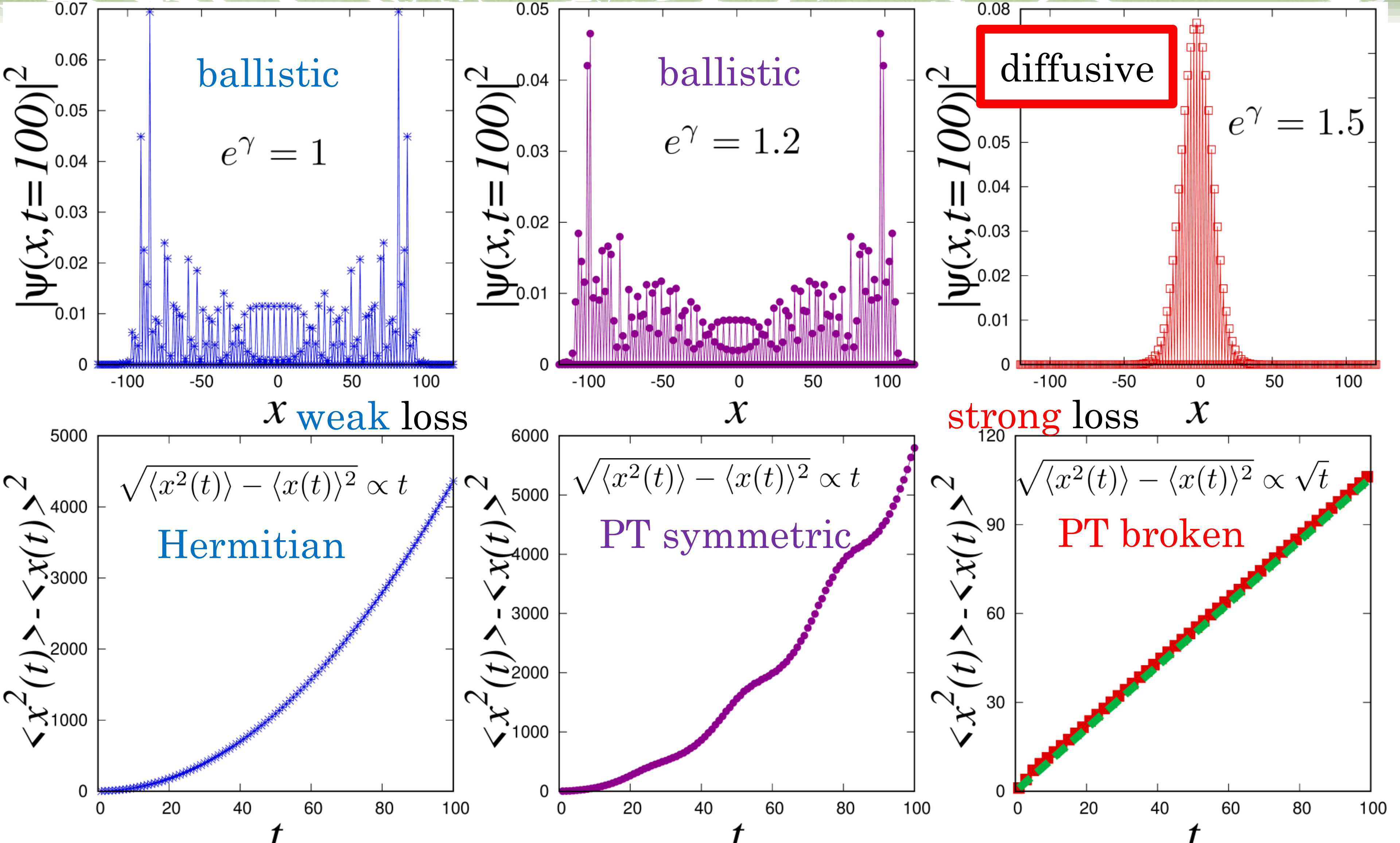
$t \geq t_{\rm d}^{\rm s/b}$: deviation from the computable distribution due to quantum interference

PT symmetry breaking enlarges the computable region

Single-particle dynamics

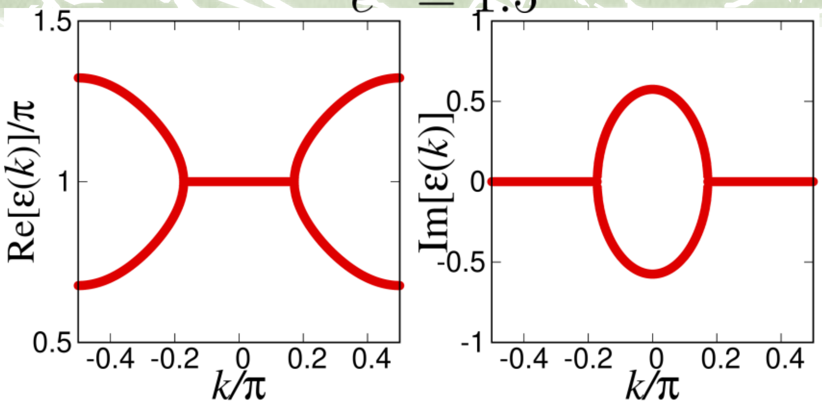
$$|\psi(t=0)\rangle = |x=0\rangle \otimes (|h\rangle + i\,|v\rangle)/\sqrt{2}$$

Ken Mochizuki, D. Kim, and H. Obuse, PRA **93**, 062116 (2016).
Ken Mochizuki, D. Kim, N. Kawakami, and H. Obuse, PRA **102**, 062202 (2020).



Analysis of the diffusive dynamics

$e^{\gamma} = 1.5$



$$U_{\rm gl}^{t}(k) = \sum_s e^{-i\varepsilon_s(k)t} |\phi_s^{\rm R}(k)\rangle \langle \phi_s^{\rm L}(k)|$$

$$\varepsilon_{\pm}(k) \simeq \varepsilon_{\pm}(k=0) \pm i\frac{D}{2}k^2,\ D>0$$

$$|\psi(t=0)\rangle = |x=0\rangle\,|\sigma_0\rangle$$ quadratic dispersion

$$N(t) = \int_{-\pi}^{\pi} dk\, \langle\sigma_0|\,[U_{\rm gl}^{\dagger}(k)]^t U_{\rm gl}^{t}(k)\,|\sigma_0\rangle$$

$$\langle x^r(t)\rangle = \frac{\sum_x x^r|\psi(x,t)|^2}{\sum_x |\psi(x,t)|^2} = \frac{(-i)^r}{N(t)}\int_{-\pi}^{\pi} dk\,\langle\sigma_0|\,[U_{\rm gl}^{\dagger}(k)]^t \frac{d^r}{dk^r} U_{\rm gl}^{t}(k)\,|\sigma_0\rangle$$

extraction of leading terms

$$\simeq \frac{(-i)^r \int_{-\infty}^{+\infty} dk e^{-\frac{Dk^2}{2}t} \frac{d^r}{dk^r} e^{-\frac{Dk^2}{2}t}}{\int_{-\infty}^{+\infty} dk e^{-Dk^2 t}} = \frac{i^r \int_{-\infty}^{+\infty} dk e^{-Dk^2 t} \left(\frac{Dt}{2}\right)^{\frac{r}{2}} H_r\left(\sqrt{\frac{Dt}{2}}k\right)}{\int_{-\infty}^{+\infty} dk e^{-Dk^2 t}}$$

extension of the integral range

$$\langle \exp[\xi x(t)]\rangle \simeq \frac{\int_{-\infty}^{+\infty} dk \exp\left(-Dk^2 t + \frac{Dt}{2}\xi^2 + i\xi Dtk\right)}{\int_{-\infty}^{+\infty} dk \exp(-Dk^2 t)} = \exp\left[\frac{\xi^2}{2}\left(\sqrt{\frac{Dt}{2}}\right)^2\right]$$

$H_r\left(\sqrt{\frac{Dt}{2}}k\right)$: Hermite Polynomial $\sum_{r=0}^{\infty} H_r(y) z^r/r! = \exp(2yz - z^2)$: generating function

Moment generating function corresponds to that of Gaussian distribution

System-size dependence

$\theta_1 = 0.65\pi,\ \theta_2 = 0.25\pi$

initial state : $\hat{b}^\dagger_{x=\frac{X}{4},h}\hat{b}^\dagger_{x=\frac{3}{4}X,v}|0\rangle$

PT-symmetric phase : ballistic dynamics

$$\sqrt{\langle x^2(t)\rangle - \langle x(t)\rangle^2} \propto t$$

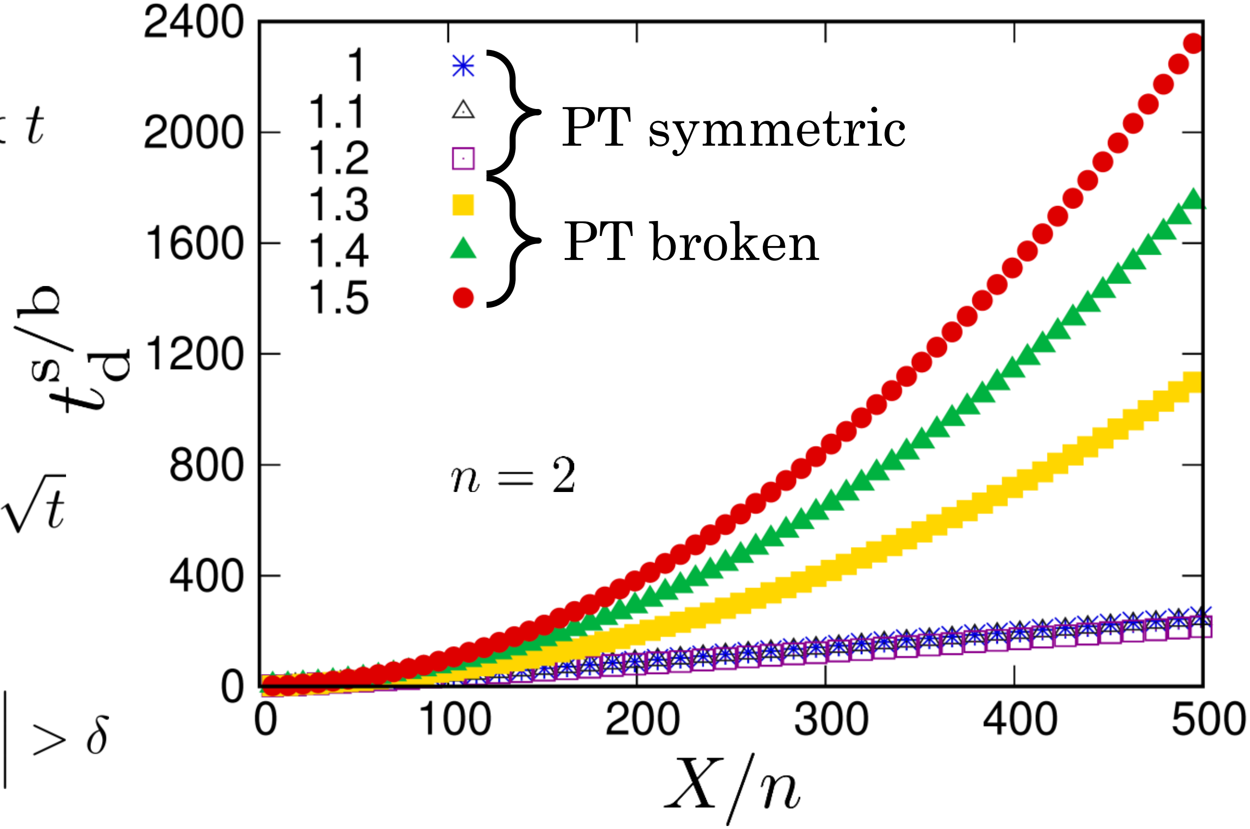
$$\Rightarrow t_{\mathrm{d}}^{\mathrm{s}} \propto X$$

PT-broken phase : diffusive dynamics

$$\sqrt{\langle x^2(t)\rangle - \langle x(t)\rangle^2} \propto \sqrt{t}$$

$$\Rightarrow t_{\mathrm{d}}^{\mathrm{b}} \propto X^2$$

$$\left|P\left(t_{\mathrm{d}}^{\mathrm{s/b}}\right) - P_{\mathrm{d}}\left(t_{\mathrm{d}}^{\mathrm{s/b}}\right)\right| > \delta$$



PT symmetry breaking leads to ballistic dynamics
⇒ quadratic system-size dependence ∴ prolongation of transition times

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Long-time dynamical complexity transition

$$U_{\rm gl} = C(\theta_1/2)SG(+\gamma)C(\theta_2)G(-\gamma)SC(\theta_1/2),\ \mathcal{U}^{\rm T}(t) = U_{\rm gl}^t$$

$$P_{\rm d}(\{n_{\rm in}\},\{n_{\rm out}\},t) = \frac{\sum_\omega \prod_{p=1}^{n} |\mathcal{U}_{{\rm in}_p {\rm out}_{\omega(p)}}(t)|^2}{N_{\rm d}(t)\prod_{j=1}^{Y} n_j^{\rm in}! n_j^{\rm out}!}$$

probability distribution of distinguishable particles

easy to computable

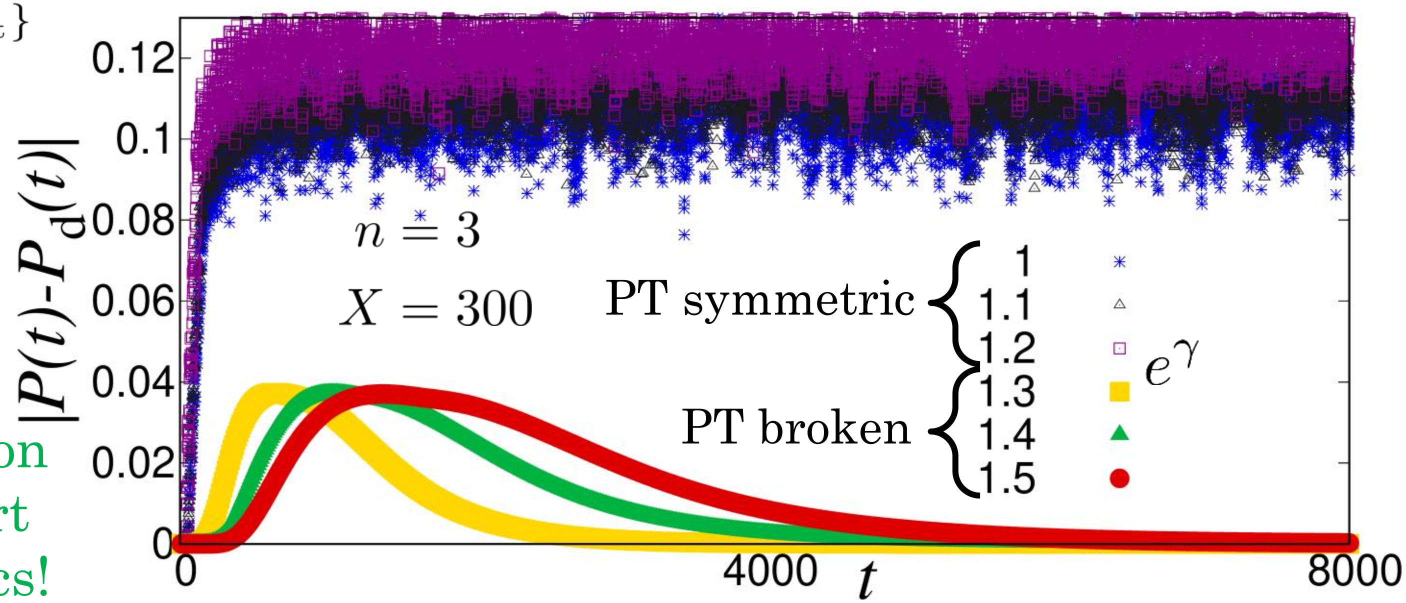
$$|P(t) - P_{\rm d}(t)| = \sum_{\{n_{\rm out}\}} |P(\{n_{\rm in}\},\{n_{\rm out}\},t) - P_{\rm d}(\{n_{\rm in}\},\{n_{\rm out}\},t)|$$

$|P(t) - P_{\rm d}(t)| < \delta$

↓

computable within δ

additional transition with no counterpart in unitary dynamics!



PT symmetry breaking ⇒ The distribution becomes computable again

Analysis based on the dominant state

$$P(\{n_{\mathrm{in}}\},\{n_{\mathrm{out}}\},t) = |\mathrm{Per}[W(t)]|^2/N(t)\prod_j n_j^{\mathrm{in}}!n_j^{\mathrm{out}}!$$

$$U_{\mathrm{gl}}|\phi_l^{\mathrm{R}}\rangle = \lambda_l|\phi_l^{\mathrm{R}}\rangle\,,\ \langle\phi_l^{\mathrm{L}}|U_{\mathrm{gl}} = \lambda_l\langle\phi_l^{\mathrm{L}}|$$

$$\mathcal{U}^{\mathrm{T}}(t) = U_{\mathrm{gl}}^t \simeq \lambda_m^t|\phi_m^{\mathrm{R}}\rangle\langle\phi_m^{\mathrm{L}}|\,,\ |\lambda_m| = \max_l|\lambda_l|$$

approximation based on the dominant eigenstate

$$W_{pq}(t) = [\mathcal{U}(t)]_{\mathrm{in}_p\mathrm{out}_q} \simeq \lambda_m^t\langle\mathrm{out}_q|\phi_m^{\mathrm{R}}\rangle^*\langle\phi_m^{\mathrm{L}}|\mathrm{in}_p\rangle^*$$

$$\mathrm{Per}[W(t)] \simeq n!\prod_{p=1}^{n}\langle\mathrm{out}_p|\phi_m^{\mathrm{R}}\rangle^*\langle\phi_m^{\mathrm{L}}|\mathrm{in}_p\rangle^*$$

in_p: p-th input state
out_q: q-th output state

$$P(\{n_{\mathrm{in}}\},\{n_{\mathrm{out}}\},t) \simeq P_{\mathrm{m}}(\{n_{\mathrm{out}}\}) = \frac{\prod_{p=1}^{n}|\langle\mathrm{out}_p|\phi_m^{\mathrm{R}}\rangle|^2}{N_{\mathrm{m}}\prod_{j=1}^{Y}n_j^{\mathrm{out}}!},\ N_{\mathrm{m}} = \frac{\langle\phi_m^{\mathrm{R}}|\phi_m^{\mathrm{R}}\rangle^n}{n!}$$

computable

In the PT broken phase, the true distribution approaches a distribution computable through the dominant eigenstate

Numerical confirmation

$$P(\{n_{\rm in}\},\{n_{\rm out}\},t) \simeq P_{\rm m}(\{n_{\rm out}\}) = \frac{\prod_{p=1}^{n} |\langle {\rm out}_p|\phi_m^{\rm R}\rangle|^2}{N_{\rm m}\prod_{j=1}^{Y} n_j^{\rm out}!},\ N_{\rm m} = \frac{\langle \phi_m^{\rm R}|\phi_m^{\rm R}\rangle^n}{n!}$$

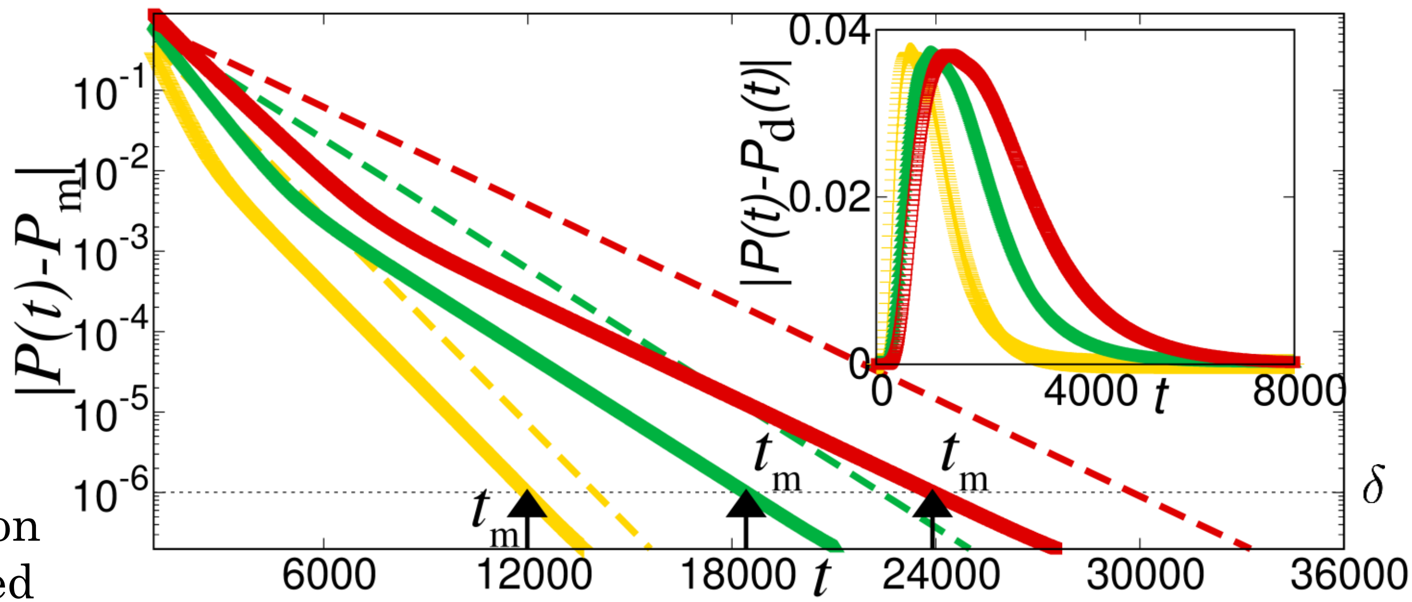
$$\mathcal{U}^{\rm T}(t) \simeq \lambda_m^t |\phi_m^{\rm R}\rangle\langle\phi_m^{\rm L}| \to P_{\rm d}(\{n_{\rm in}\},\{n_{\rm out}\},t) \simeq P_{\rm m}(\{n_{\rm out}\})$$ computable

$$|P(t)-P_{\rm m}| = \sum_{\{n_{\rm out}\}} |P(\{n_{\rm in}\},\{n_{\rm out}\},t) - P_{\rm m}(\{n_{\rm out}\}|$$

$|P(t)-P_{\rm m}| < \delta$

↓

The true distribution can be approximated within the precision δ



transition time $t_{\rm m}: |P(t_{\rm m}) - P_{\rm m}| < \delta$

Summary

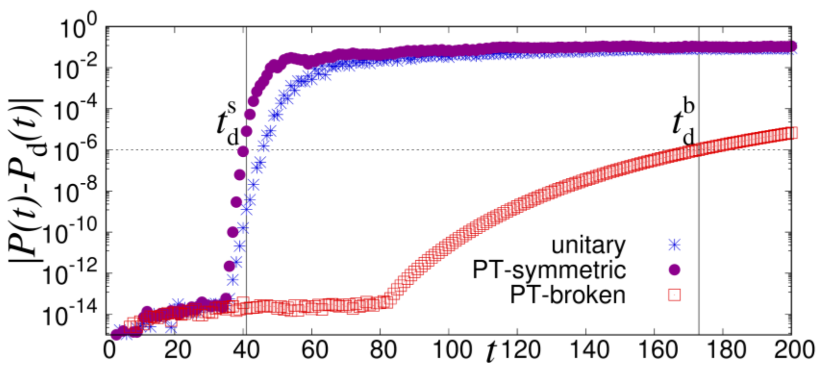
Ken Mochizuki and R.Hamazaki, in preparation.

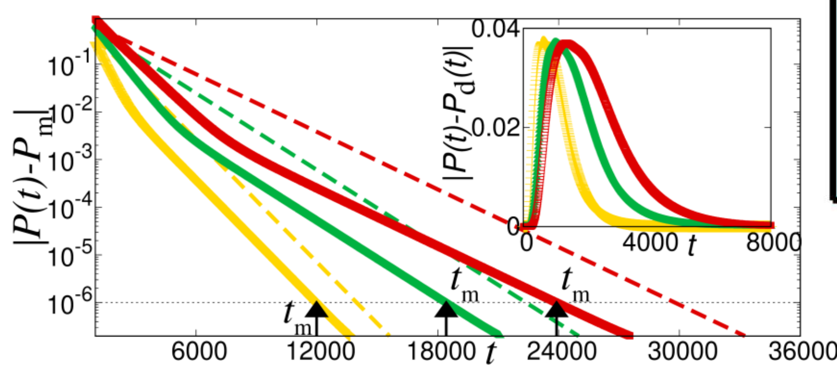
extending single-photonic non-Hermitian system into many-photon system

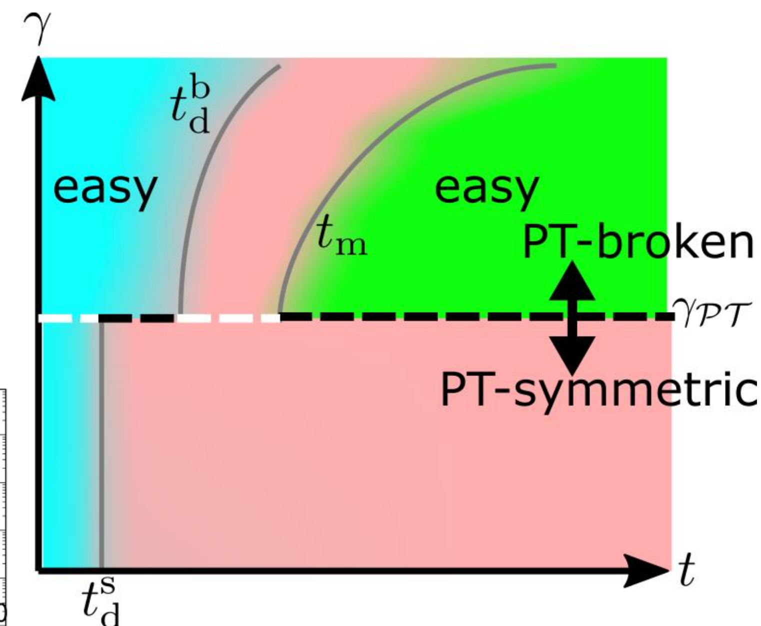
➡ computational complexity of boson sampling problem

PT symmetry breaking

➡
- prolongation of the threshold for the short-time complexity transition
- long-time complexity transition unique to non-Hermitian systems







easy ⇔ classically computable