Bulk-edge correspondence in non-Hermitian systems & physics sensitive vs. insensitive to the boundary condition

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The outline

1) Bulk-edge correspondence (BEC) in *Hermitian topological* systems: BEC-1

sensitive

2) BEC in *non-Hermitian* topological systems: BEC-1'

3) A new type of BEC in non-Hermitian systems exhibiting *skin effect*: BEC-2

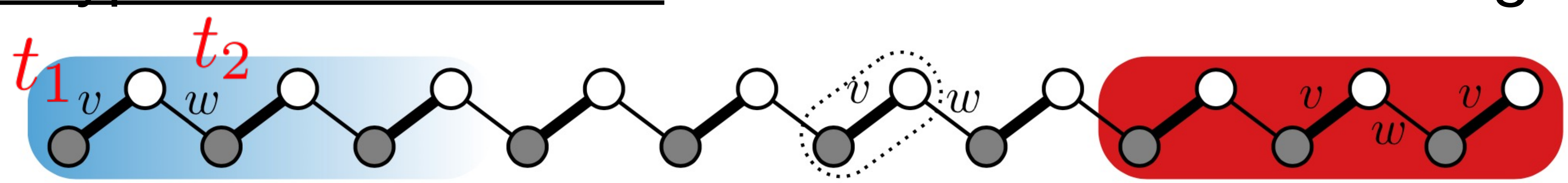
4) Non-Hermitian wave-packet *dynamics* ← insensitive

Physics sensitive vs. insensitive to the boundary condition

in non-Hermitian systems

1) Bulk-edge correspondence in *Hermitian topological* systems: BEC-1

A prototype: the SSH model   or the Su-Schrieffer-Heeger model

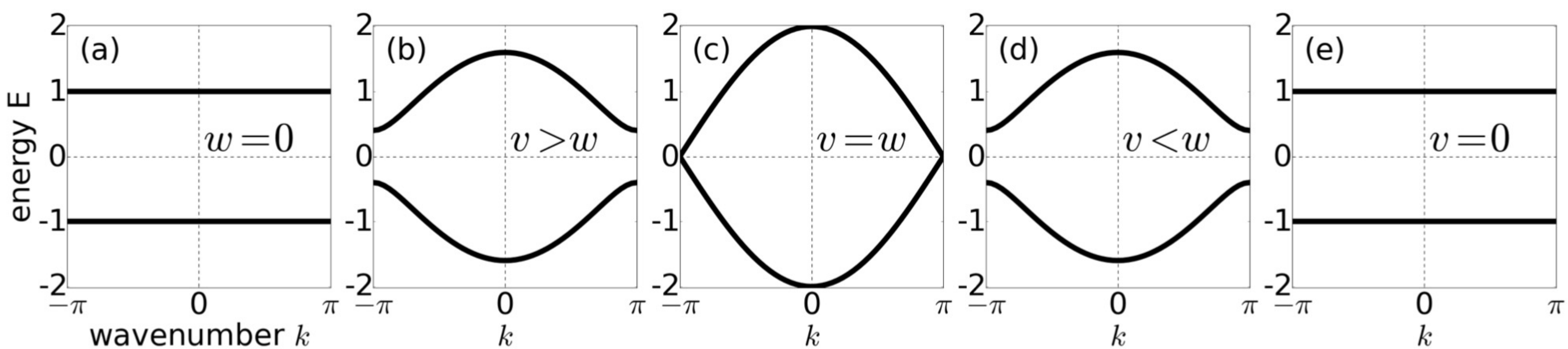


$$\hat{H} = \underset{t_1}{v} \sum_{m=1}^{N} \left(|m,B\rangle \langle m,A| + h.c. \right) + \underset{t_2}{w} \sum_{m=1}^{N-1} \left(|m+1,A\rangle \langle m,B| + h.c. \right)$$ obc, in real space

$$H(k) = \begin{pmatrix} 0 & v + we^{-ik} \\ v + we^{ik} & 0 \end{pmatrix}; \qquad E(k) = \left| v + e^{-ik} w \right| = \sqrt{v^2 + w^2 + 2vw\cos k}.$$

Bloch Hamiltonian in k-space, pbc;
energy spectrum/dispersion

(Bulk) energy bands: **pbc**

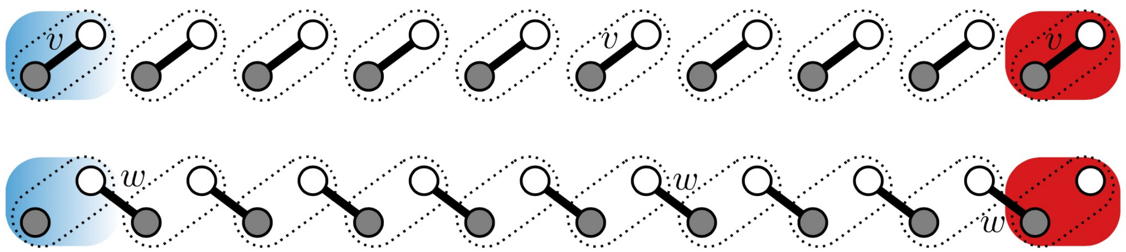


Asboth, et al., Lecture Notes in Physics, 919 (2016)

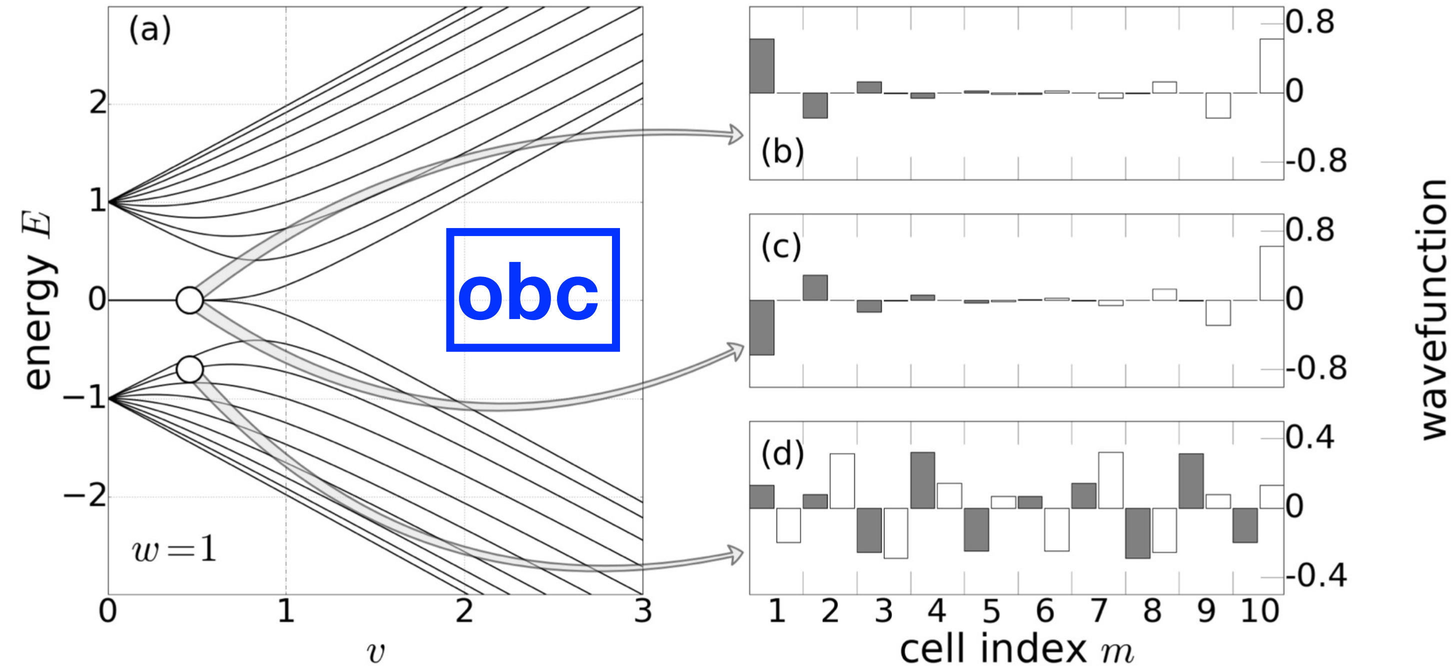
The bulk-boundary correspondence

Edge states, (topological) phase transition

$v > w$ no edge state

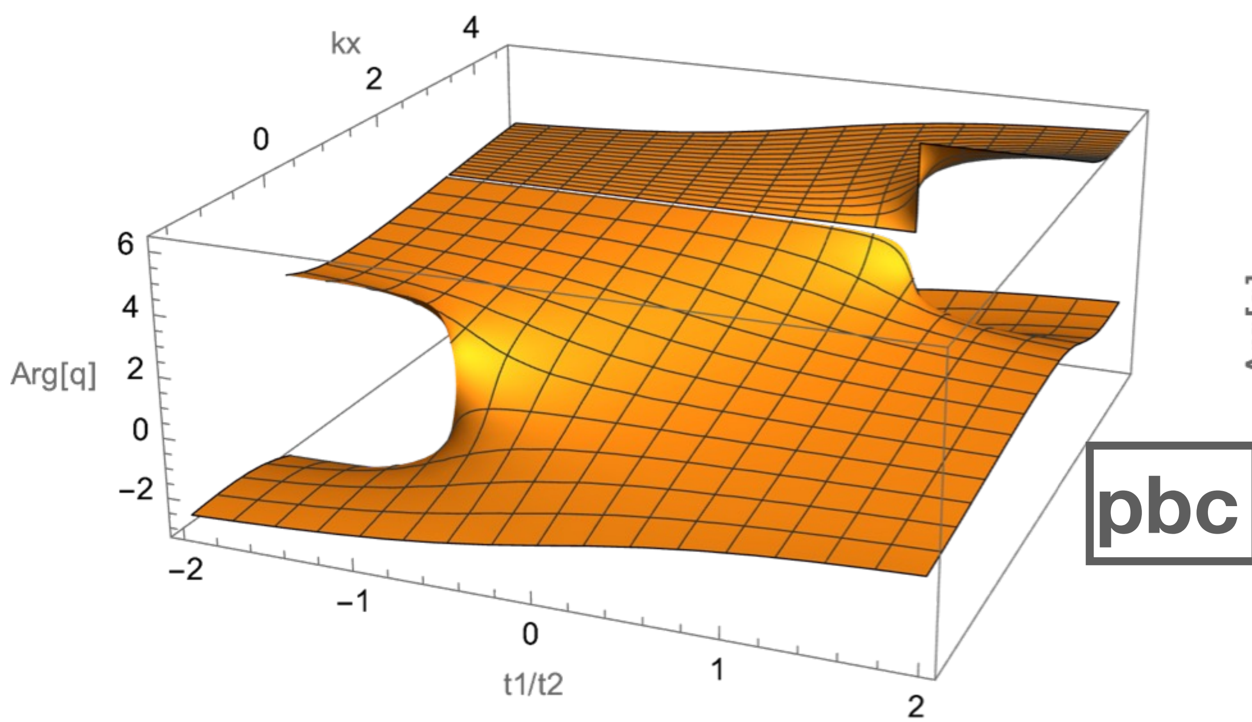


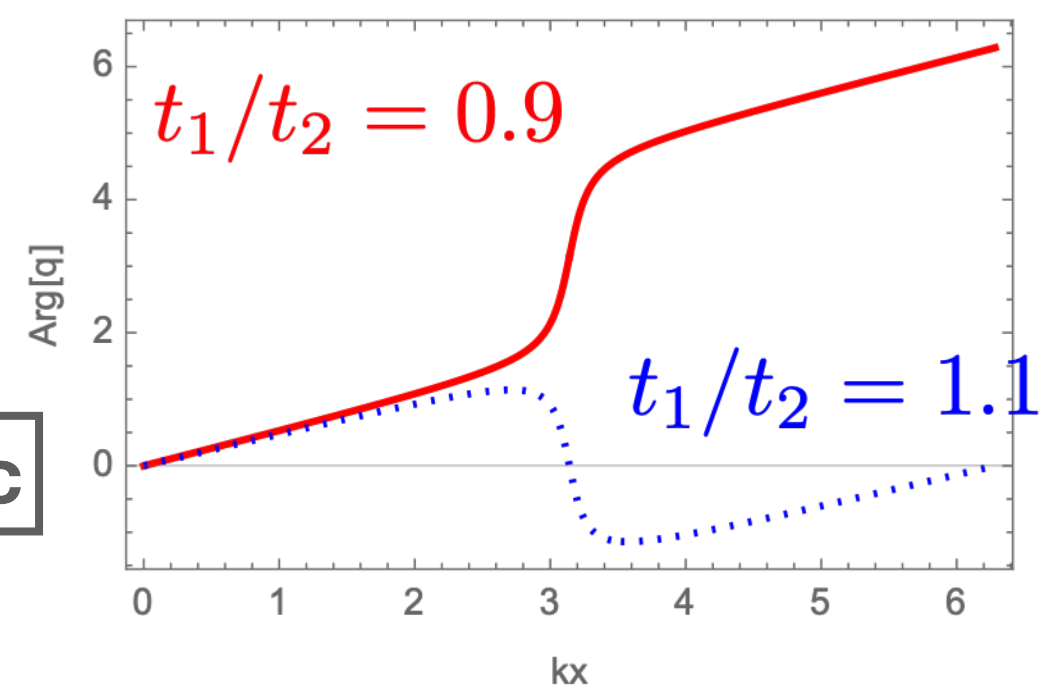
$v < w$ (a pair of) states



The winding number: ~ Berry/geometric phase

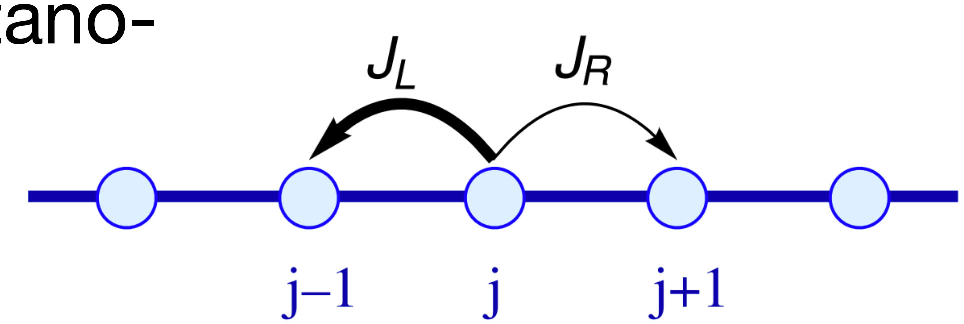
$$\nu = \frac{1}{2\pi i}\int_{-\pi}^{\pi} dk \frac{d}{dk} \log h(k) \qquad H(k) = \begin{pmatrix} 0 & h(k) \\ h^*(k) & 0 \end{pmatrix}; \qquad \nu = \begin{cases} 1 \text{ for } & -|t_2| < t_1 < |t_2| \\ 0 \text{ for } & |t_1/t_2| > 1 \end{cases}$$





2) BBC in *non-Hermitian* topological systems: BBC-1'

- Non-Hermitian system (in particular, the Hatano-Nelson type model) is sensitive to boundary conditions; so is a topological insulator…



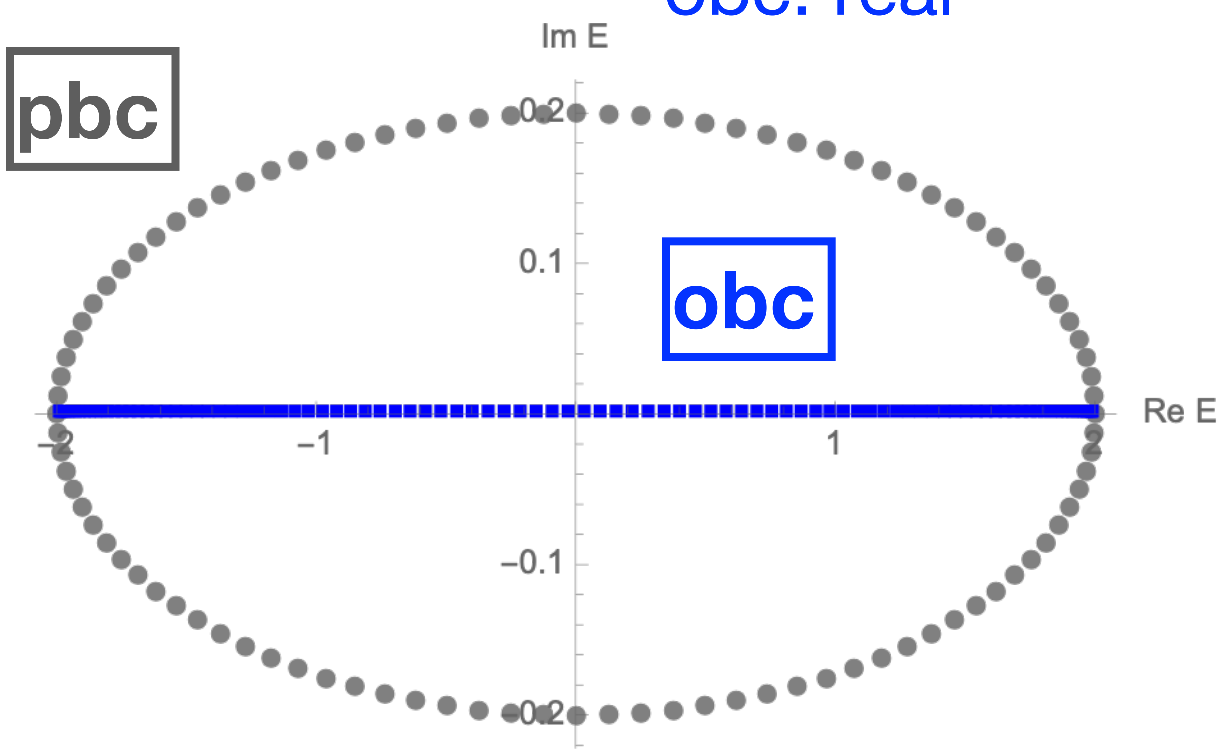
A prototype: Hatano-Nelson model

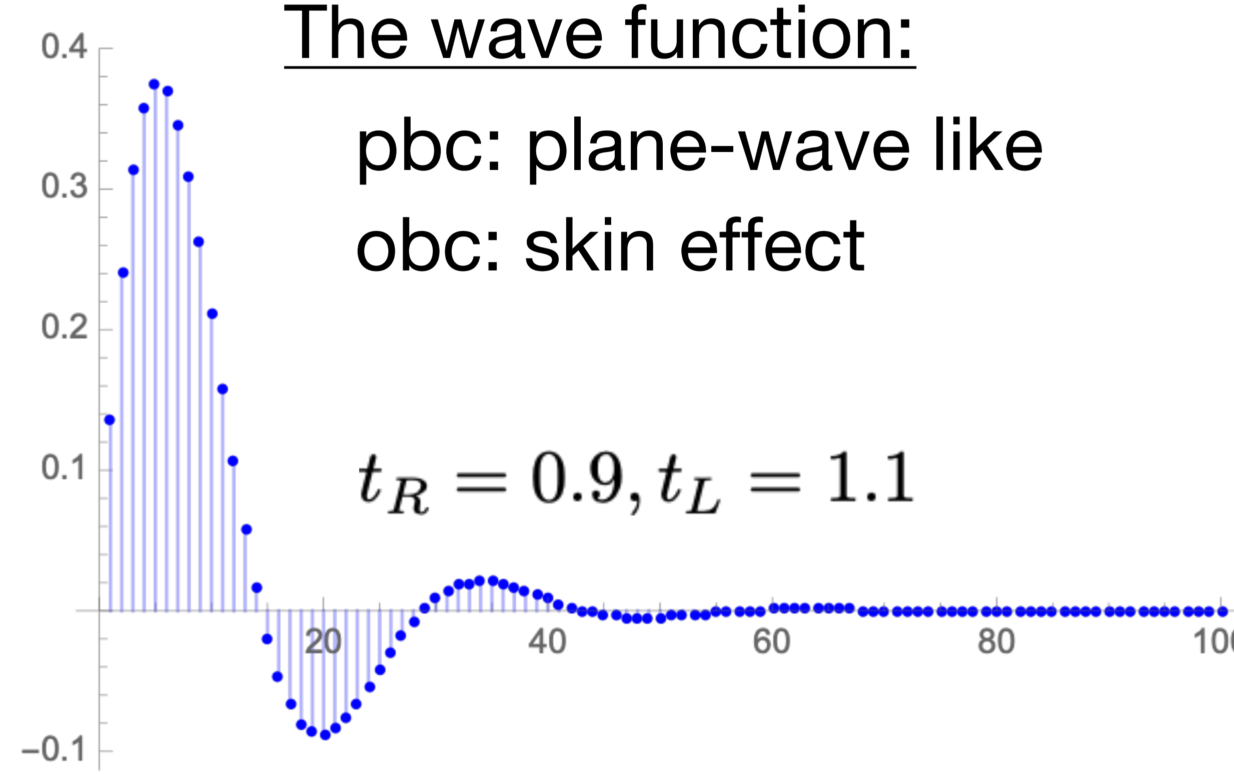
$$H_{\rm HN} = \sum_j (t_R |j+1\rangle\langle j| + t_L |j\rangle\langle j+1| + W_j |j\rangle\langle j|) \qquad W_j \in [-W/2, W/2]$$

anisotropic/non-reciprocal hopping: $t_R \neq t_L$

The spectrum:

pbc: complex
obc: real



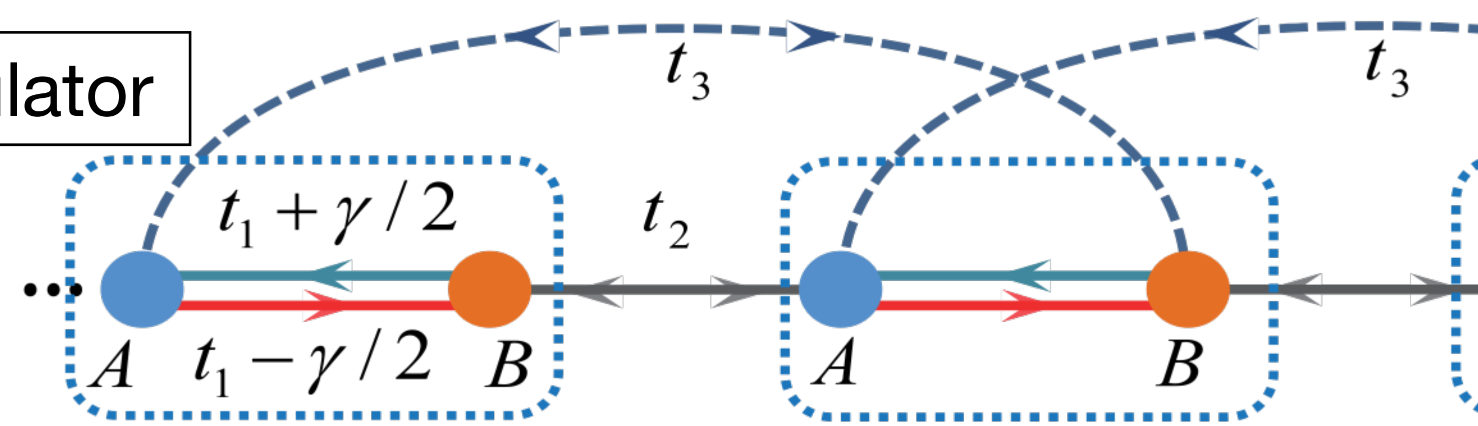


The wave function:

pbc: plane-wave like
obc: skin effect

$t_R = 0.9, t_L = 1.1$

Non-Hermitian topological insulator



A paradigmatic example: SSH×Hatano-Nelson model

$$H_{\rm NN}=\sum_n \left[t_1^-|nB\rangle\langle nA|+t_1^+|nA\rangle\langle nB| + t_2^-|n+1,A\rangle\langle n,B|+t_2^+|nB\rangle\langle n+1,A|\right]$$

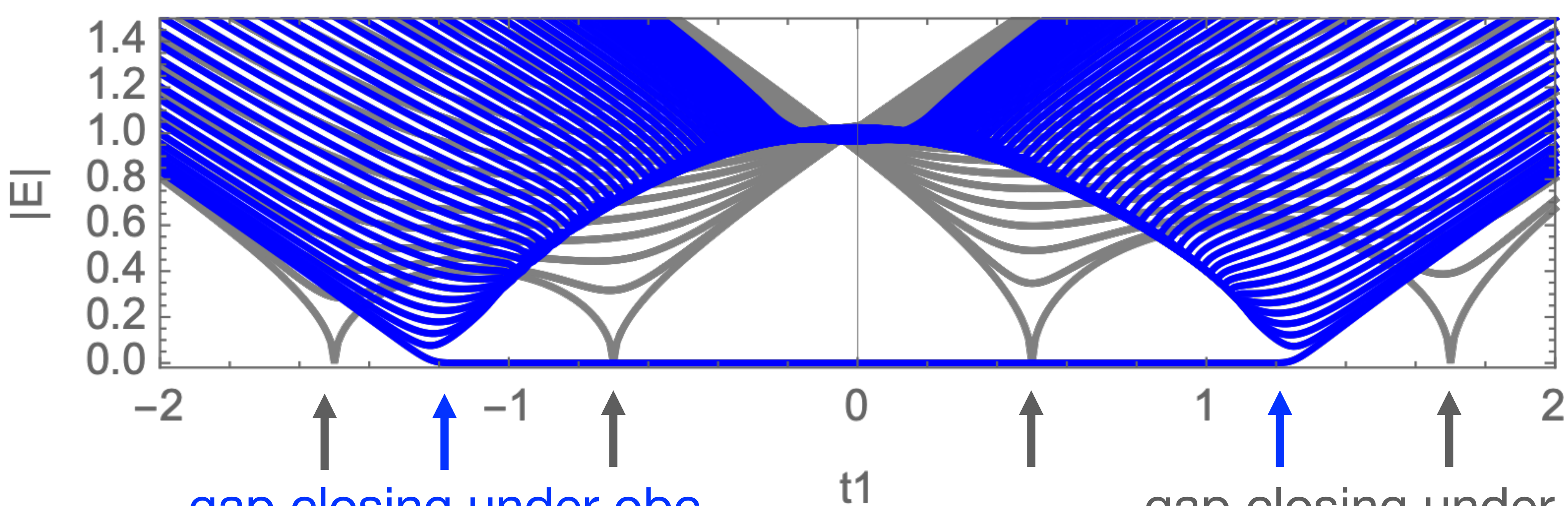
$$H_{\rm 3NN}=\sum_n \left[t_3^-|n+1,B\rangle\langle nA|+t_3^+|nA\rangle\langle n+1,B|\right]$$

$$H=H_{\rm NN}+H_{\rm 3NN}+\cdots$$

$$t_1^\pm=t_1\pm\gamma_1,\, t_2^\pm=t_2\pm\gamma_2$$

- anisotropic/non-reciprocal hopping (Hatano-Nelson)
- alternating hopping (SSH)

(Apparent) breakdown of the bulk-edge correspondence:



gap closing under obc

gap closing under obc

OBC ≠ PBC

The underlying reason: non-hermitian skin effect

Yao & Wang, PRL 2018

To recover the bulk-edge correspondence in non-Hermitian topological systems

two recipes are known:

<u>Recipe 1 (our recipe):</u> *Imura & Takane, PRB 2019; PTEP 2020*

Problem: the bulk (at least, under the standard pbc) was "wrong" ↓ $\neq$ boundary/edge

- choose to modify the bulk ⟷ pbc
- modify/generalize the periodic boundary condition

→ generalized pbc (gpbc)

<u>Recipe 2: an alternative scenario</u>

- the so-called "non-Bloch" approach
- abandons pbc
- completes a BEC within the obc

Yao & Wang, PRL 2018
Yokomizo & Murakami, PRL 2019

~ the underlying idea ~

The generalized PBC:

$$H_{gpbc} = H_{obc} + \underline{H_{bd}(b)}$$

The boundary Hamiltonian:

$$H_{bd}(b) = b^{-L} t_1 |1\rangle\langle L| + b^L t_2 |L\rangle\langle 1|$$

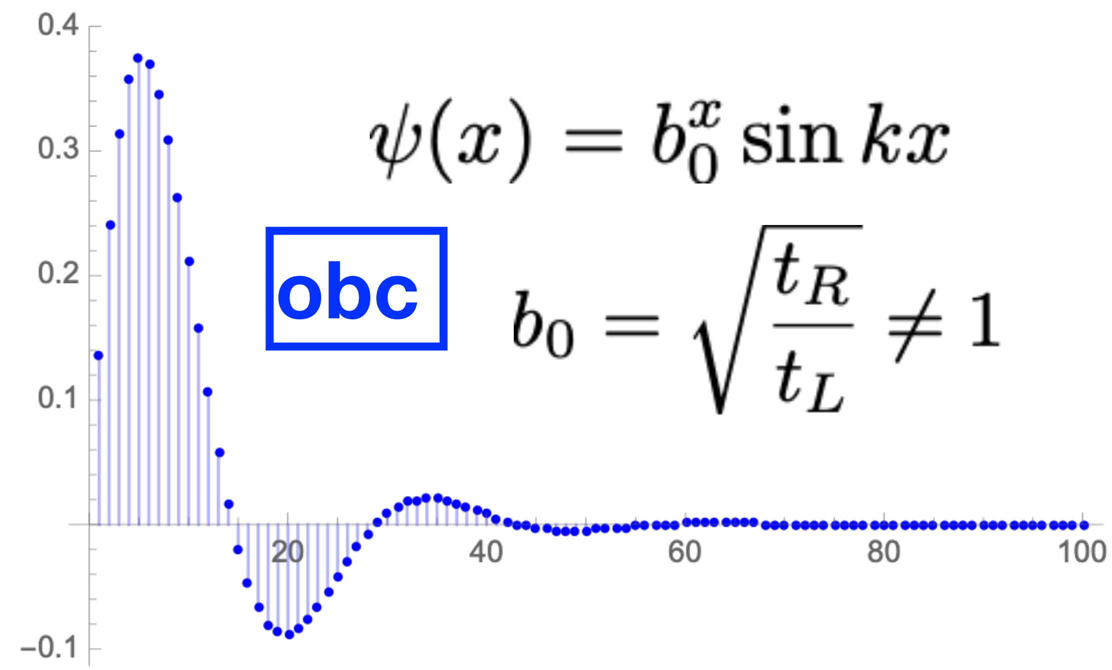
e.g., case of Hatano-Nelson model

$$H_{obc} = \sum_{x=1}^{L} (t_R |x+1\rangle\langle x| + t_L |x\rangle\langle x+1|)$$

→ Correspondingly, generalized Bloch Hamiltonian, generalized Bloch states, and the generalized Brillouin zone

$$|\beta\rangle = \sum_x \beta^x |x\rangle \qquad \beta = b e^{ik}$$

$$b \neq 1$$



$$\psi(x) = b_0^x \sin kx$$

obc

$$b_0 = \sqrt{\frac{t_R}{t_L}} \neq 1$$

Remarks:

- Eigen wave functions under obc are composed of solutions with

$$b \sim b_0 \neq 1$$

- The (naive) periodic boundary condition: b=1 automatically selects those solutions with b=1

inadequate for describing topological phase transitions under obc

Application to our paradigmatic example:

the SSH×Hatano-Nelson model

The generalized PBC:

b: free parameter

$$H_{gpbc} = H_{obc} + \underline{H_{bd}(b)} \qquad H_{obc} = H_{\rm NN} + H_{\rm 3NN} \qquad b \neq 1$$

The boundary Hamiltonian:

$$H_{bd}(b) = H_{bd}^{\rm NN} + H_{bd}^{\rm 3NN}$$

$$H_{bd}^{\rm NN} = b^{-L} t_2^- |1, A\rangle\langle L, B| + b^L t_2^+ |L, B\rangle\langle 1, A|$$
$$H_{bd}^{\rm 3NN} = b^{-L} t_3^- |1, B\rangle\langle N, A| + b^L t_3^+ |N, A\rangle\langle 1, B|$$

Generalized Bloch Hamiltonian, generalized Bloch states:

$$H_{gpbc}|\beta\rangle = E(\beta)|\beta\rangle \qquad |\beta\rangle = \sum_{j=0}^{L-1} \beta^j (c_A|jA\rangle + c_B|jB\rangle)$$

$$\begin{bmatrix} 0 & R_+(\beta) \\ R_-(\beta) & 0 \end{bmatrix} \begin{bmatrix} c_A \\ c_B \end{bmatrix} = E \begin{bmatrix} c_A \\ c_B \end{bmatrix}$$

$$\boxed{\beta = b e^{ik}, b \neq 1}$$

generalized Brillouin zone

$$R_+(\beta) = t_1^+ + t_2^- \beta^{-1} + t_3 \beta = t_1 + \gamma_1 + (t_2 - \gamma_2)\beta^{-1} + t_3 \beta$$
$$R_-(\beta) = t_1^- + t_2^+ \beta + t_3 \beta^{-1} = t_1 - \gamma_1 + (t_2 + \gamma_2)\beta + t_3 \beta^{-1}$$

To identify different topological phases

⟶ a pair of chiral winding numbers (w_+, w_-)

where

$$w_\pm(b) = \frac{1}{2\pi}[\arg R_\pm(\beta)]_{k=0}^{2\pi}$$

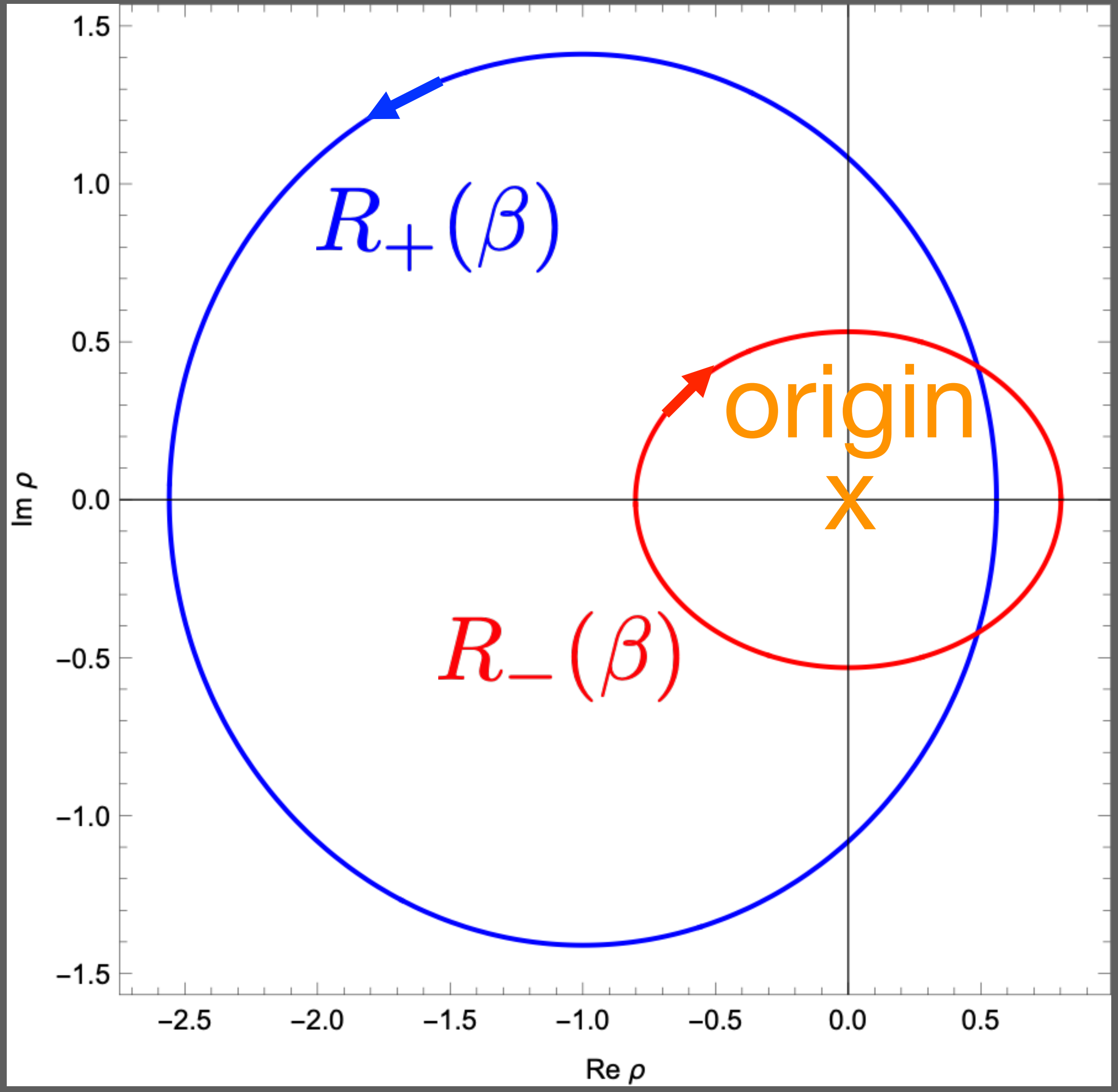
Yao & Wang, PRL 2018

- Mapping: $\boxed{\beta = be^{ik}}$

$\beta \to \rho_\pm$ $= \rho_\pm$

- The trajectory of $\rho = R_\pm(\beta)$

Measure of how many times winding around the origin



- Hermitian limit: $w_- = -w_+$

$$w = \frac{w_+ - w_-}{2}$$

TI: w=1 ⟷ A pair of E=0 edge states

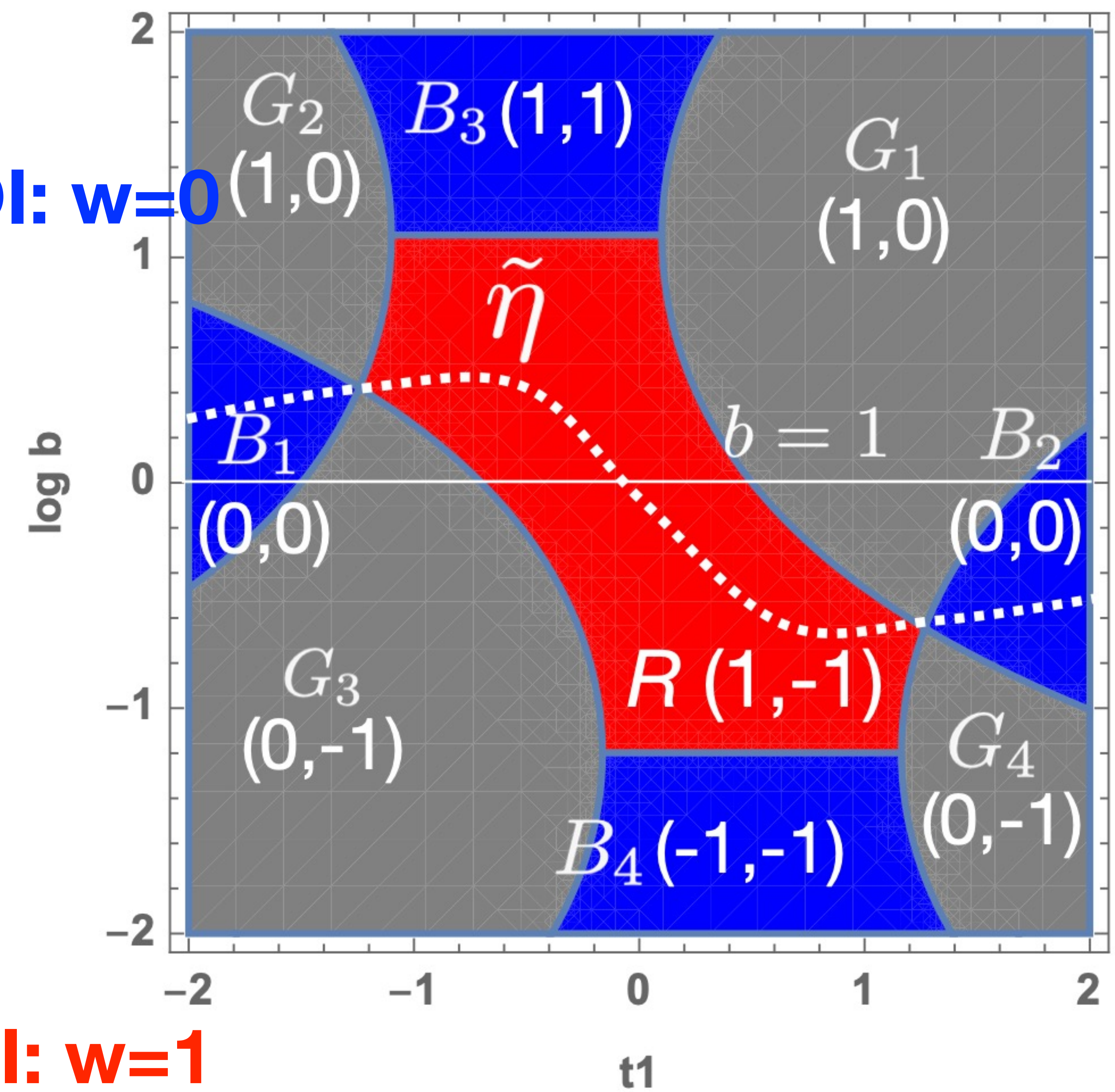
OI: w=0 ⟷ No edge state

Bulk-edge correspondence

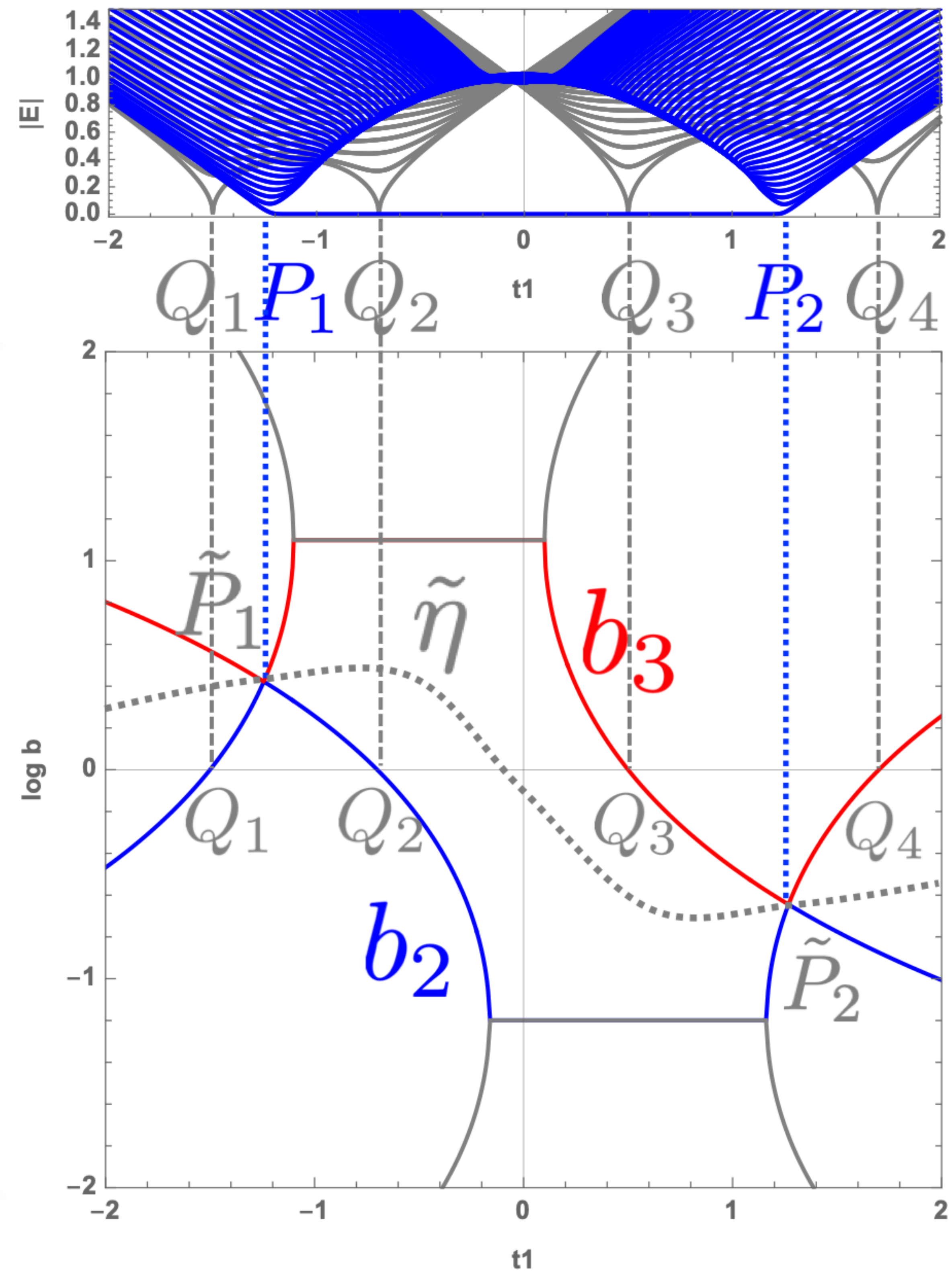
Phase diagram

b: free parameter

OI: w=0



TI: w=1



$\tilde{\eta}$: using the free parameter b chosen on this path, the BEC is recovered !!

Remarks: <u>meaning of the phase boundaries</u>

Phase boundaries are at the zeros: $R_{\pm}(\beta) = 0$

cf. $w_{\pm}(b) = \frac{1}{2\pi}[\arg R_{\pm}(\beta)]_{k=0}^{2\pi}$

Are they all physically meaningful?

⟷ Do they all correspond to the physical phase boundary under obc?

obc: open boundary condition

⤡

Bulk gap closing under obc

At the gap closing, bulk states appear at E=0

The bulk condition: $b_2 = b_3$ where $\beta = be^{ik}$

satisfied at E=0

Yao & Wang, PRL 2018

Yokomizo & Murakami, PRL 2019

Spectrum:
obc vs. pbc

Phase boundaries in the *generalized parameter space*:

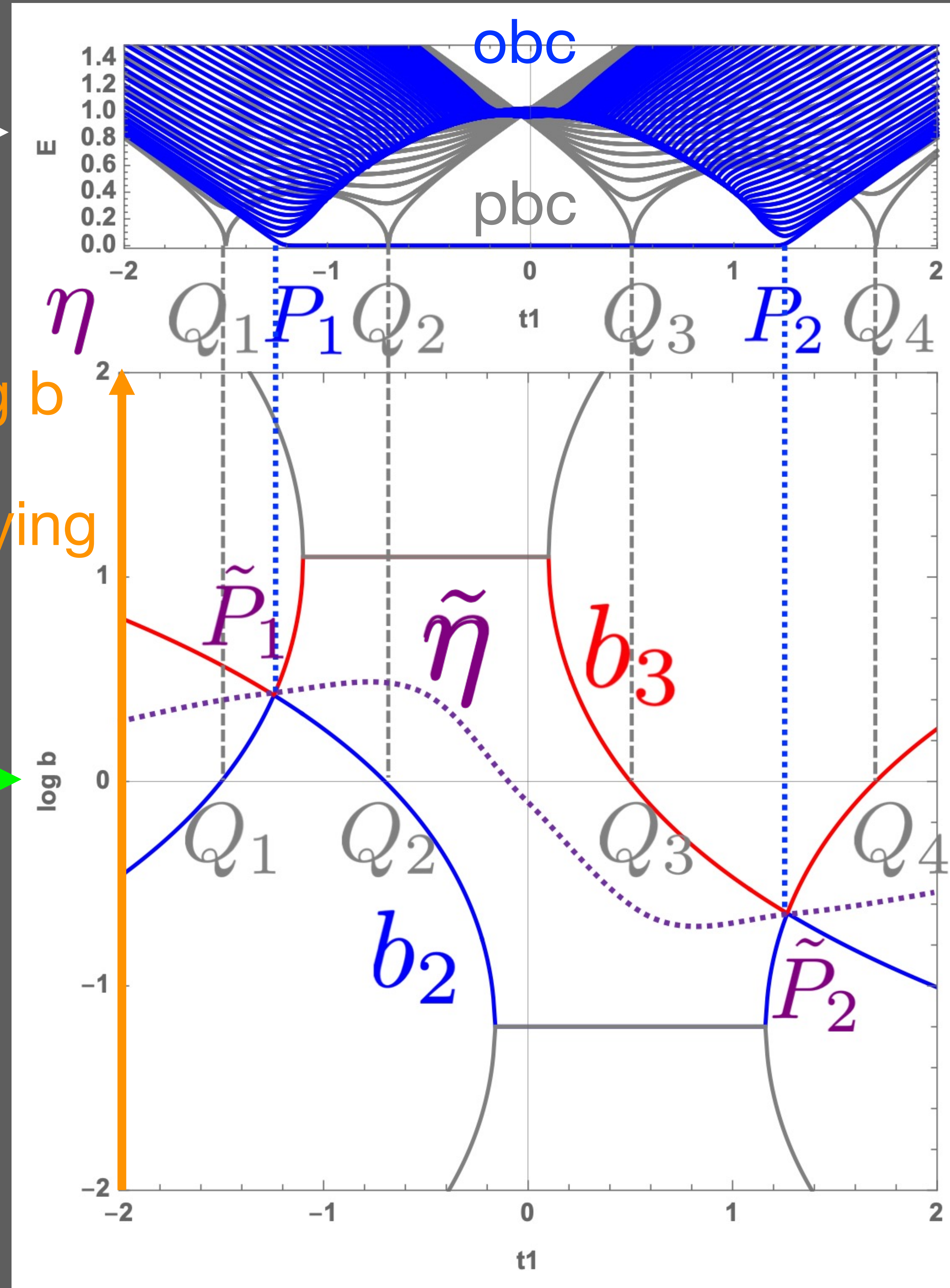
- The gap closing under obc are at

$$b_2 = b_3$$

$\longrightarrow \tilde{P}_1, \tilde{P}_2$

- A spinout: gap closing under pbc

$\longrightarrow Q_1, Q_2, Q_3, Q_4$



An alternative scenario: the so-called non-Bloch approach

Generalized Bloch Hamiltonian: $$H_{gpbc}(\beta) = \begin{bmatrix} 0 & R_+(\beta) \\ R_-(\beta) & 0 \end{bmatrix}$$

$$\boxed{\beta = be^{ik}}$$

- secular equation: $\det[H(\beta) - E1_2] = 0$ → quartic equation for β: $R_+(\beta)R_-(\beta) = E^2$

- fundamental solutions: $\beta = \beta_1, \beta_2, \beta_3, \beta_4$

where $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq |\beta_4|$

($|\beta_2|, |\beta_3|$: bulk solutions; $|\beta_1|, |\beta_4|$: evanescent modes)

The eigenstates under obc are superpositions of the fundamental solutions:

$$|\psi_{obc}\rangle = c_1|\beta_1\rangle + c_2|\beta_2\rangle + c_3|\beta_3\rangle + c_4|\beta_4\rangle$$

→ The "bulk condition": $|\beta_2| = |\beta_3|$

$b_m = b_{m+1}$ at m=2 $\boxed{b_2 = b_3}$

Yao & Wang, PRL 2018

Yokomizo & Murakami, PRL 2019

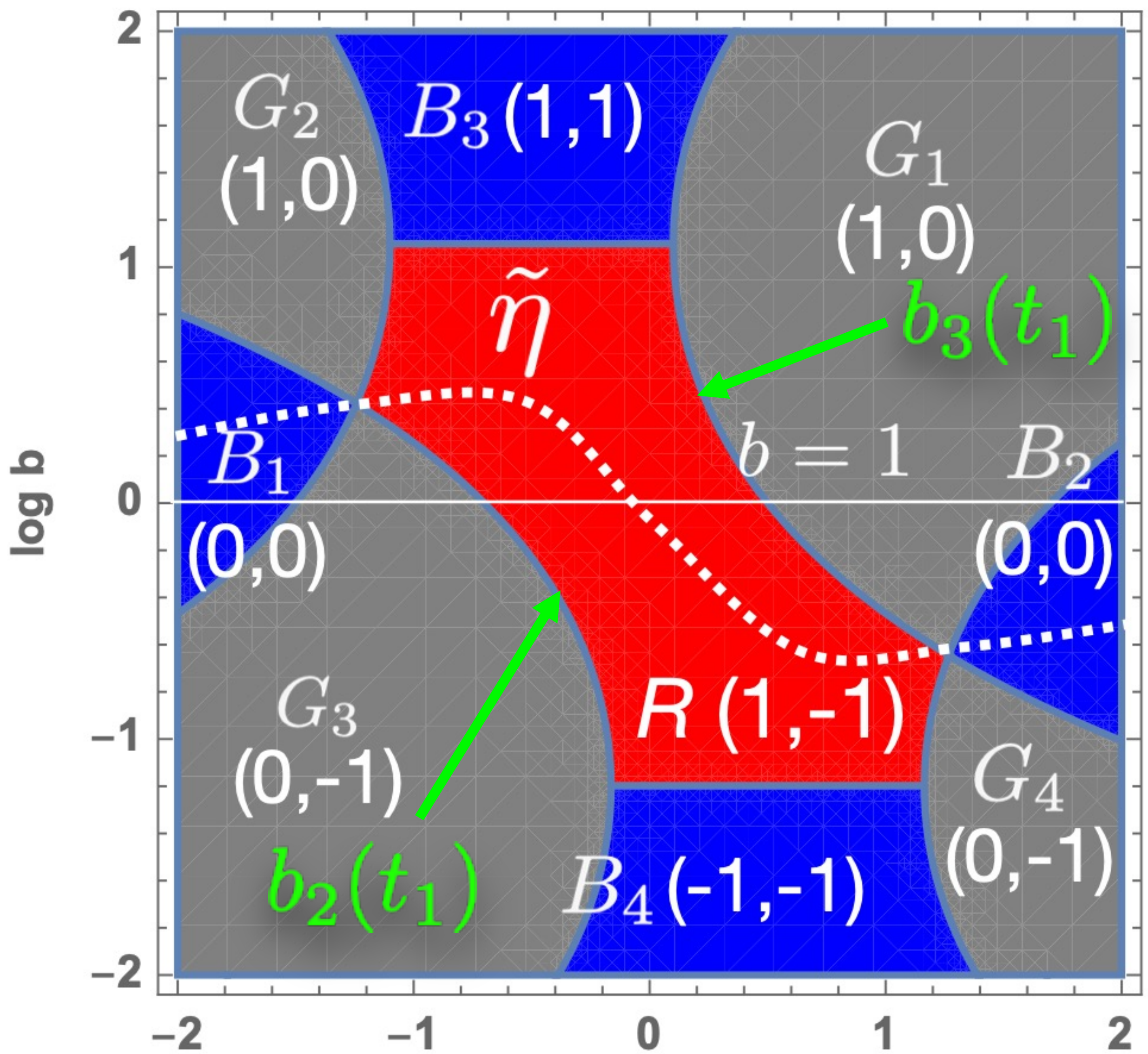
The generalized PBC

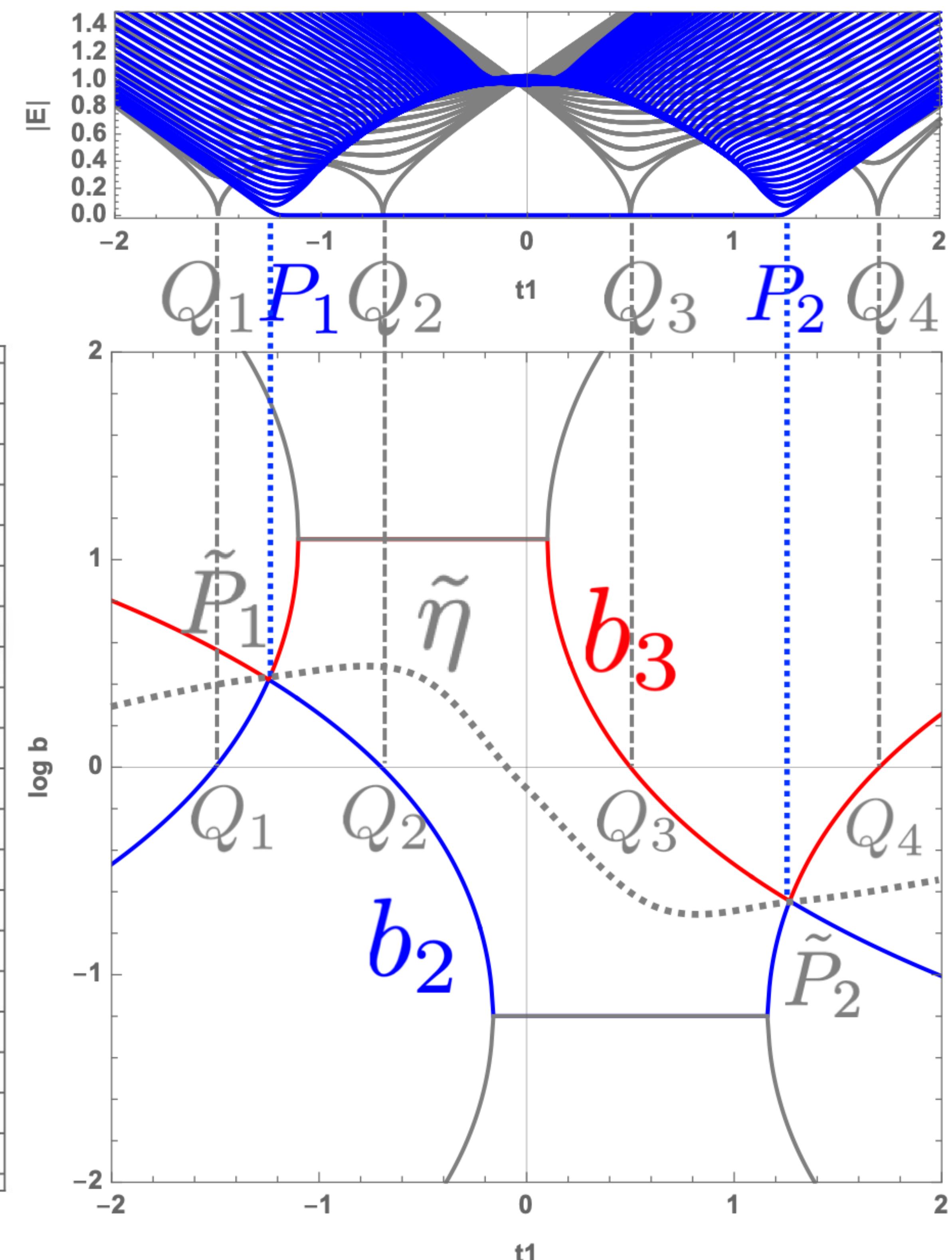
~ the recipe ~

Imura & Takane, PRB 2019; PTEP 2020

To recover the BEC in the paradigmatic non-Hermitian model,

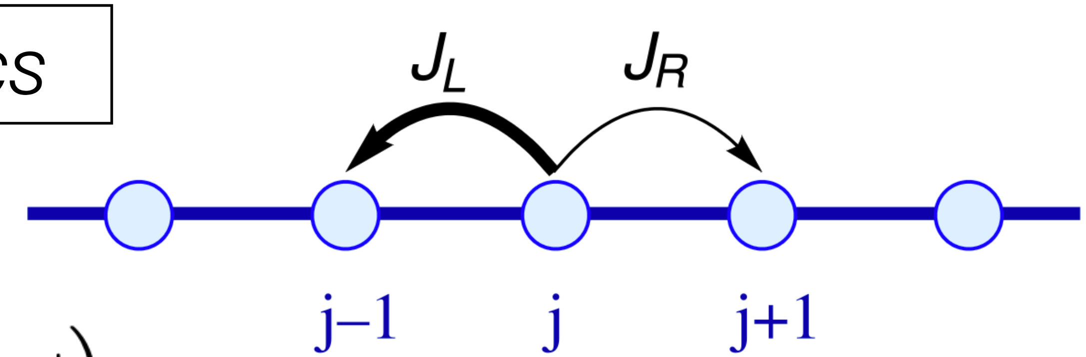
> $\tilde{\eta}$ specifying the parameter: $b(t_1)$ must be chosen so as to satisfy
>
> $$b_2(t_1) < b(t_1) < b_3(t_1)$$





Physics sensitive vs. insensitive to the boundary condition:

4) Non-Hermitian wave-packet *dynamics*



Let us consider again the Hatano-Nelson model:

$$H = -\sum_{j=0}^{L-1} \Big(\Gamma_R |j+1\rangle\langle j| + \Gamma_L |j\rangle\langle j+1| \Big)$$

- asymmetric/non-reciprocal hopping: $\Gamma_L = e^{g}\Gamma_0,\ \Gamma_R = e^{-g}\Gamma_0$

- Hatano-Nelson×Aubry-Andre model:

$$H = -\sum_{j=0}^{L-1} \Big(\Gamma_R |j+1\rangle\langle j| + \Gamma_L |j\rangle\langle j+1| \Big) + W_j \sum_{j=0}^{L-1} |j\rangle\langle j|,$$

Aubry-Andre model: quasi-periodic disorder

Aubry & Andre, AIPS '80

$W_j = W\cos(2\pi\theta j + \theta_0)$, θ : an irrational constant

e.g., (chosen typically to be) $\theta = \dfrac{\sqrt{5}-1}{2}$

θ_0 : disorder configuration ⟶ sample average

- Wave-packet dynamics:

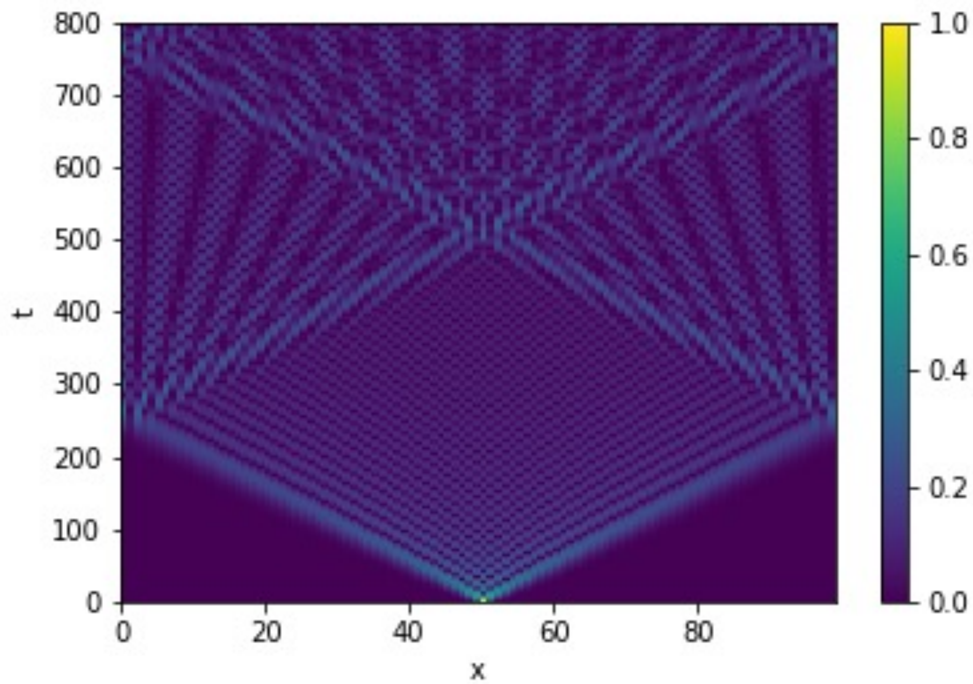
$$|\psi(t)\rangle = \sum_j \psi_j(t)|j\rangle = \sum_n c_n e^{-i\epsilon_n t}|n\rangle,$$

- initial wave packet:

$|\psi(t=0)\rangle = |j_0\rangle$ localized at a single site

Simulation of the wave-packet dynamics:
(a) Hermitian case

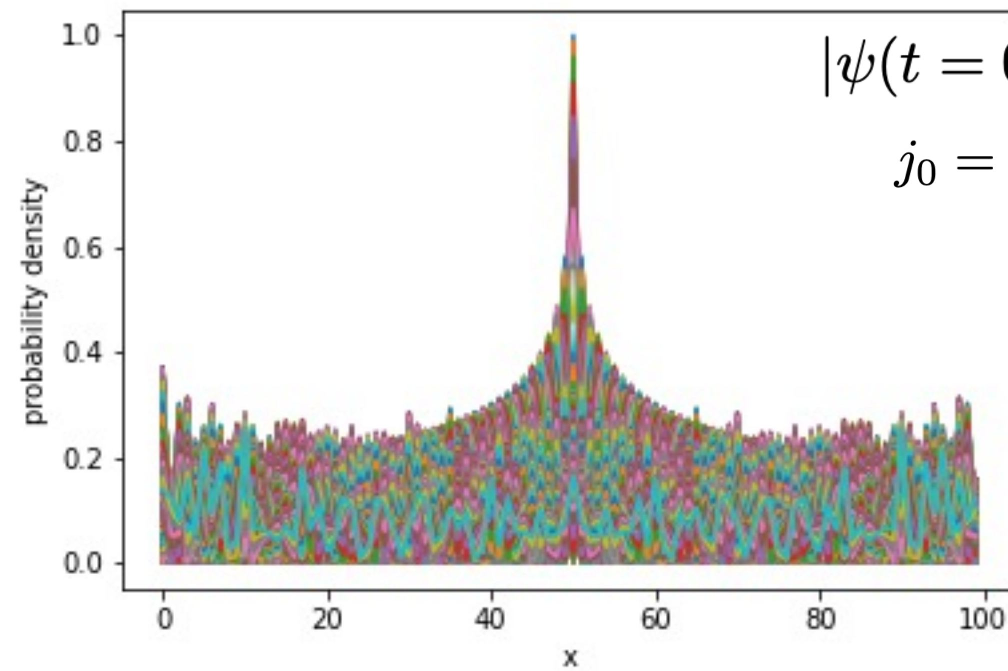
- clean limit: W=0



- initial wave packet:

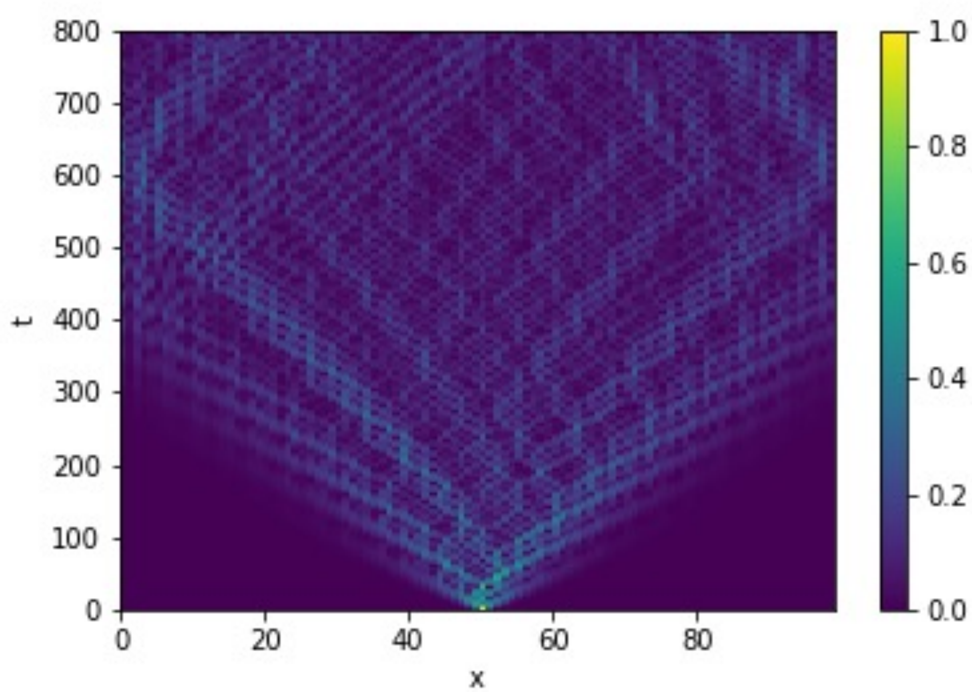
$$|\psi(t=0)\rangle = |j_0\rangle$$

$$j_0 = L/2, L = 100$$



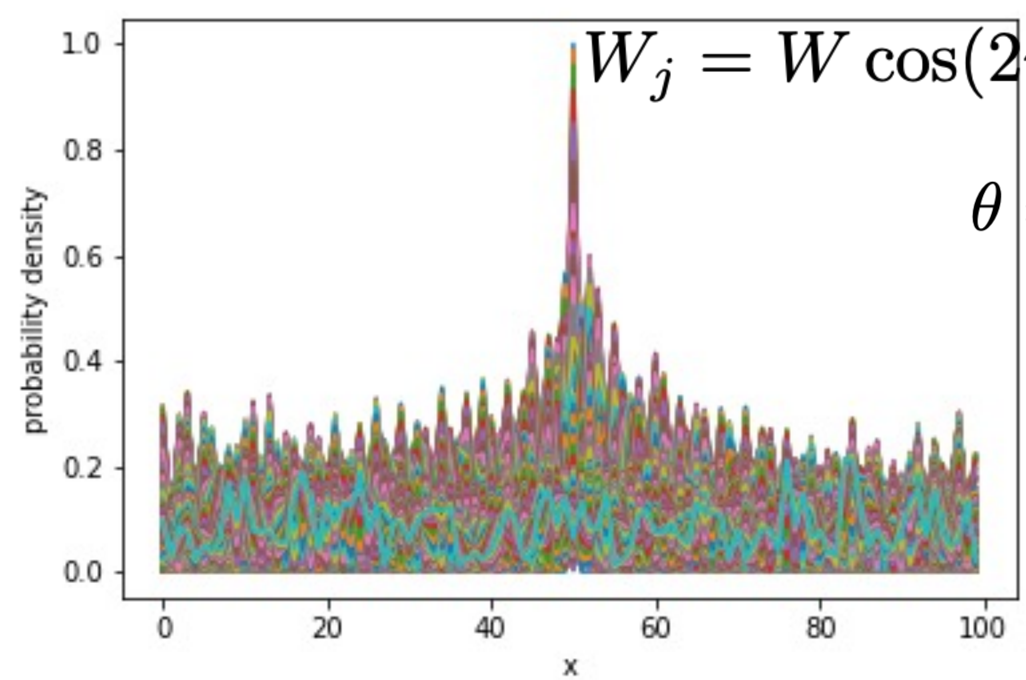
- weakly disordered case: W=1.0

[type of disorder: quasi-periodic potential disorder (AA model)]



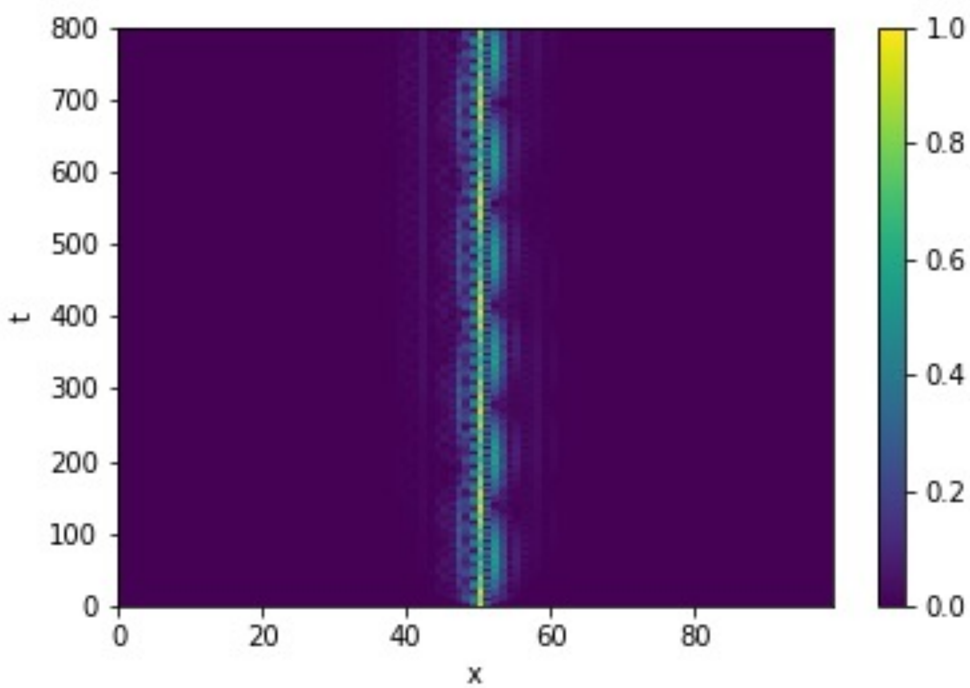
$$W_j = W\cos(2\pi\theta j + \theta_0),$$

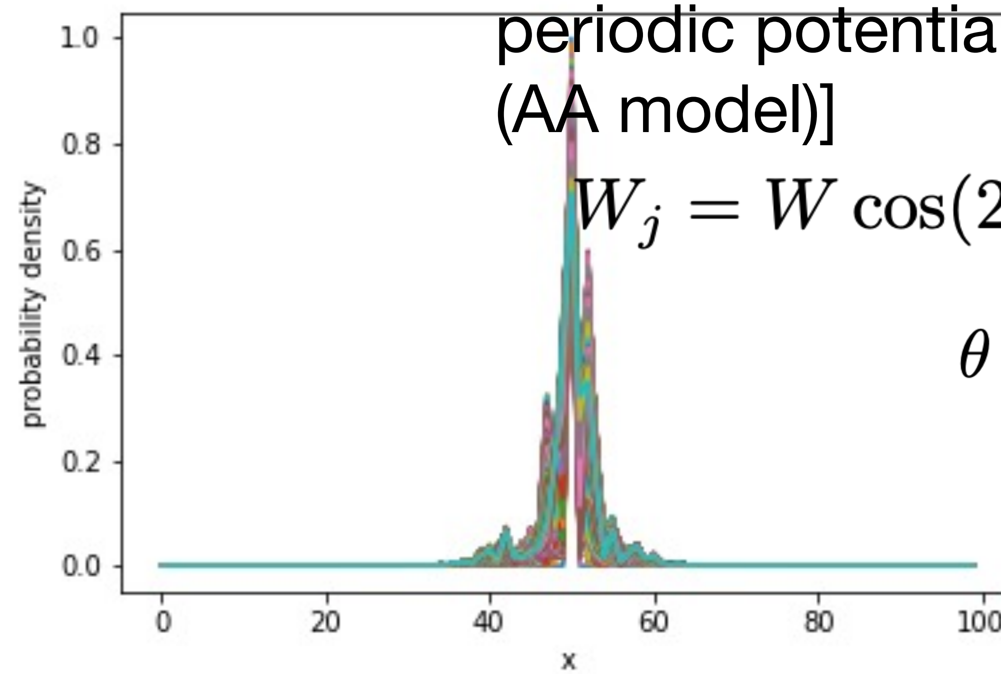
$$\theta = \frac{\sqrt{5}-1}{2}$$



Simulation of the wave-packet dynamics: (a) Hermitian case (continued)

- Case of stronger disorder: W=3.0 (localized phase)





[type of disorder: quasi-periodic potential disorder (AA model)]

$$W_j = W \cos(2\pi\theta j + \theta_0),$$

$$\theta = \frac{\sqrt{5}-1}{2}$$

Notes on Localization-delocalization transition in the AA model

- in 1D such a transition usually does not occur; cf. Anderson ’58
- Aubry-Andre model is an exceptional case:

Localization length:

$$\xi^{-1} \simeq \log \frac{W}{2\Gamma} \longrightarrow \frac{W_c}{2\Gamma_0} = 1.$$

$\xi \to \infty$ at $W = W_c$

cf. Longhi, PRB ’21

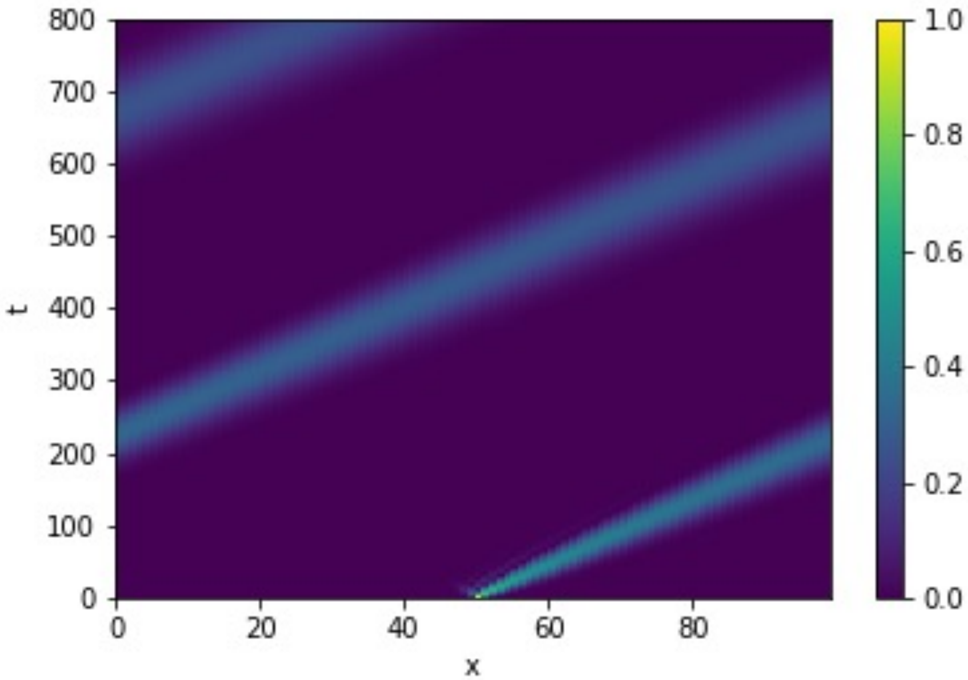
Simulation of the wave-packet dynamics: (b) non-Hermitian case

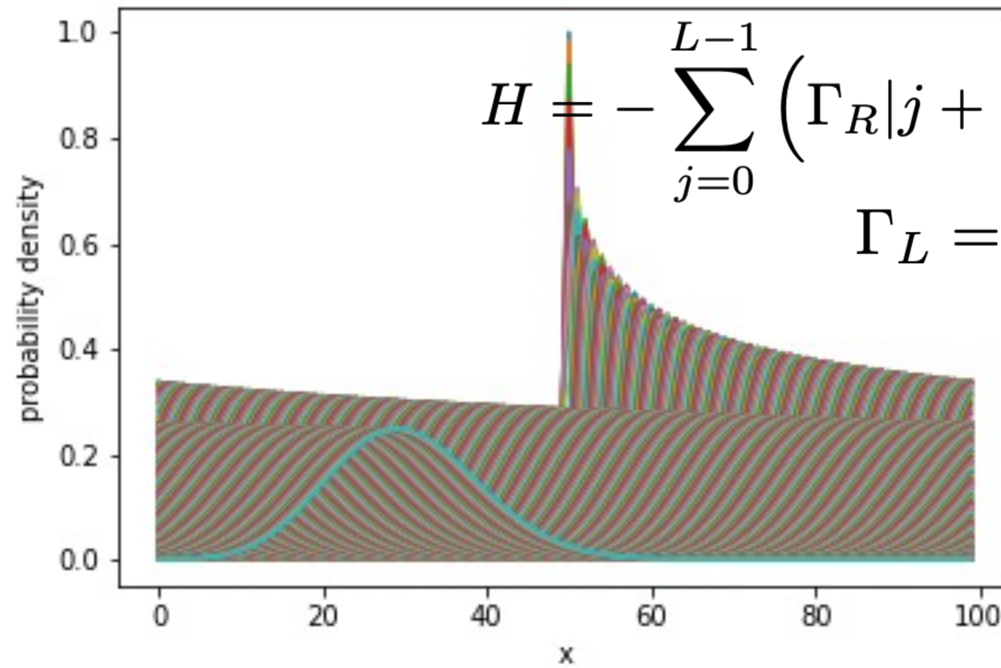
model: Hatano-Nelson × Aubry-Andre model;

$$H = -\sum_{j=0}^{L-1} \left(\Gamma_R |j+1\rangle\langle j| + \Gamma_L |j\rangle\langle j+1| \right)$$

$$\Gamma_L = e^{g}\Gamma_0,\ \Gamma_R = e^{-g}\Gamma_0$$

- clean limit: W=0



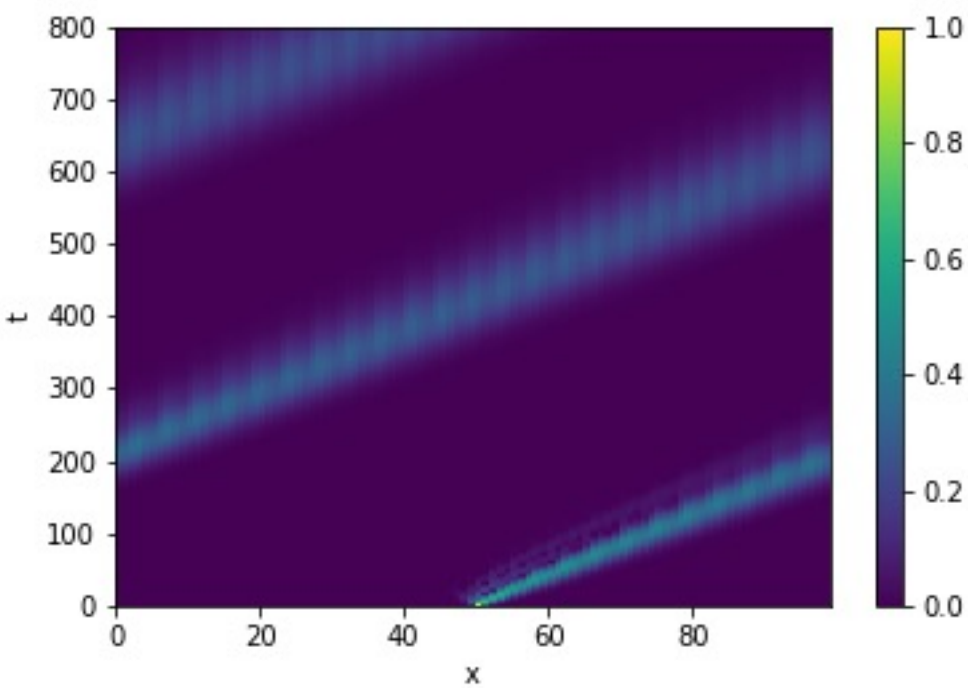


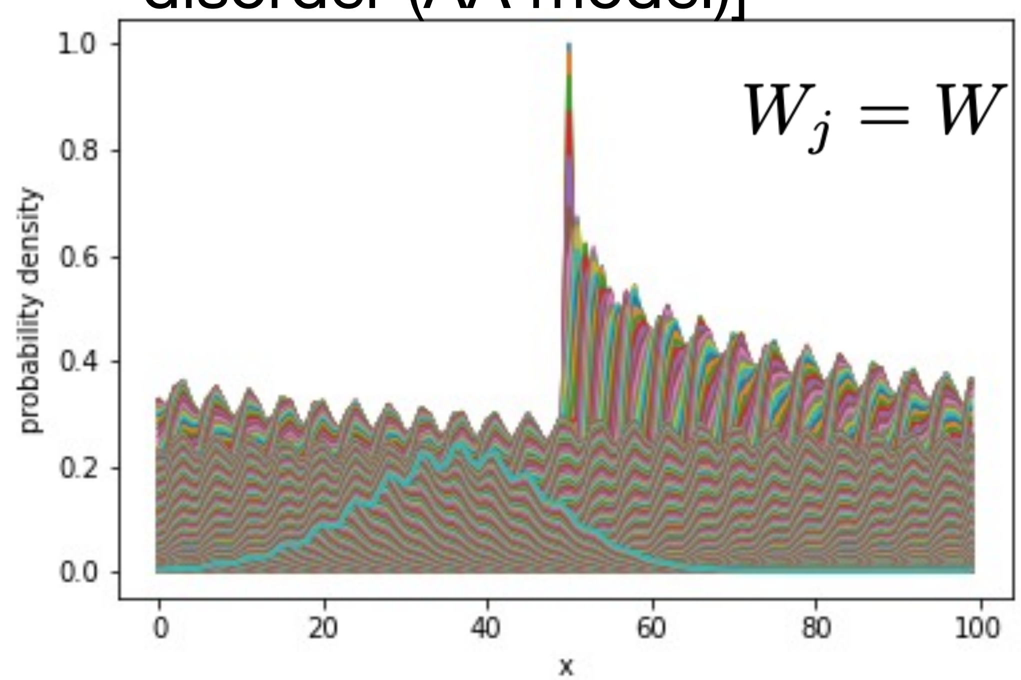
- (weakly) disordered case: W=1.0

[type of disorder: quasi-periodic potential disorder (AA model)]

$$W_j = W\cos(2\pi\theta j + \theta_0),$$

$$\theta = \frac{\sqrt{5}-1}{2}$$



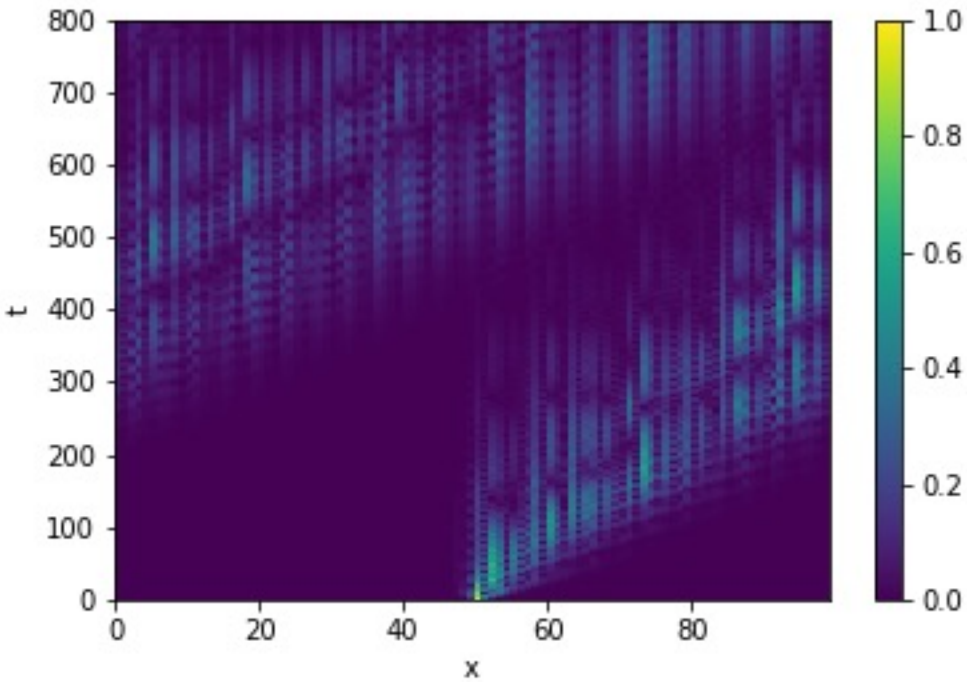


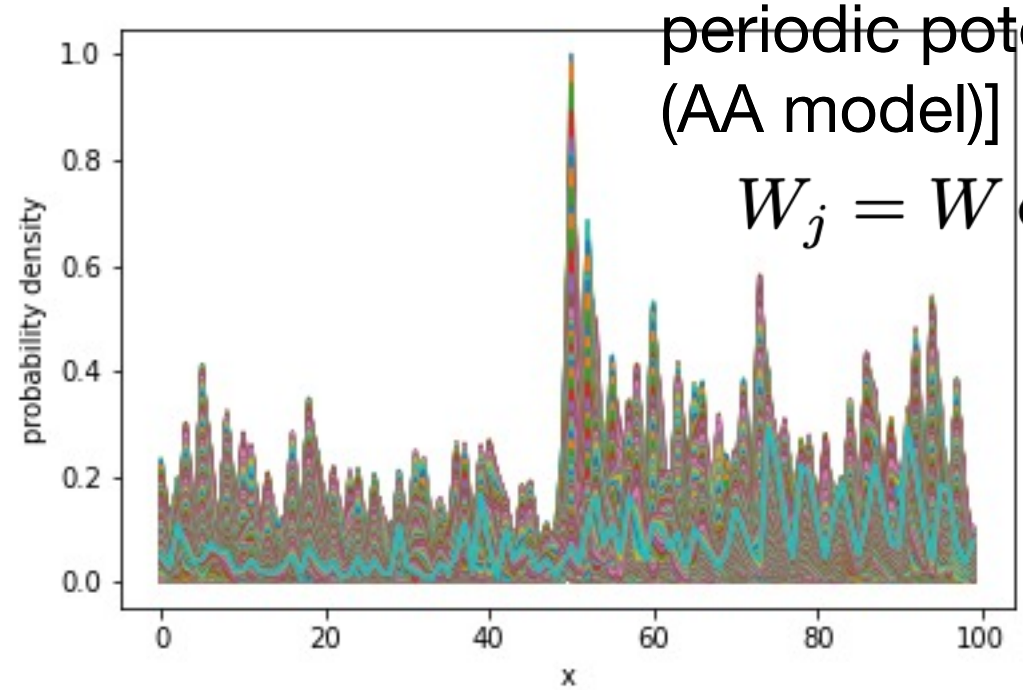
pbc

periodic boundary condition

Simulation of the wave-packet dynamics: (b) non-Hermitian case (continued)

- Case of stronger disorder: W=3.0 (still in the extended phase)



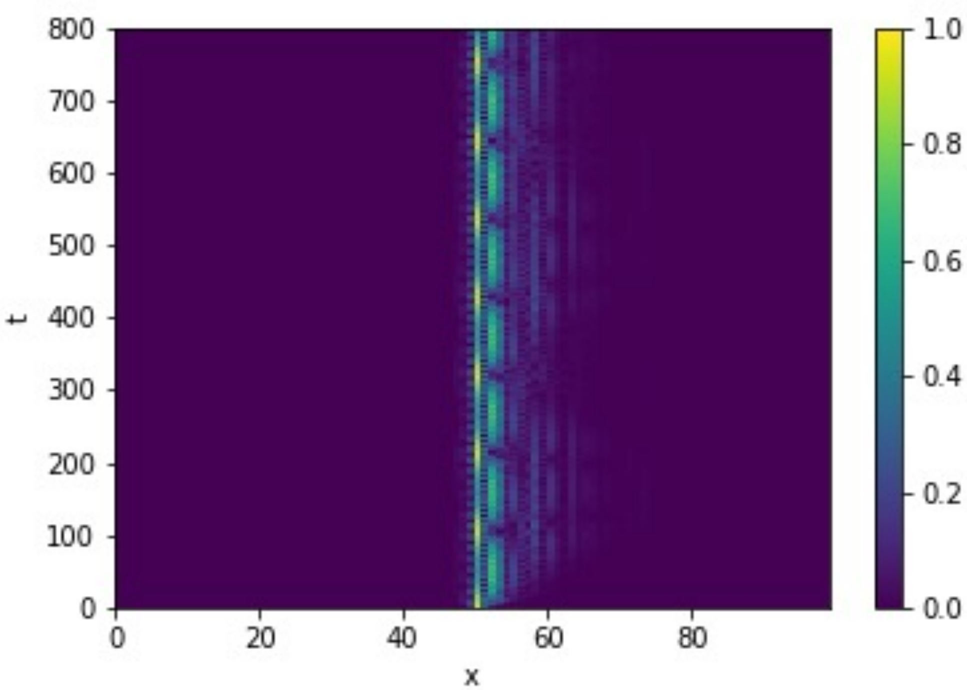


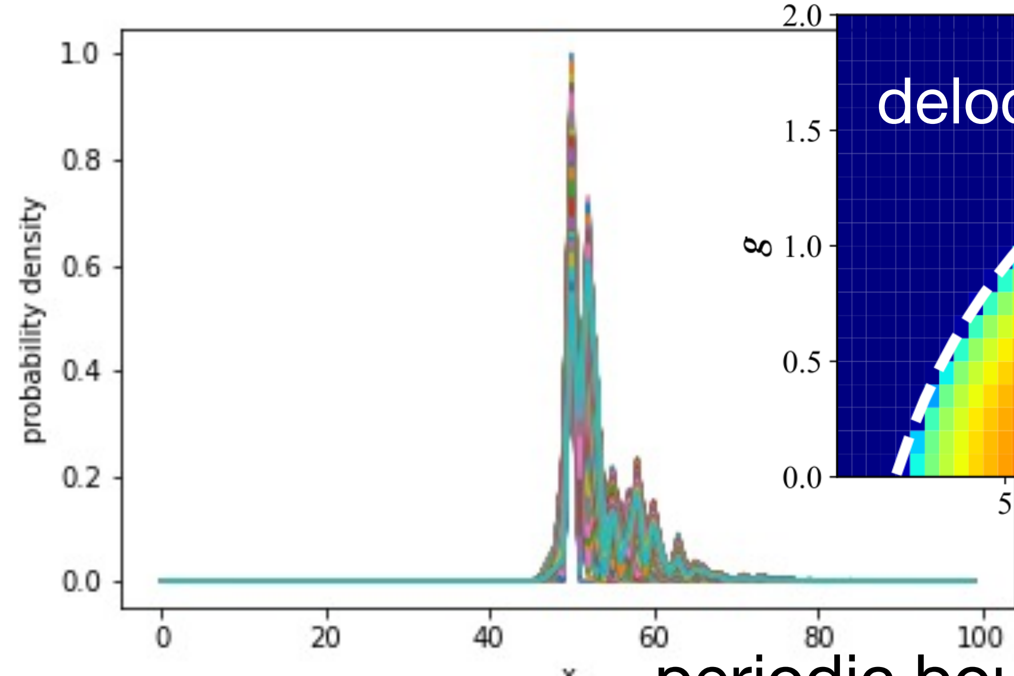
[type of disorder: quasi-periodic potential disorder (AA model)]

$$W_j = W\cos(2\pi\theta j + \theta_0),$$

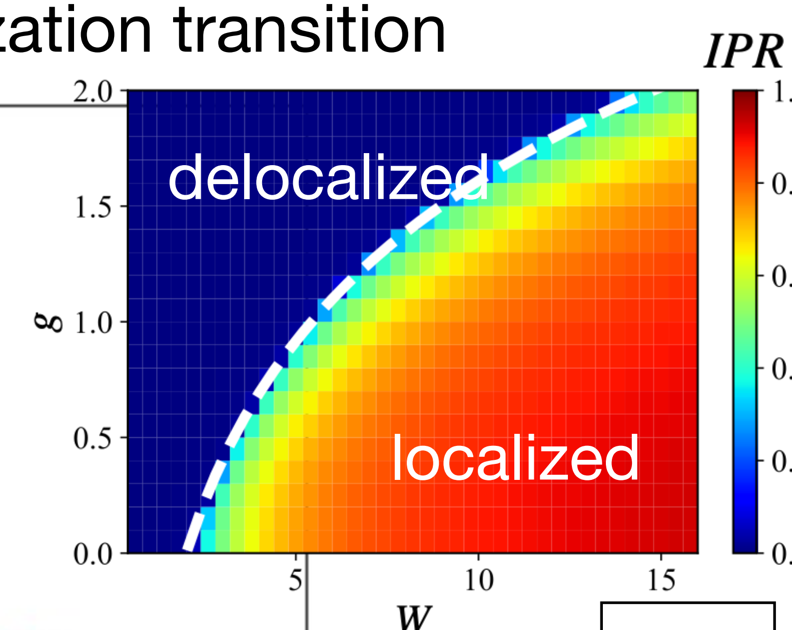
$$\theta = \frac{\sqrt{5}-1}{2}$$

- Case of even stronger disorder (localized phase): W=4.0





- phase diagram: delocalization-localization transition



pbc

periodic boundary condition

Localization-delocalization transition in the Hatano-Nelson×Aubry-Andre model

wave function:		spectrum (eigenvalues):
delocalized	⟷	complex
localized	⟷	real

Hatano & Nelson, PRL '96

- Localization length in the Aubry-Andre model: $\xi^{-1} \simeq \log \frac{W}{2\Gamma}$

*in the Hermitian case: $\xi \to \infty$ at $W = W_c$ ⟶ $\frac{W_c}{2\Gamma_0} = 1.$

- in the Non-Hermitian case:

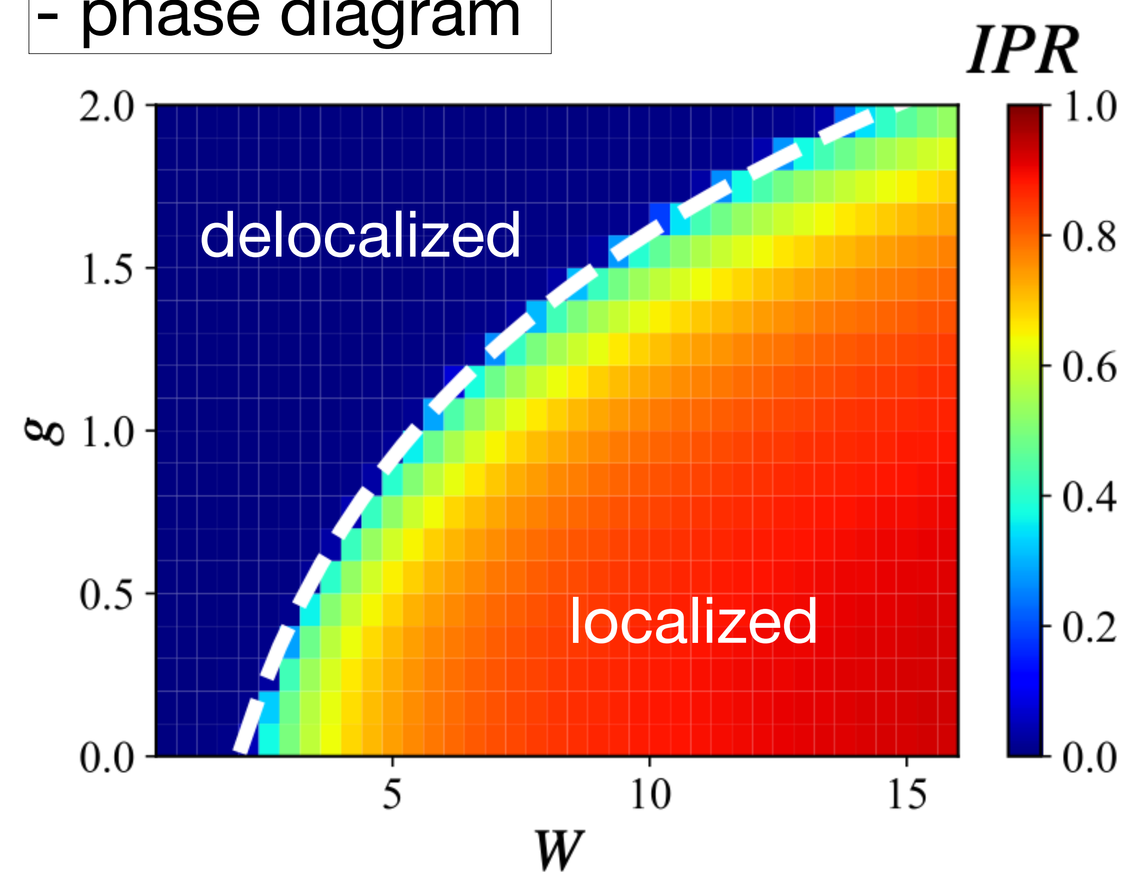
$$\psi^{L,R}(x) \sim \exp\left(-\frac{|x - x_c|}{\xi} \mp g(x - x_c)\right),$$

delocalization point: $\xi^{-1} = g > 0.$

⟶ $W = W_c = 2\Gamma_0\, e^g = 2\Gamma_L$

$$g = \log \frac{W}{2}$$

- phase diagram



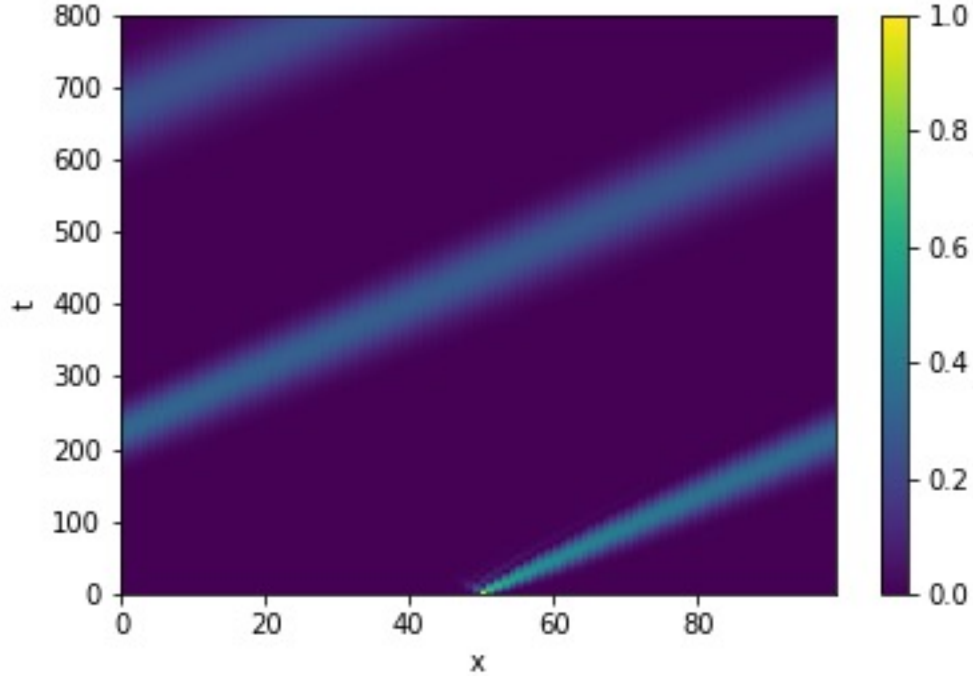
IPR: inverse participation ratio

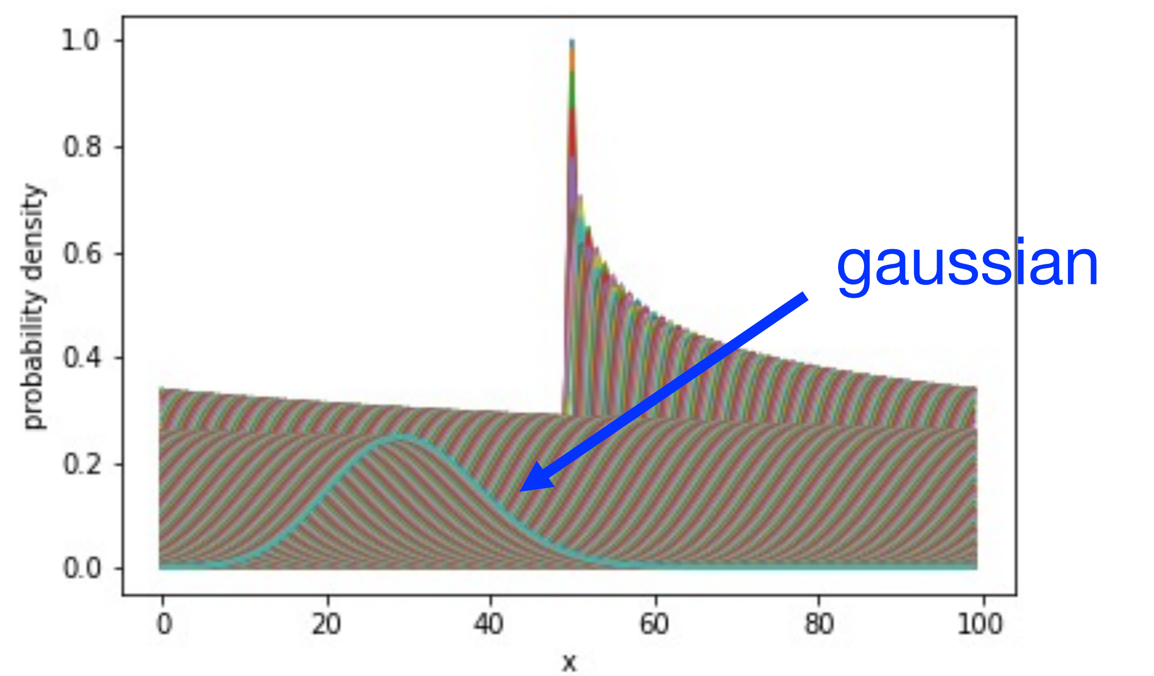
Comparison of periodic vs. open boundary conditions

case of pbc:
the *periodic* boundary conditions

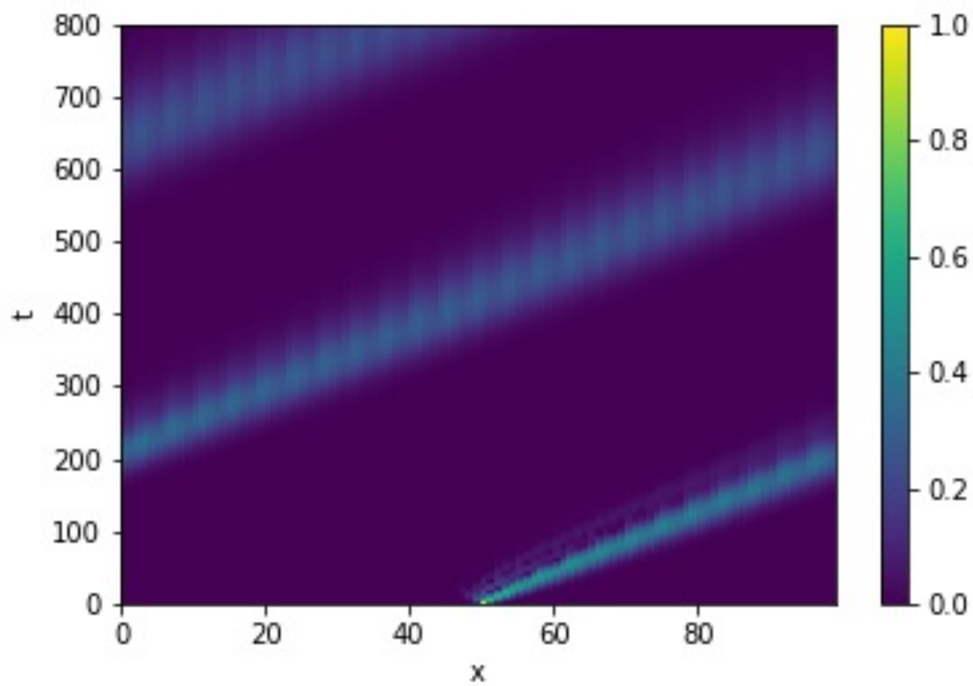
Simulation of the wave-packet dynamics:
(b) non-Hermitian case

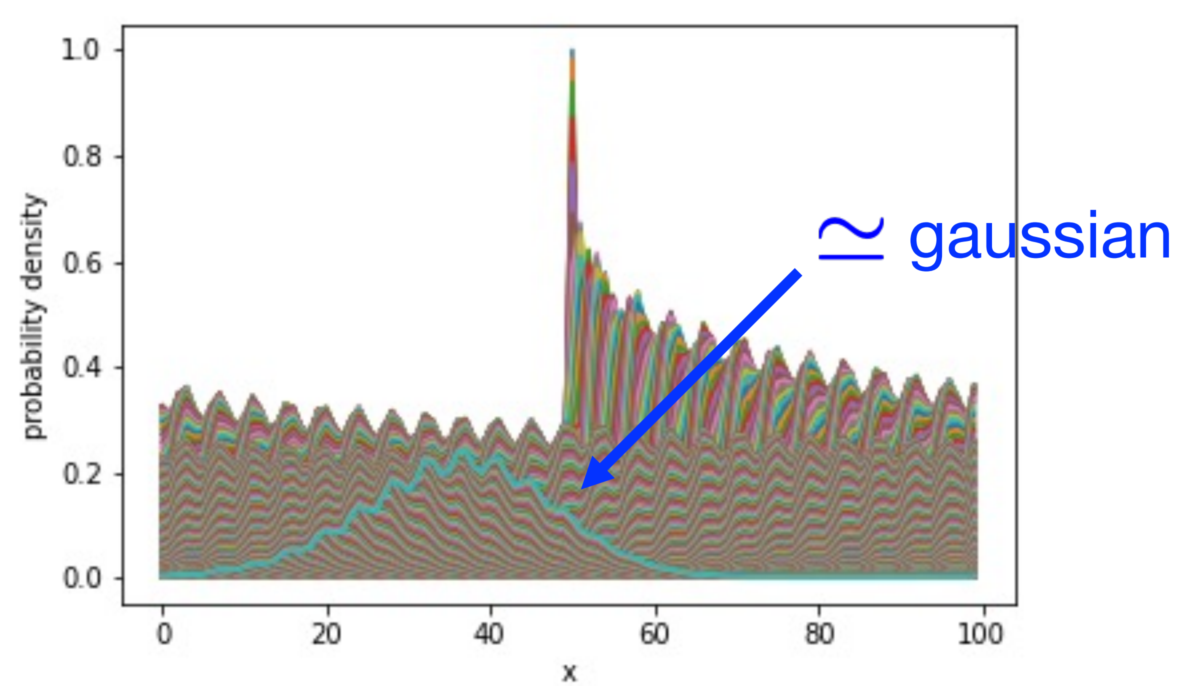
- clean limit: W=0





- case of weak disorder: W=1.0



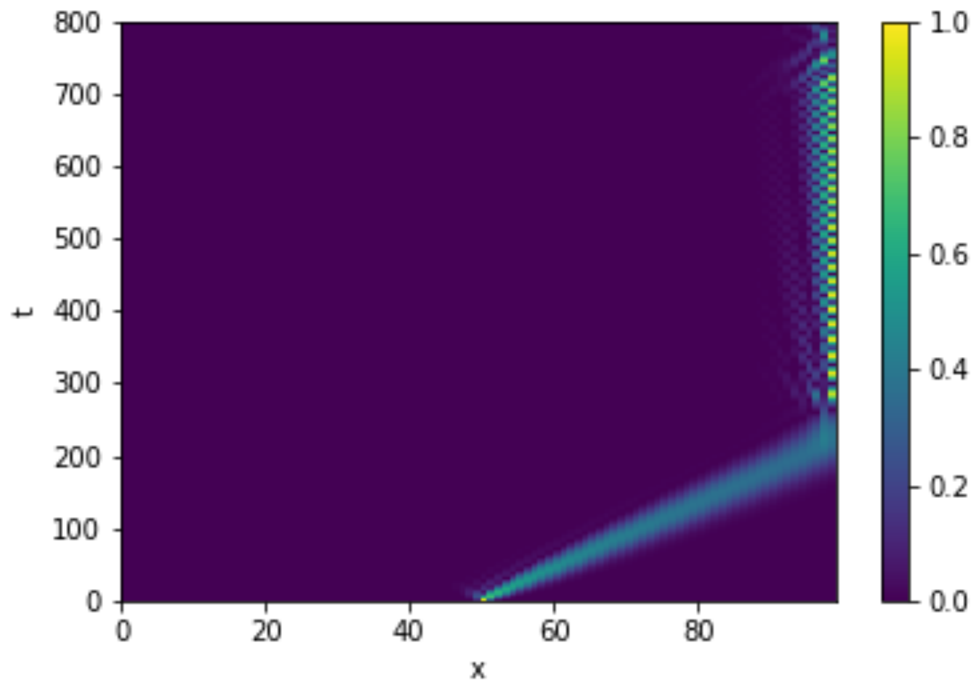


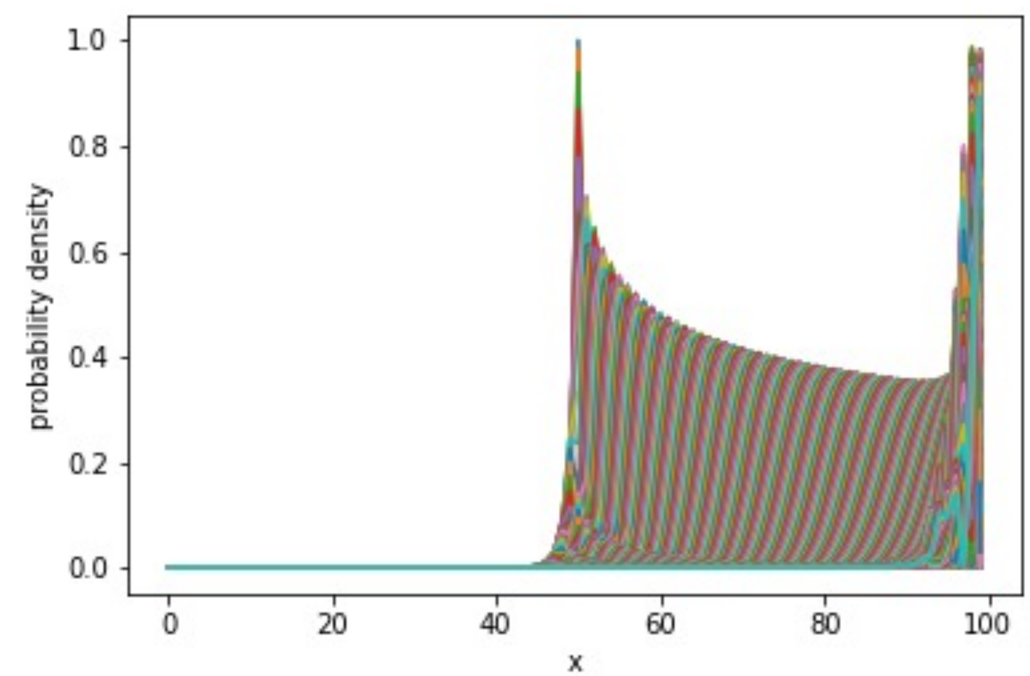
Comparison of periodic vs. open boundary conditions

Simulation of the wave-packet dynamics:
(b) non-Hermitian case

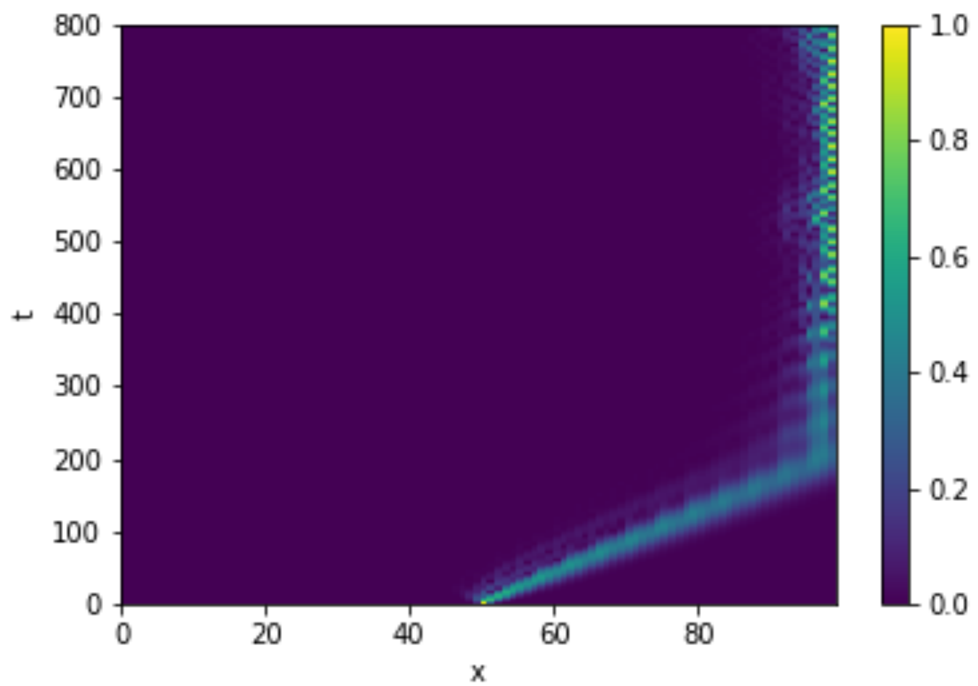
case of obc:
the *open* boundary conditions

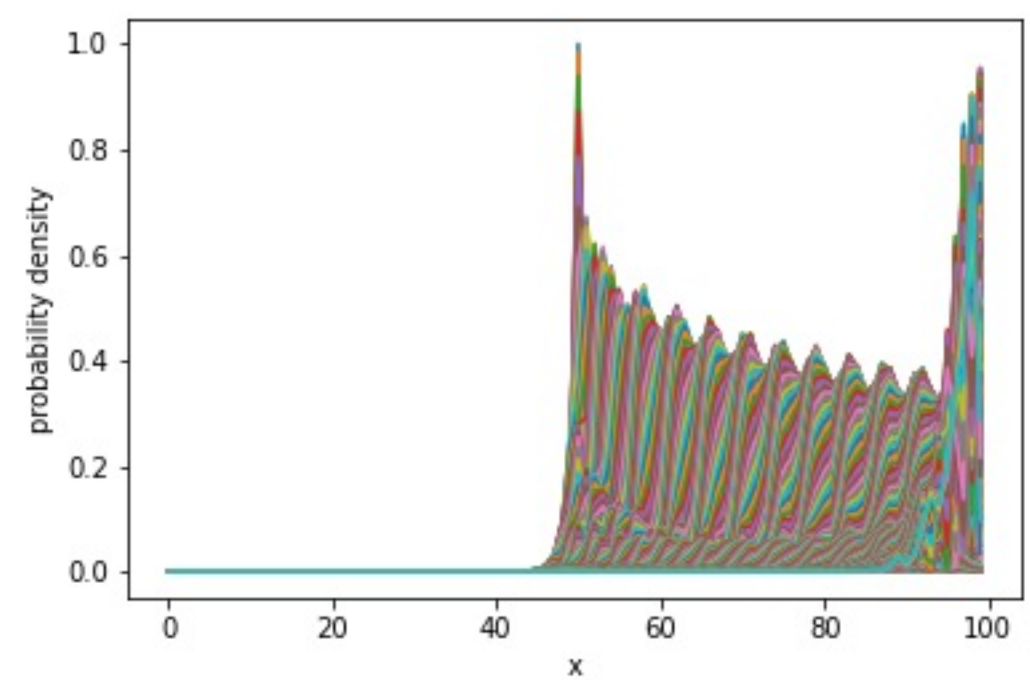
- clean limit: W=0





- case of weak disorder: W=1.0



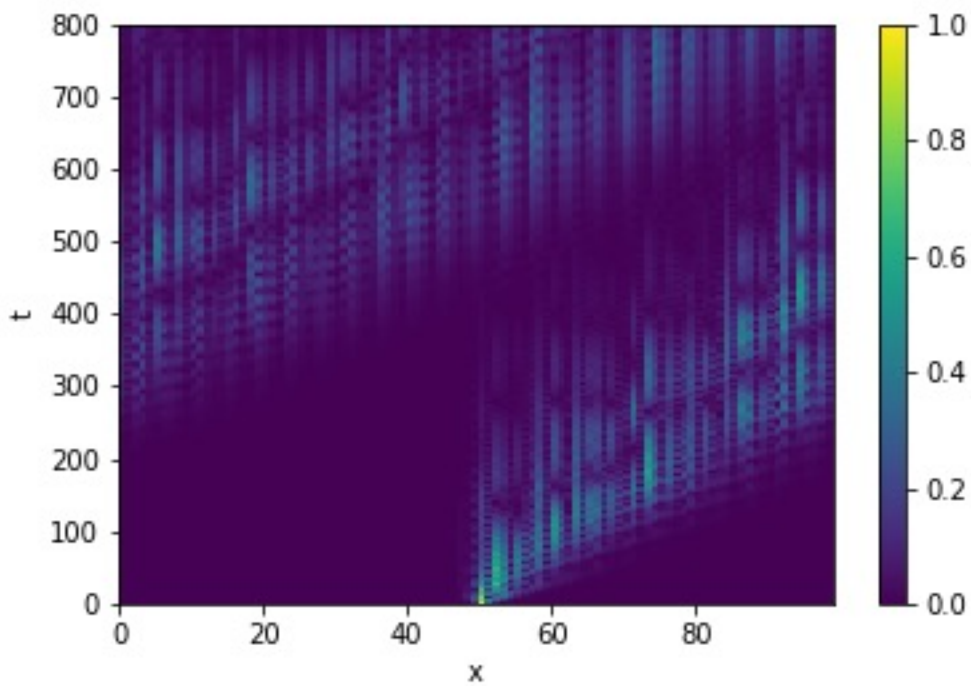


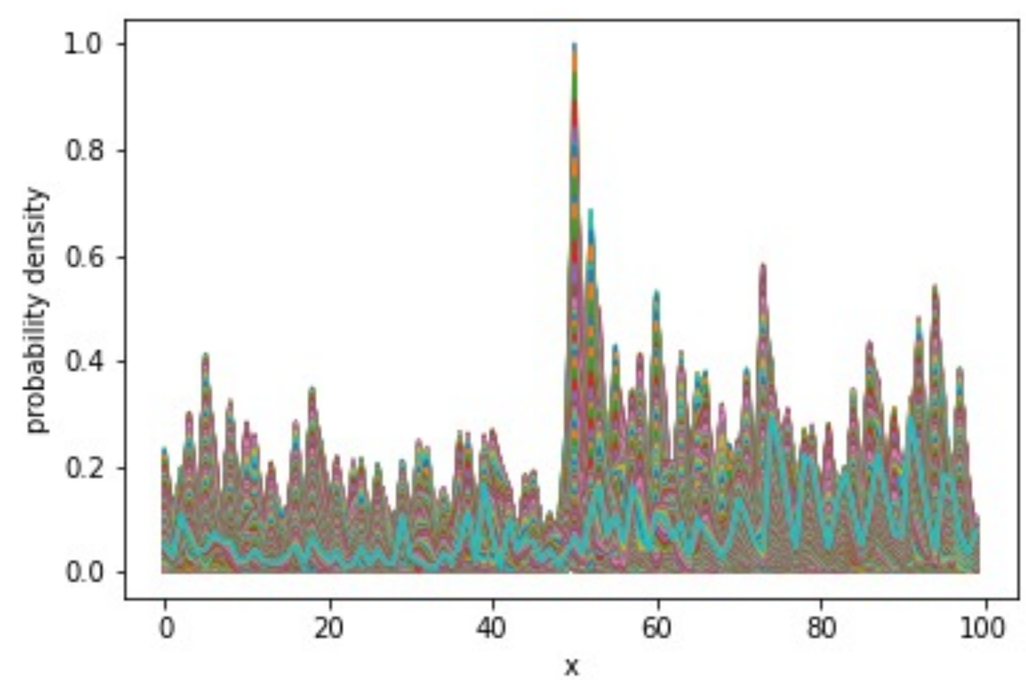
Simulation of the wave-packet dynamics:
(b) non-Hermitian case

<u>pbc vs. obc</u>
(continued)

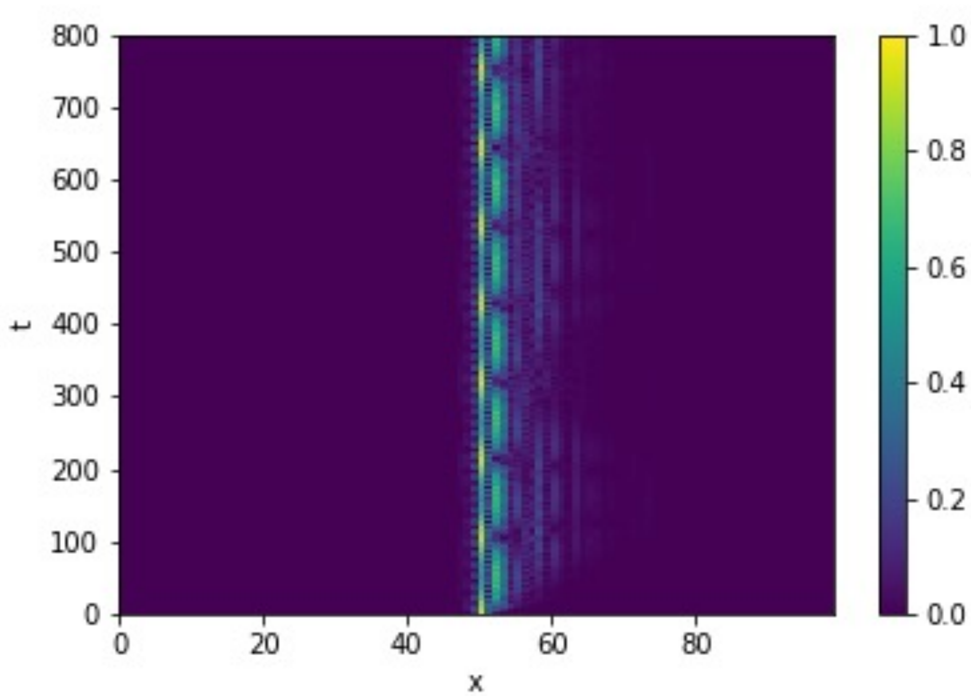
case of pbc:
the *periodic* boundary conditions

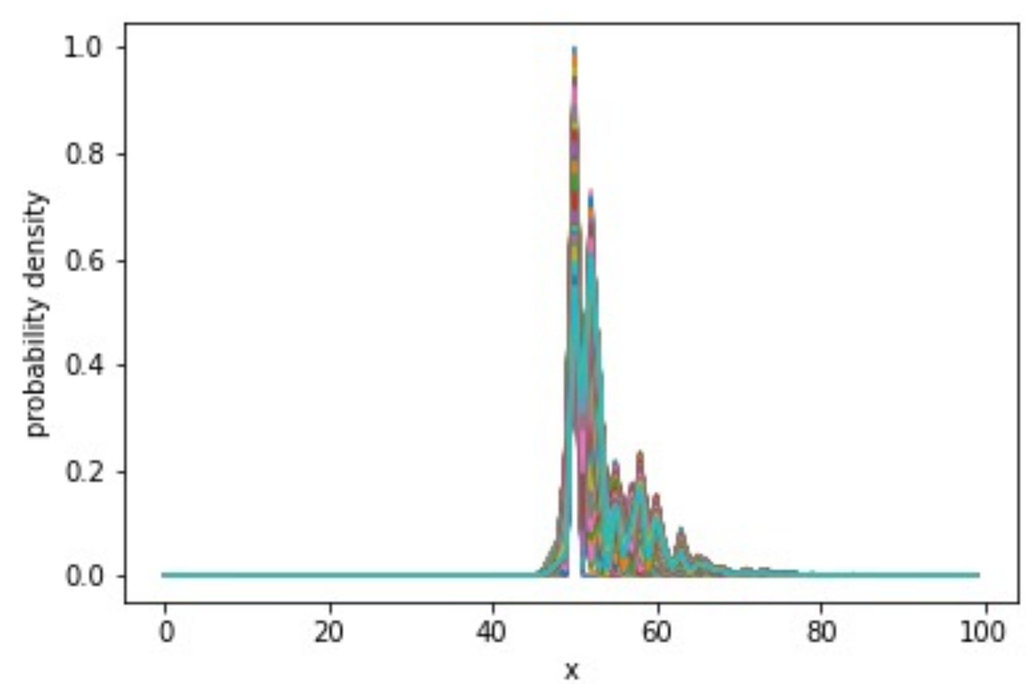
- Case of stronger disorder: W=3.0 (still in the extended phase)





- Case of even stronger disorder: W=4.0 (localized phase)



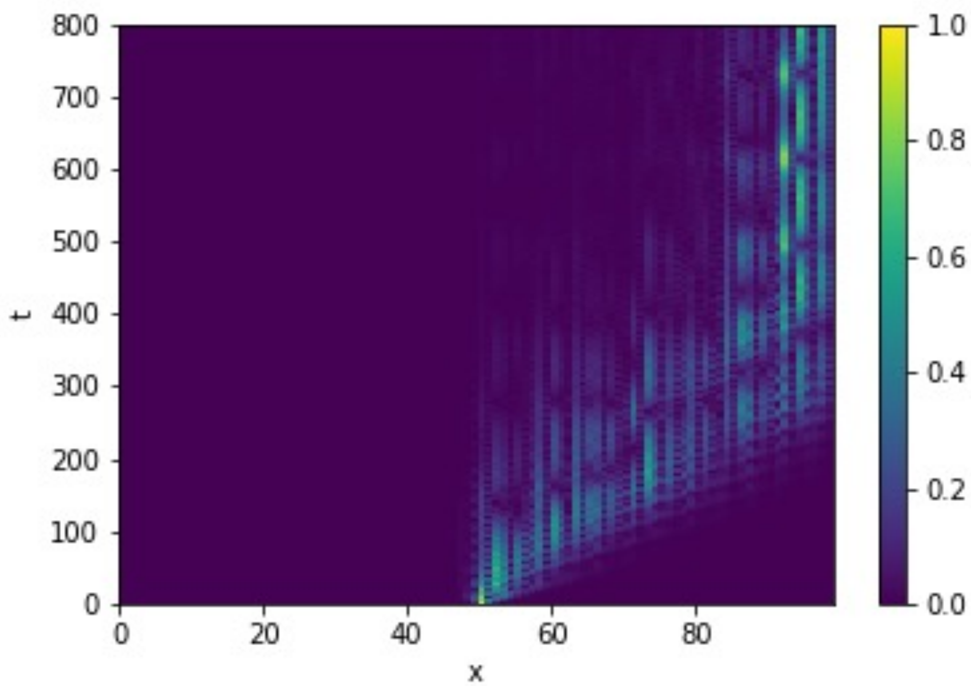


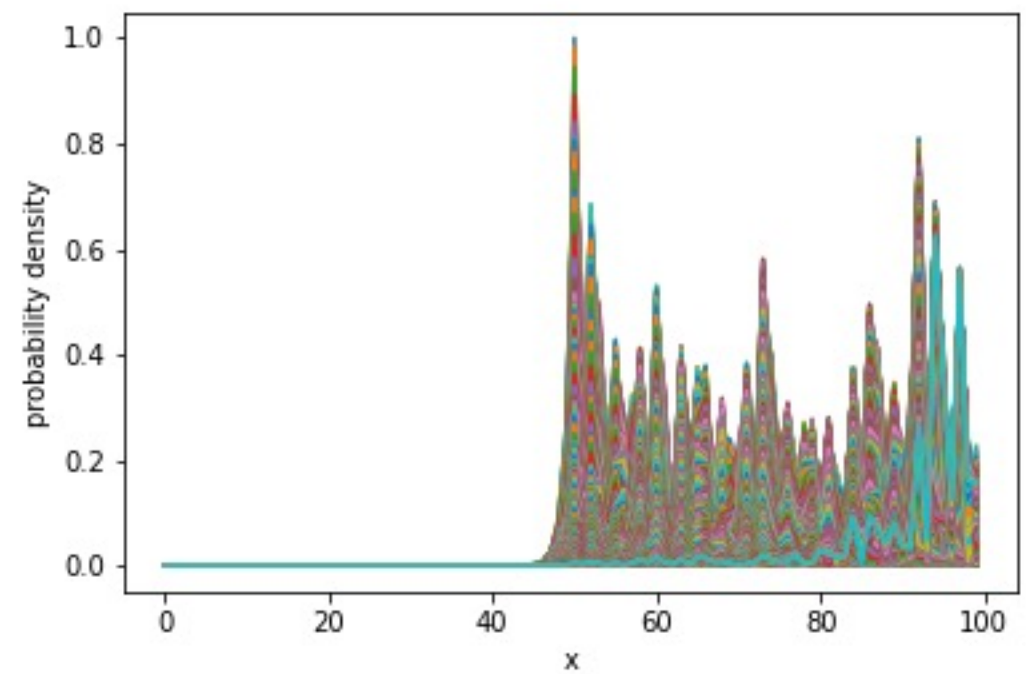
Simulation of the wave-packet dynamics:
(b) non-Hermitian case

<u>pbc vs. obc</u>
(continued)

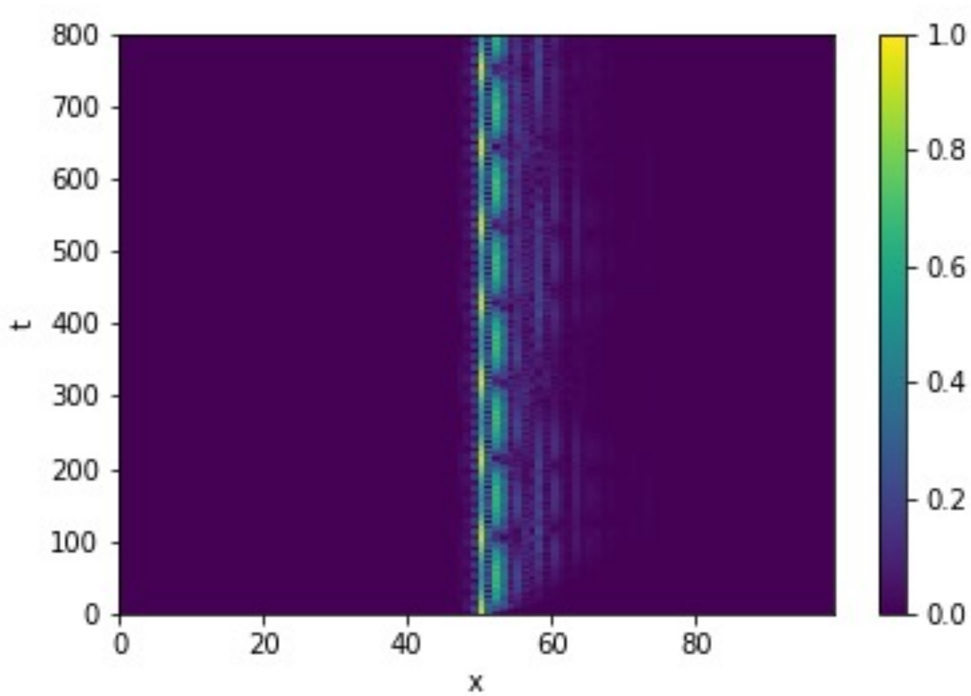
case of obc:
the *open* boundary conditions

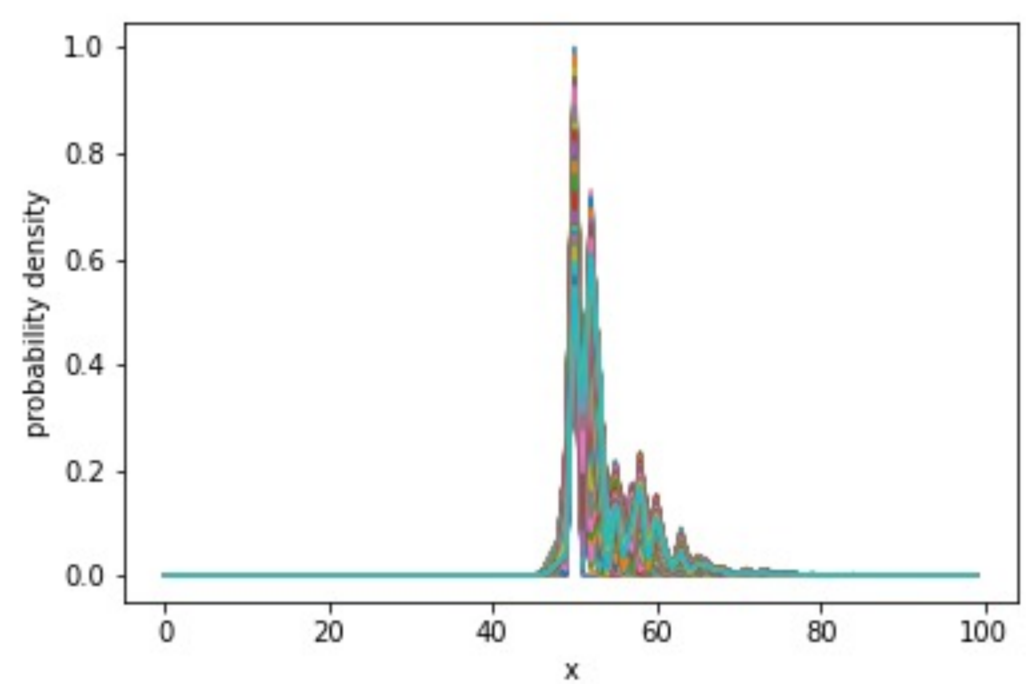
- Case of stronger disorder: W=3.0 (still in the extended phase)





- Case of even stronger disorder: W=4.0 (localized phase)





Summary

1) Bulk-edge correspondence (BEC) in *Hermitian topological* systems: BEC-1

2) BEC in *non-Hermitian* topological systems: BEC-1'

3) A new type of BEC in non-Hermitian systems exhibiting *skin effect*: BEC-2

sensitive

4) Non-Hermitian wave-packet *dynamics*

insensitive

Physics sensitive vs. insensitive to the boundary condition

in non-Hermitian systems

Summary/conclusions:

Physics sensitive vs. insensitive to the boundary condition

1) The topological insulator:

- the underlying mechanism: bulk-boundary correspondence (BBC)
- BBC superficially breaks down in non-Hermitian systems;

$\longrightarrow$ two possible way out: one *modifies*, the other *abandons* pbc

2) The Hatano-Nelson model:

- 1D tight-binding model with non-reciprocal (asymmetric) hopping
- Localization/delocalization transition in 1D
- Non-Hermitian skin effect; a topological interpretation

<u>Take-home message</u>:

Nature of the localization/delocalization transition in Hatano-Nelson model *multifaceted/upgraded*

i) complex vs. real spectra;
ii) delocalization-localization transition;
iii) skin effect vs. no skin effect (obc)
iv) topological transition;

A quadruple phase transition!

From the viewpoint of bulk-boundary correspondence:

- Bulk-boundary correspondence (BBC) multi-faceted:

obc vs. pbc

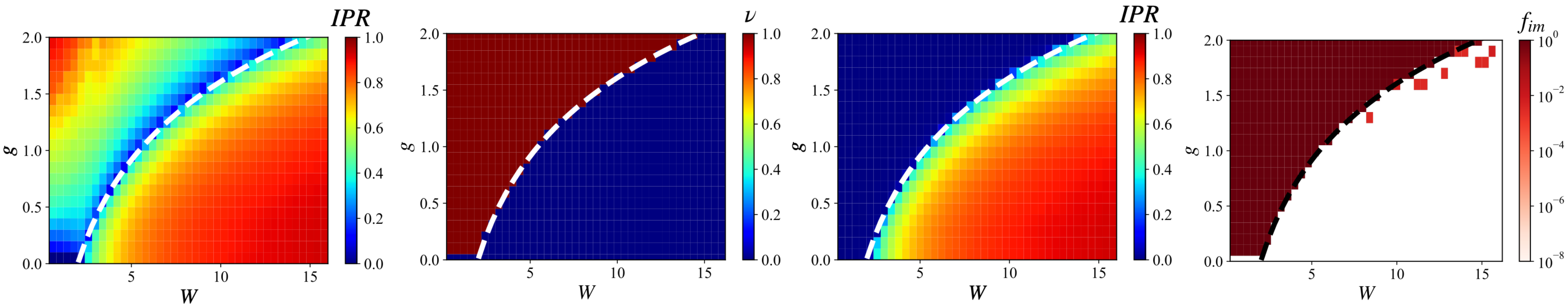
the original BBC doublet

skin effect

winding number

IPR: inverse participation ratio

ratio of complex eigenvalues



The Hatano-Nelson quadruplet

The triplet of bulk phase transitions

(Hatano-Nelson bulk triplet)

3) Wave-packet dynamics

- Hermitian case: quantum interference
- Non-Hermitian case

i) Clean limit: pseudo-classical diffusive dynamics

ii) Disorder enhances quantum interference

4) On-going studies on :

i) the effects of interaction; many-body case
ii) entanglement/entropy dynamics

Orito & Imura, Phys. Rev. B ***105****, 024303 (2022)*

Considering also…

+ wave-packet & entanglement/entropy dynamics

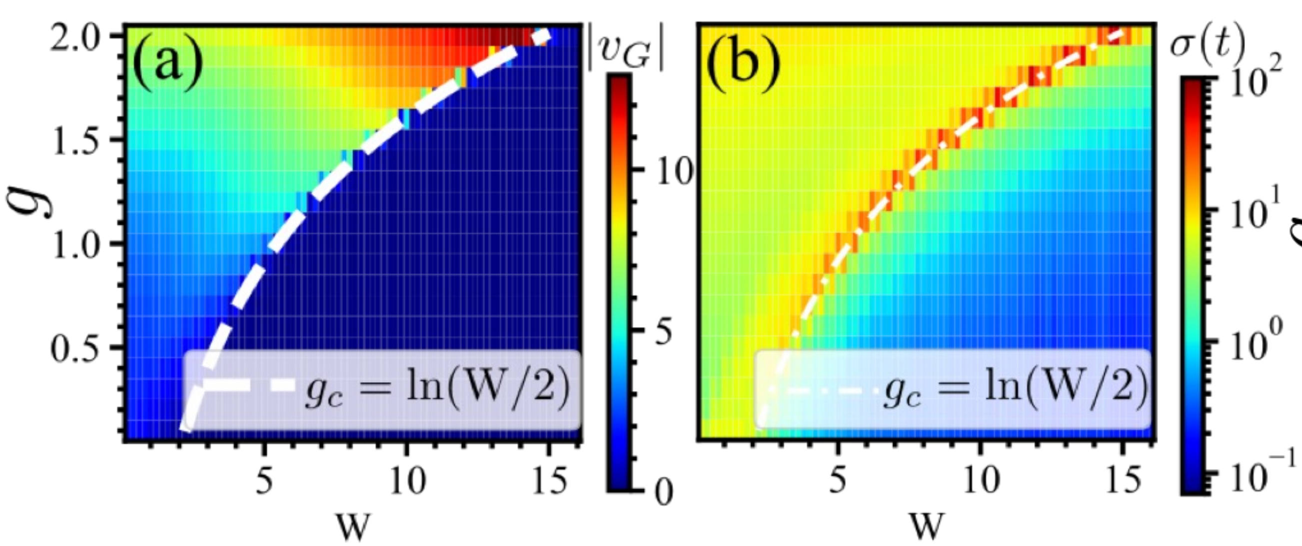
i) wave-packet dynamics

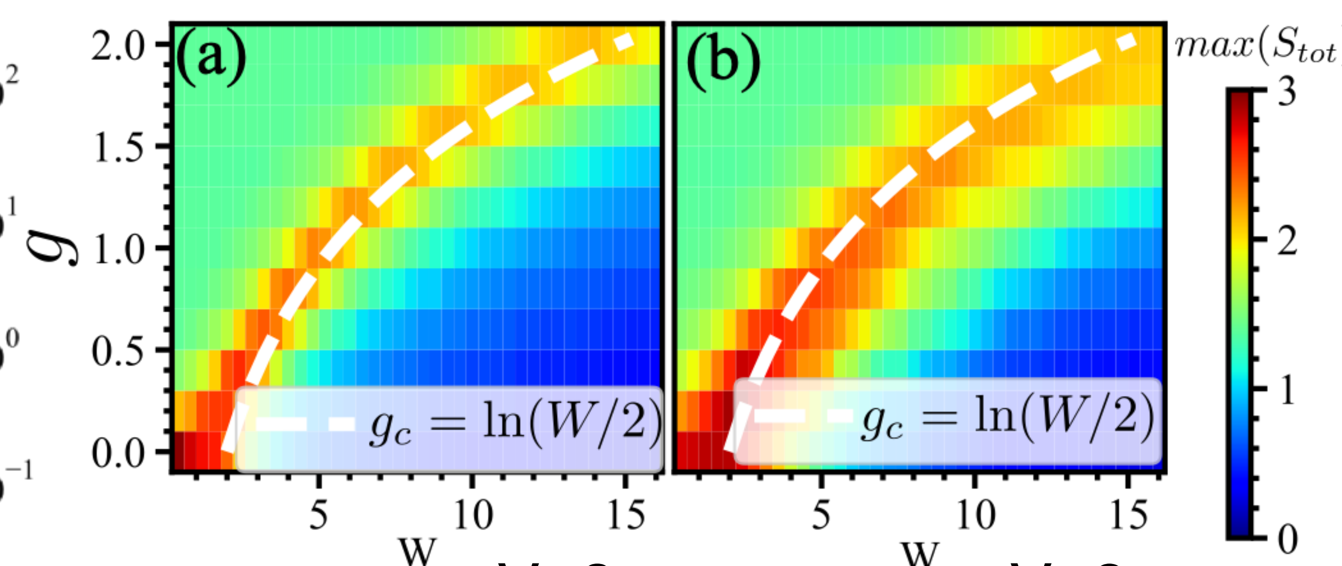
ii) entanglement/entropy dynamics

COM velocity

wave packet spreading

Maximal entanglement entropy





V=0

V=2

The Hatano-Nelson "sextuplet"

The modified periodic boundary condition

specified by a real positive parameter: b

$$H_{\mathrm{NN}}^{mpbc} = b^{-L} t_2^- |1, A\rangle\langle L, B| + b^L t_2^+ |L, B\rangle\langle 1, A|$$
$$H_{3\mathrm{NN}}^{mpbc} = b^{-L} t_3^- |1, B\rangle\langle L, A| + b^L t_3^+ |L, A\rangle\langle 1, B|$$

to be added to $H_{obc} = H_{\mathrm{NN}} + H_{3\mathrm{NN}}$

<u>In a transformed basis:</u>

$$H_{mpbc} = H_{obc} + H_{\mathrm{NN}}^{mpbc} + H_{3\mathrm{NN}}^{mpbc}$$

- A similarity transformation: $\tilde{H}_{mpbc} = S^{-1} H_{mpbc} S$

$$S = \mathrm{diag}[1, 1, b, b, b^2, b^2, \cdots, b^{L-1}, b^{L-1}]$$

- Under the transformed basis:

$$\tilde{H}_{pbc} = \tilde{H}_{obc} + \underline{\tilde{H}_{\mathrm{NN}}^{pbc} + \tilde{H}_{3\mathrm{NN}}^{pbc}} \qquad \tilde{H}_{pbc}|k\rangle = E(k)|k\rangle$$

standard pbc

The meaning of the modified periodic boundary condition (continued)

- back in the original basis: $H_{mpbc}|\beta\rangle = E(\beta)|\beta\rangle$

$$|\beta\rangle = \sum_{j=0}^{L-1} \beta^j (c_A|jA\rangle + c_B|jB\rangle)$$

$$\boxed{\beta = be^{ik}}$$

The coefficients c_A, c_B are determined by the eigenvalue equation:

$$\underbrace{\begin{bmatrix} 0 & R_+(\beta) \\ R_-(\beta) & 0 \end{bmatrix}}_{H_{mpbc}(\beta)} \begin{bmatrix} c_A \\ c_B \end{bmatrix} = E \begin{bmatrix} c_A \\ c_B \end{bmatrix}$$

where

$$\begin{aligned} R_-(\beta) &= t_1^- + t_2^+\beta + t_3\beta^{-1} \\ &= t_1 - \gamma_1 + (t_2 + \gamma_2)\beta + t_3\beta^{-1} \\ R_+(\beta) &= t_1^+ + t_2^-\beta^{-1} + t_3\beta \\ &= t_1 + \gamma_1 + (t_2 - \gamma_2)\beta^{-1} + t_3\beta \end{aligned}$$

Consistency of the two approches:

1) our modified periodic boundary condition

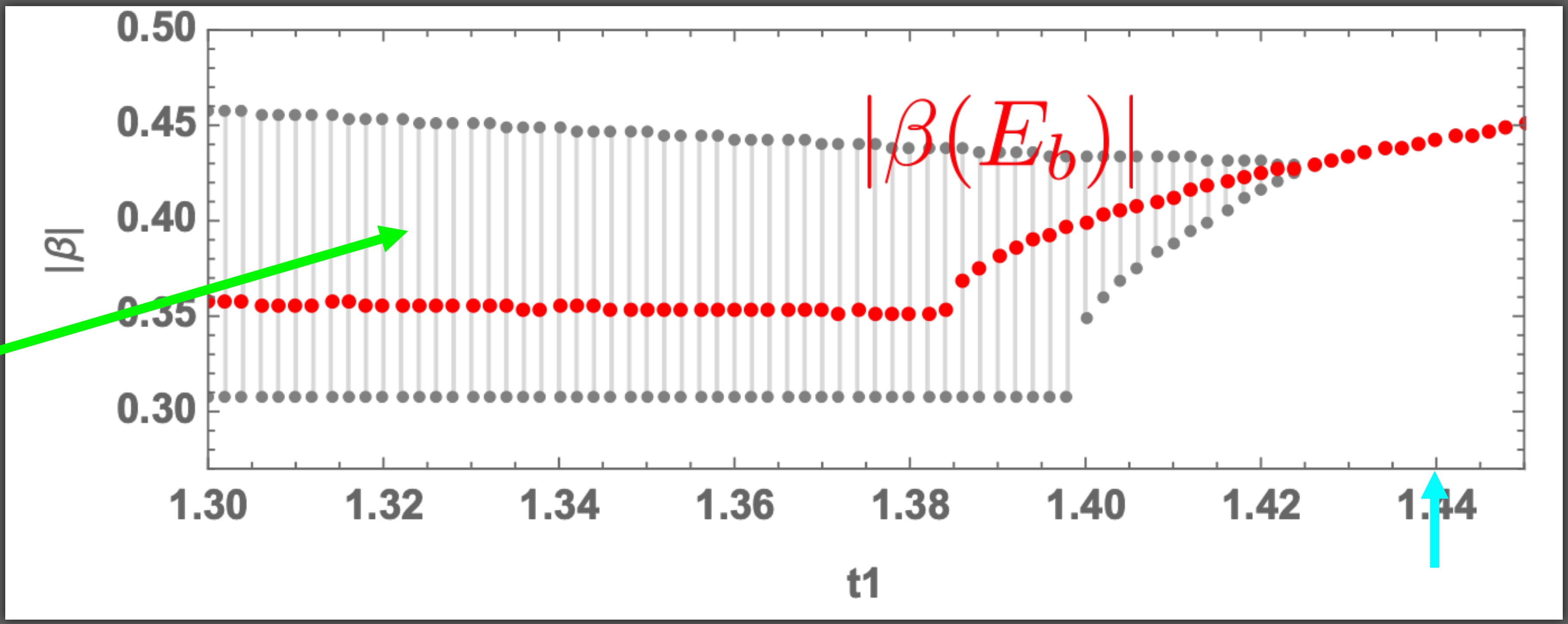
vs.

with 2) the so-called generalized Brillouin zone approach

Yokomizo & Murakami, arXiv:1902…

- So far, b has been a free parameter
- We can choose our b to be the one corresponding to the generalized Brillouin zone *at the bottom of the band*

$$b = |\beta(E_b)|$$



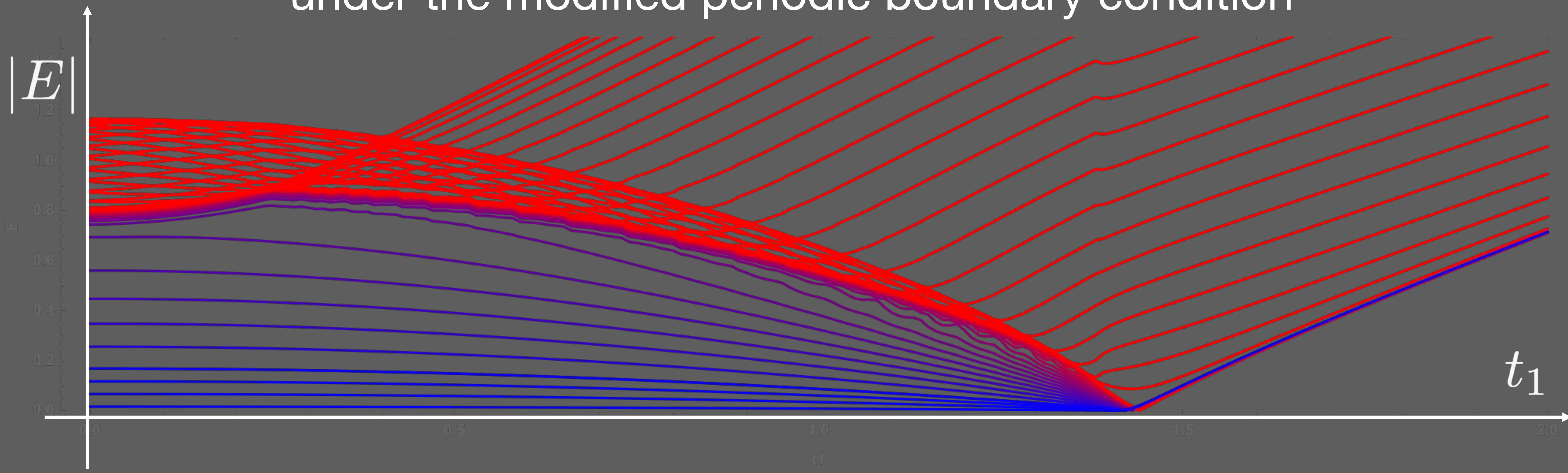
our

w=1 region

gap closing

Bulk-edge correspondence *visualized*

under the modified periodic boundary condition



- The modified periodic boundary condition (mpbc):

$$H = H_{obc} + \delta H_{mpbc}$$

- continuous change of the boundary condition:

$$\delta = 1 \to 0$$

mpbc: purely bulk spectrum — obc: bulk+edge spectrum

$\delta = 1$ ⟶ $\delta = 0$

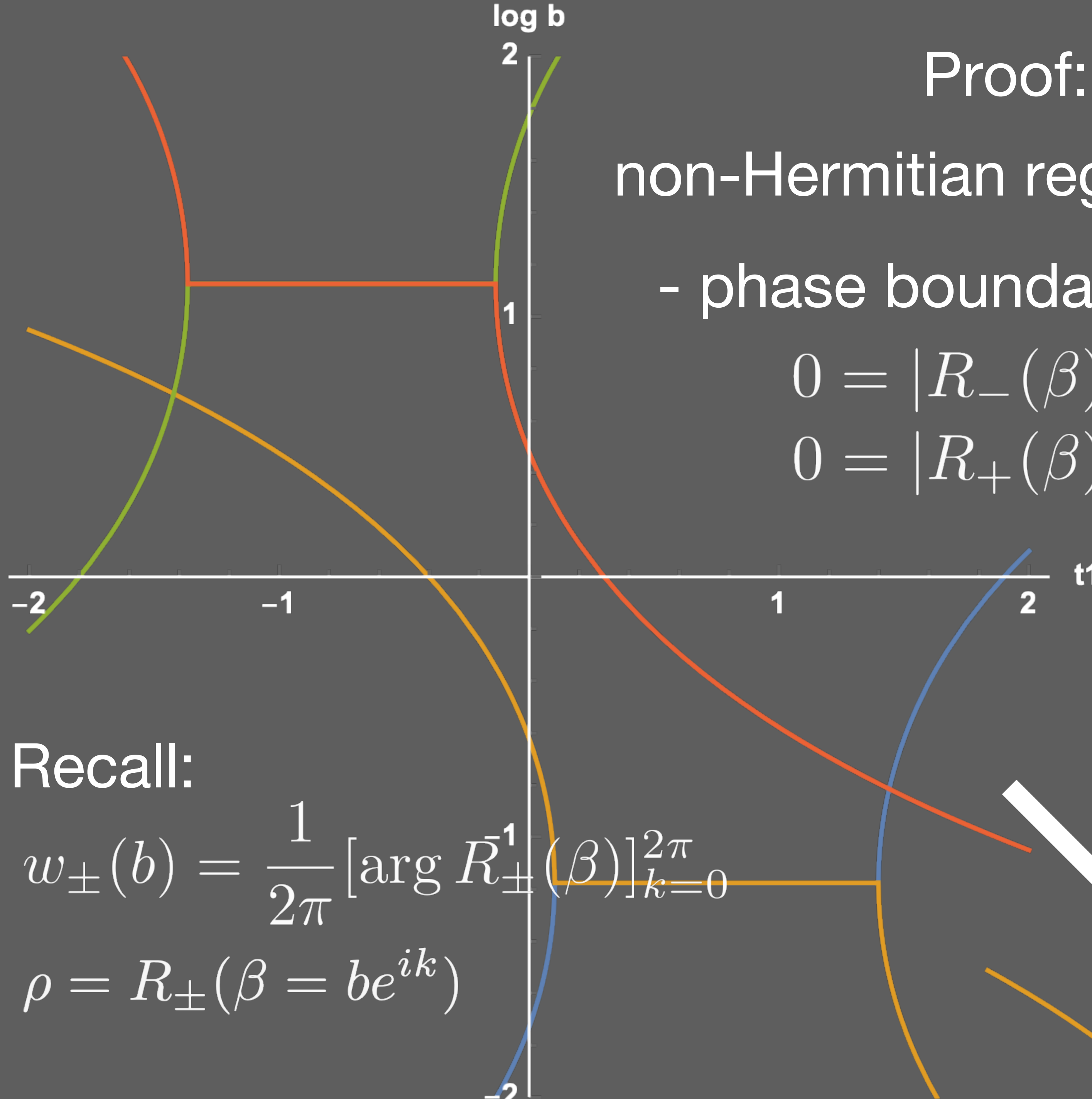
smooth evolution = BEC

Proof: smoothly connected

non-Hermitian regime ⟷ Hermitian limit

- phase boundaries in the non-hermitian regime:

$$0 = |R_-(\beta)| = |t_1 - \gamma_1 + (t_2 + \gamma_2)\beta + t_3\beta^{-1}|$$
$$0 = |R_+(\beta)| = |t_1 + \gamma_1 + (t_2 - \gamma_2)\beta^{-1} + t_3\beta|$$

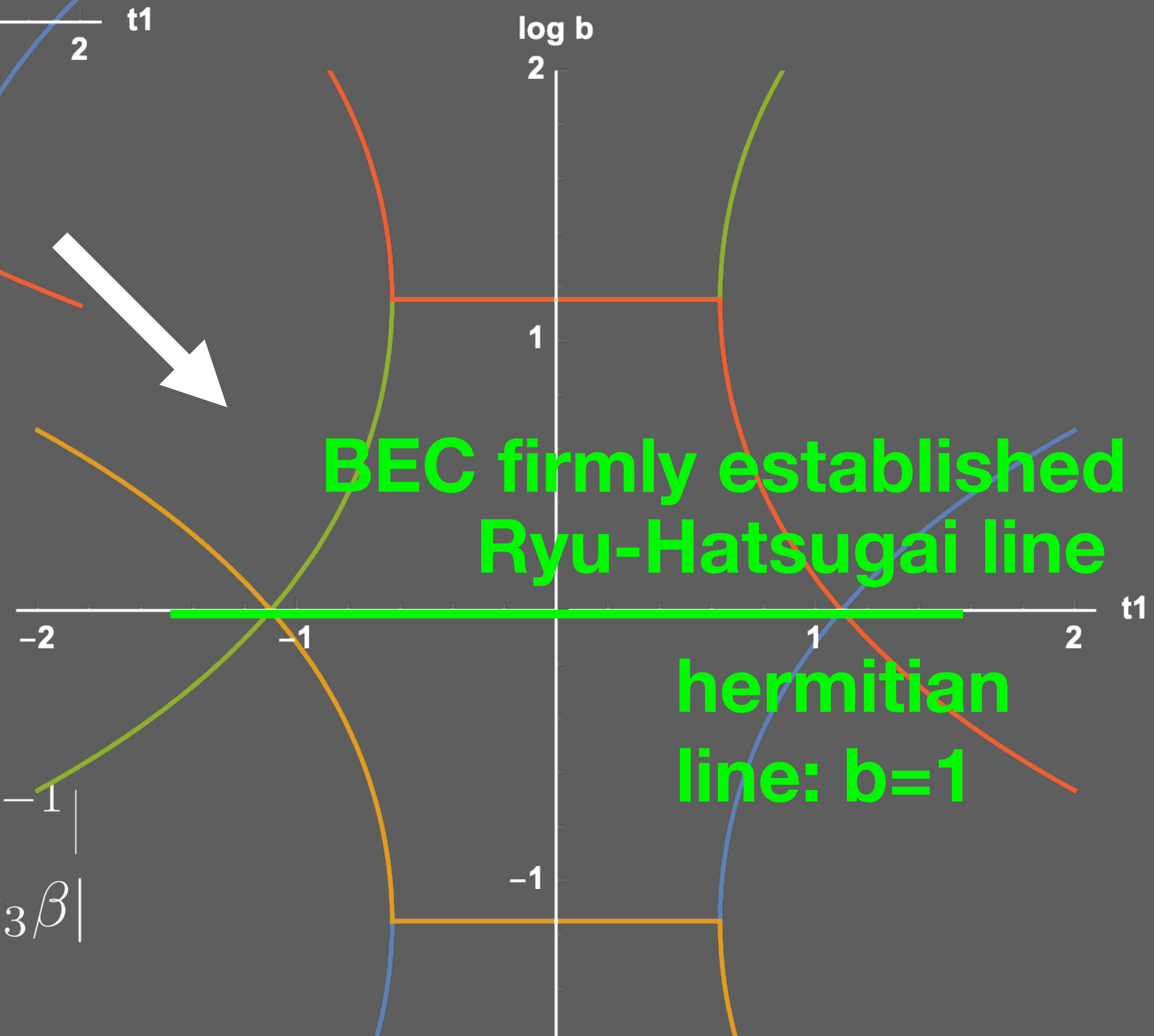


Recall:

$$w_\pm(b) = \frac{1}{2\pi}[\arg R_\pm(\beta)]_{k=0}^{2\pi}$$
$$\rho = R_\pm(\beta = be^{ik})$$

- The phase boundaries in the hermitian limit:

$$0 = |R_-(\beta)| = |t_1 + t_2\beta + t_3\beta^{-1}|$$
$$0 = |R_+(\beta)| = |t_1 + t_2\beta^{-1} + t_3\beta|$$



BEC firmly established

Ryu-Hatsugai line

hermitian line: b=1