The role of PT phase transition in biology and ecology

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Stochastic unravelling problem

Finding a wave-function-valued random process $|\psi_t\rangle$ such that the corresponding density matrix $\hat{\rho} = \mathbb{E}[|\psi_t\rangle\langle\psi_t|]$ obeys the Lindblad equation

$$\partial_t \hat{\rho} = -\mathrm{i}[\hat{H}, \hat{\rho}] + \frac{1}{4}\sigma^2 \left[\hat{L}\hat{\rho}\hat{L} - \frac{1}{2}\left(\hat{L}^2 \rho + \hat{\rho}\hat{L}^2 \right) \right] \tag{1}$$

is the stochastic unravelling problem.

A well-known solution is given by the stochastic Schrödinger equation

$$\mathrm{d}|\psi_t\rangle = -\mathrm{i}\hat{H}\,|\psi_t\rangle\,\mathrm{d}t + \tfrac{1}{2}\sigma\big(\hat{L} - \langle\hat{L}\rangle_t\big)\,|\psi_t\rangle\,\mathrm{d}W_t - \tfrac{1}{8}\sigma^2\big(\hat{L} - \langle\hat{L}\rangle_t\big)^2|\psi_t\rangle\,\mathrm{d}t, \tag{2}$$

where

$$\langle\hat{L}\rangle_t = \langle\psi_t|\hat{L}|\psi_t\rangle, \tag{3}$$

and $\{W_t\}$ denotes a Brownian motion.

The solution to the stochastic Schrödinger equation (2) can be found by considering an underlying signal detection problem.

A signal-detection problem

Consider the following problem in signal processing.

We have an unknown quantity of interest represented by the random variable L, that takes the discrete values $\{l_i\}$ with the probabilities $\{p_i\}$.

The true value of L is unknown to the observer, who merely receives information about the value of the signal L that is obscured by noise.

Assume that noise is additive and is modelled by a Brownian motion $\{B_t\}$, and that the signal is revealed in time at a constant rate σ.

Then the noisy observation of the signal is characterised by the information process

$$\xi_t = \sigma L t + B_t. \tag{4}$$

The best estimate of L that minimises the quadratic error, given the observed time series $\{\xi_s\}_{s\leq t}$ up to time t, is given by the conditional expectation

$$\langle L\rangle_t = \sum_i l_i \, \mathbb{P}\left(L = l_i | \{\xi_s\}_{s\leq t}\right). \tag{5}$$

Writing $\pi_{it} = \mathbb{P}(L = l_i|\{\xi_s\}_{s\leq t})$, we have

$$\pi_{it} = \frac{\mathbb{P}(L = l_i)\, \rho(\xi_t|L = l_i)}{\sum_j \mathbb{P}(L = l_j)\, \rho(\xi_t|L = l_j)}, \tag{6}$$

where $\mathbb{P}(L = l_j) = p_i$ and

$$\rho(\xi|L = l_i) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(\xi - \sigma l_i t)^2}{2t}\right). \tag{7}$$

Therefore, we deduce that

$$\pi_{it} = \frac{p_i \, \exp\left(\sigma l_i \, \xi_t - \frac{1}{2}\sigma^2 l_i^2 t\right)}{\sum_j p_j \, \exp\left(\sigma l_j \, \xi_t - \frac{1}{2}\sigma^2 l_j^2 t\right)}. \tag{8}$$

Taking the stochastic differential of the conditional probability process, we find

$$\mathrm{d}\pi_{it} = \sigma\pi_{it}\big(l_i - \langle L\rangle_t\big)\left(\mathrm{d}\xi_t - \sigma\langle L\rangle_t\,\mathrm{d}t\right). \tag{9}$$

Introducing the process $\{W_t\}$ according to

$$W_t = \xi_t - \sigma\int_0^t \langle L\rangle_s\,\mathrm{d}s, \tag{10}$$

we can show that $\{W_t\}$ thus defined is a standard Brownian motion.

In signal detection, $\{W_t\}$ is called the innovations process, and it reveals the arrival of new information.

Now if we define $\phi_{it} = \sqrt{\pi_{it}}$, then we find

$$\mathrm{d}\phi_{it} = \tfrac{1}{2}\sigma\big(l_i - \langle L\rangle_t\big)\,\phi_{it}\,\mathrm{d}W_t - \tfrac{1}{8}\sigma^2\big(l_i - \langle L\rangle_t\big)^2\phi_{it}\,\mathrm{d}t. \tag{11}$$

Signal processing quantum dynamics

With this expression in mind, consider a Hilbert space $\mathcal{H}$ associated with a physical system, on which an operator $\hat{L}$ is defined, whose eigenvalues are $\{l_i\}$ and eigenstates are $\{|l_i\rangle\}$.

Assume that the Hamiltonian $\hat{H}$ of the system, whose eigenvalues are $\{E_i\}$, commutes with $\hat{L}$.

Let the initial state $|\psi_0\rangle$ of the system be pure, and let $p_i = |\langle l_i|\psi_0\rangle|^2$.

Let the process $|\psi_t\rangle$ be given by

$$|\psi_t\rangle = \sum_i \phi_{it}\, \mathrm{e}^{\mathrm{i}(\theta_i - E_i t)}\, |l_i\rangle, \tag{12}$$

where $\{\theta_i\}$ are arbitrary constant and ϕ_{it} satisfies (11).

Quantum dynamics for motion tracking

This feature of quantum dynamics resembles, at least at an intuitive level, the behaviour of biological systems.

With this in mind we attempt to model motions of biological systems – plants in this case – in response to changing environments using stochastic Schrödinger equations.

To characterise the tracking motion we consider the model:

$$\mathrm{d}\psi_t(x) = -\mathrm{i}\mu \hat{P}\,\psi_t(x)\,\mathrm{d}t + \tfrac{1}{2}\sigma\big(\hat{Q} - \langle\hat{Q}\rangle_t\big)\psi_t(x)\,\mathrm{d}W_t - \tfrac{1}{8}\sigma^2\big(\hat{Q} - \langle\hat{Q}\rangle_t\big)^2\psi_t(x)\,\mathrm{d}t. \tag{13}$$

The idea that we propose here therefore is to regard the squared wave function $\pi_t(x) = |\psi_t(x)|^2$ as representing the probability distribution of the location of the light source, as "perceived" by the plant.

Hence the integral $\int x\,\pi_t(x)\,\mathrm{d}x$ represents the best estimate of the location of the light source.

For simplicity let us assume that the initial state is a standard Gaussian state: $\psi_0(x) = (2\pi)^{-1/4}\exp(-\frac{1}{4}x^2)$.

The signal detection solution to (13) is then as follows.

We let X be a normal random variable with mean zero and variance one.

Consider the "signal" process $X_t = X + \mu t$.

The observation of the signal, however, is obscured by a Brownian noise, giving rise to the information process

$$\xi_t = \sigma \int_0^t X_s \, \mathrm{d}s + B_t. \tag{14}$$

Then we find that the full solution to (13) is given by

$$\psi_t(x) = \left(\frac{1+\sigma^2 t}{2\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{\left[(1+\sigma^2 t)x - \sigma\xi_t - \mu t(1+\frac{1}{2}\sigma^2 t)\right]^2}{4(1+\sigma^2 t)}\right), \tag{15}$$

where the innovations Brownian motion $\{W_t\}$ in (13) is related to the information process $\{\xi_t\}$ according to

$$W_t = \xi_t - \sigma \int_0^t \frac{\sigma\xi_s + \mu s(1+\frac{1}{2}\sigma^2 s)}{1+\sigma^2 s}\,\mathrm{d}s. \tag{16}$$

The best estimate for the position of the light source is

$$\int_{-\infty}^{\infty} x\,\pi_t(x)\,\mathrm{d}x = \frac{\sigma\xi_t + \mu t(1+\frac{1}{2}\sigma^2 t)}{1+\sigma^2 t}. \tag{17}$$

Quantifying the information extracted

We are interested in the question on how much information extraction is needed so as to deduce the location of the unknown moving target $\{X_t\}$.

To quantify the information extraction, we consider the reduction $S_0 - S_\tau$ of the Shannon-Wiener entropy

$$S_t = -\int \pi_t(x)\log \pi_t(x)\mathrm{d}x, \quad \pi_t(x) = |\psi_t(x)|^2. \tag{18}$$

Because the entropy process $\{S_t\}$ is stochastic, we are interested in the averaged entropy reduction $\Delta S = S_0 - \mathbb{E}[S_\tau]$ for the information gain.

In the present example, we find that

$$\Delta S = \frac{1}{2}\log 2, \tag{19}$$

and henace that the amount of information processed to track the motion is at most of order few bits.

Information erasure and heat production

It seems reasonable to assume that much of the information processed for plant movements are not stored in the plant indefinitely.

Information erasure is necessarily accompanied by energy consumption and hence heat production, resulting in an increase of environmental entropy.

The minimum amount of energy consumption required for the erasure of one bit of information at temperature T is $k_{\rm B}T \log 2$.

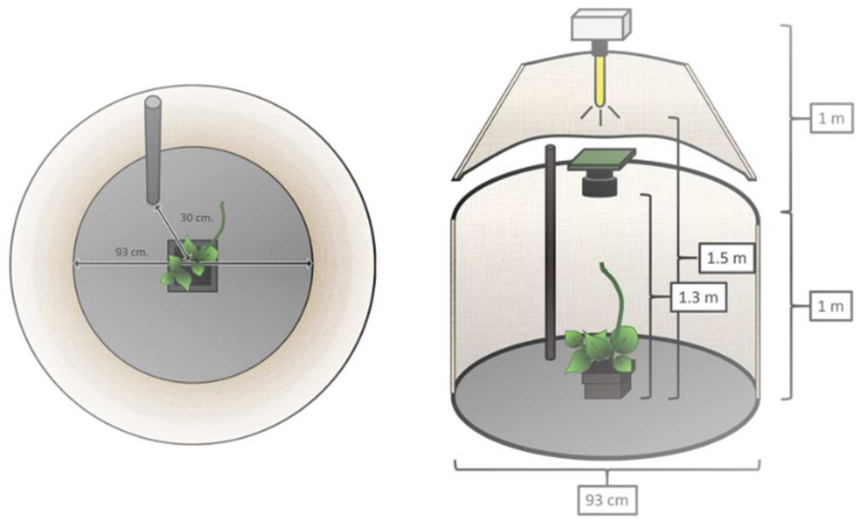
Based on our estimate, a daisy flower at summer-night temperature (say, 290 K) will thus consume at least of order 10^{-2} eV of energy per each information-encoding unit.

This in turn results in heat production.

To get a better intuition for the scale involved, consider the circumnutation of a common bean plant, assuming that each cell contains at most a single such unit.

To identify the location of an object within a $\pm 10^\circ$ angular window with 95% confidence, it must process between 15 and 20 bits of information.

If the object is removed, this information must be erased.



Raja, *et al.* *Scientific Reports* **10**, 19465 (2020)

If the plant can process information relatively close to the Landauer limit, say, 10^4 times the limit, then the cost of erasure is of order 10^4 eV; about 1% of the total energy consumed by the cell.

If information is processed significantly less efficiently, say at the level of our everyday computers, then the erasure cost becomes too high for survival.

Phase transition in stochastic Schrödinger equation

So far we have considered the case in which either $[\hat{H}, \hat{L}] = 0$ or $[\hat{H}, \hat{L}] \propto \mathbb{1}$.

In more general situations, the system can exhibit a phase transition, depending on the overall magnitudes of $\hat{H}$ and $\hat{L}$.

This is the transition associated with the reality of the eigenvalue structures of the Liouville operator – like a PT phase transition (Brody & Longstaff, PRR **1**, 033127, 2019).

In biological or ecological context, $\hat{L}$ represents adaptation, while $\hat{H}$ represents changing environmental conditions.

Thus, to model the survival of species or the breakdown of ecological systems, manifestly ‘quantum mechanical’ models are needed.

Second law in biology

It is tempting to conjecture that biological systems operate by extracting information from their environments, processing them, and arriving at the best estimate of the state of the environment for the purpose of adaptation.

This results in the gain of information, and thus reduction in entropy.

However, some, or much of the processed information is lost, resulting in increasing entropy.

The process of information erasure then must lie at the heart of ageing – or arrow of time – in biological systems.

Work based on:
Brody, D.C.
"Open quantum dynamics for plant motions"
Scientific Reports 12, 3042 (2022)