MASTER THESIS

Models of Universal Power-Law Distributions in Natural and Social Sciences

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Abstract

To date, many power-law distributions have been found in various fields of natural and social sciences. In particular, Zipf's law, the power-law distribution with the power-law exponent 1, is reported very frequently. This universality of Zipf's law, however, has not been well explained theoretically. Until now, we have only models specific to each problem.

In the present thesis, we first introduce a simple and generic model that reproduces Zipf's law. We can regard this model both as the time evolution of the population of cities and that of the asset distribution. We show that our model is very robust against various variations.

Next, we explain theoretically why our model reproduces Zipf's law. By considering the time-evolution equation of our model, we see that the essence of Zipf's law is an asymmetric random walk in a logarithmic scale.

Finally, we extend our model by introducing an additional asymmetry. We show that the extended model reproduces various power-law exponents. By extending the theoretical argument for Zipf's law, we find a simple equation of the power-law exponent.

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1 Introduction

Zipf's law is the observation that the frequency n of the occurrence of various events is an inverse power-law function $n \propto R^{-1}$, where the rank R is determined by the descending order of the frequency of each event. Zipf [1] first made this observation for the frequency of occurrence of English words in literature; that is, the most frequent word "the" (rank $R = 1$) appears twice as many times as the second most frequent word "of"(rank $R = 2$). Since then, Zipf's law has been found in various fields, including the population of cities [2] and the asset distribution of companies [3, 4]. This universality of Zipf's law, however, has not been well explained theoretically. To date, we have only models specific to each problem [2]. The purpose of the present thesis is to explain the mechanism of Zipf's law and thereby its universality.

In various fields of natural and social sciences, there are also many phenomena that exhibit power laws with other exponents, $n \propto R^{-\kappa}$ with $\kappa \neq 1$, including internet [5], traffic flow [6], economics [7, 8, 9] and family names [10]. We explain the mechanism of these power laws as well by extending the argument for Zipf's law.

In section 2, we introduce a simple and generic model that reproduces Zipf's law. We can regard this model both as the time evolution of the population of cities and that of the asset distribution. In section 3, we explain theoretically why this model reproduces the power law. In section 4, we extend our model so as to reproduce power laws with other power-law exponents. Our explanation shows that the power-law of our model is very robust.

Before introducing our model, let us make a tutorial remark on the relation among three types of distribution appearing in the present thesis: the ranked distribution, the cumulative distribution function and the probability distribution function. The ranked distribution, including Zipf's law, is a cumulative distribution function with the horizontal and vertical axes being flipped. A ranked distribution

$$n \propto R^{-\kappa} \tag{1}$$

means that there are, say, R cities whose population is greater than or equal to n. In other words, the probability that the population of an arbitrarily chosen city is greater than or equal to n is given by

$$P(\geq n) = \frac{R}{N}, \tag{2}$$

where N is the total number of cities. Changing the variable R to n on the right-hand side, we have

$$P(\geq n) \propto n^{-1/\kappa} = n^{-b}, \tag{3}$$

where b is the power-law exponent of the cumulative distribution function $P(\geq n)$. Thus we obtain the relation between the power-law exponents of the cumulative distribution function and the ranked distribution.

Next, the cumulative distribution function $P(\geq x)$ is related to the probability distribution function $p(x)$ in the form

$$P(\geq x) = \int_x^\infty p(x)dx. \tag{4}$$

This means that

$$p(x) \propto x^{-a} \Leftrightarrow P(\geq x) \propto x^{-(a-1)} = x^{-b}. \tag{5}$$

where a is the power-law exponent of the probability distribution function.

Consequently we arrive at the relation among the three power-law exponents as follows:

$$\frac{1}{\kappa} = b = a - 1. \tag{6}$$

In the following sections, we introduce models that reproduce power laws in the form of the cumulative distribution function $P(\geq x) \propto x^{-b}$. It corresponds to the power laws of the probability distribution function and the ranked distribution in the forms $p(x) \propto x^{-(b+1)}$ and $n \propto R^{-1/b}$, respectively. In particular, the case $b = 1$ is equivalent to Zipf's law in the form of the ranked distribution $n \propto R^{-1}$.

In the present thesis, we claim that the origin of the power laws is diffusion in the logarithmic scale. For this reason, we also make a remark on the variable transformation of Eq. (5). To see Eq. (5) in the logarithmic scale, we change the variable to be $\xi = \log x$. The probability distribution function $p(x)$ is transformed to

$$p(x)dx \sim x^{-a}dx = e^{-a\xi}e^{\xi}d\xi \equiv \tilde{p}(\xi)d\xi. \tag{7}$$

In other words, we define the probability distribution function with respect to ξ as

$$\tilde{p}(\xi) = e^{-(a-1)\xi} = e^{-b\xi}. \tag{8}$$

It shows that the exponent of the exponential law of the probability distribution function in the logarithmic scale coincides with the power-law exponent of the cumulative distribution function.

2 Model and simulation results

In this section we introduce our model and show the results of its simulation. The model that we introduce here evolves as follows: First, we consider a set of positive values $\{x_i\}$ with N entities. Then we repeat the following procedures T times:

1. Choose an entity i randomly from $1 \leq i \leq N$.

2. Add or subtract randomly the amount αx_i from the chosen entity with the probability 1/2:

$$x_i \to x_i \pm \alpha x_i, \tag{9}$$

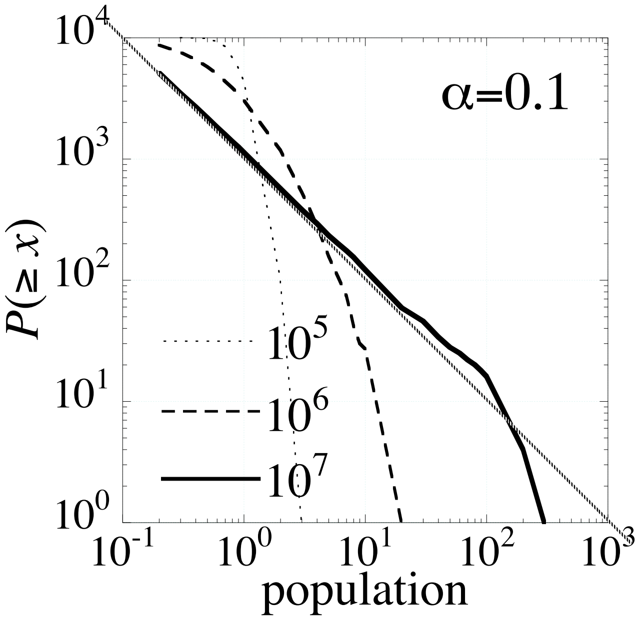
where α is a constant with $0 < \alpha < 1$. For simplicity, we assume that the initial values of $\{x_i\}$ are all equal. However, this initial condition is irrelevant in the long-time limit $T \to \infty$.

If we regard the values $\{x_i\}$ as the population of N cities, an explanation of this model from the viewpoint of the population distribution is that bigger cities tend to have greater population movement. During the time evolution we require the boundary condition that x_i should not be less than a lower bound $x_{\text{l.b.}}(> 0)$; When an operation such as $x_i \to x_i - \alpha x_i < x_{\text{l.b.}}$ is chosen in the step 2, this operation is cancelled. This boundary condition is required in order to prevent each entity x_i from decreasing limitlessly. This is quite a plausible requirement; In realities, the population of cities, for example, cannot be less than $x_{\text{l.b.}} = 1$.

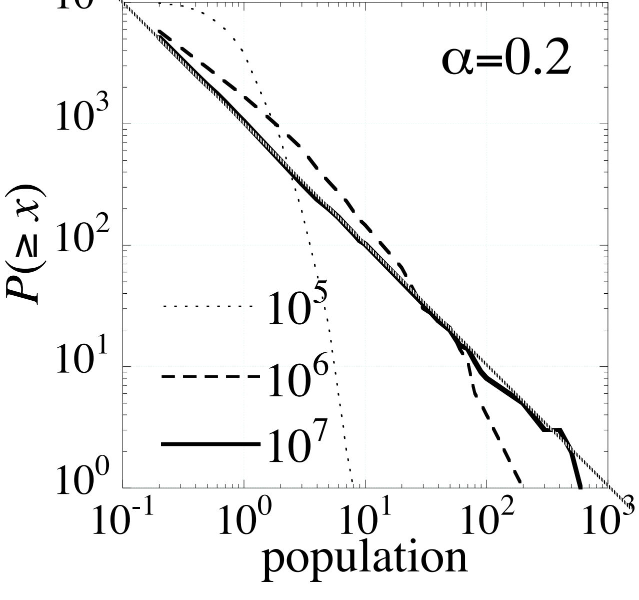
Figure 1 shows results of the simulation of the above procedures for various values of α in the form of the cumulative distribution function $P(\geq x) \propto x^{-b}$. (Here the number of the entities is $N = 10^4$.) We can clearly see that the distribution converges to power laws in all cases in the long-time limit $T \to \infty$. Figure 2 shows the power-law exponents estimated from the results in Fig. 1. (We estimated the exponents by the least-squares method.) The exponent b is obviously close to unity for any values of α. We note that the parameter α is related only to the rapidity of the convergence to Zipf's law. Because of the relation (6), we obtain Zipf's law in the form of the ranked distribution $n \propto R^{-1}$ by flipping the horizontal and vertical axes of Figure 1; The model thus reproduces the universality of Zipf's law as $T \to \infty$.

To show the robustness of the model, we modify it from the viewpoint of the asset distribution and still reproduce Zipf's law. Assume that a company tends to trade with a company of a similar size. Then we repeat the following procedures T times:

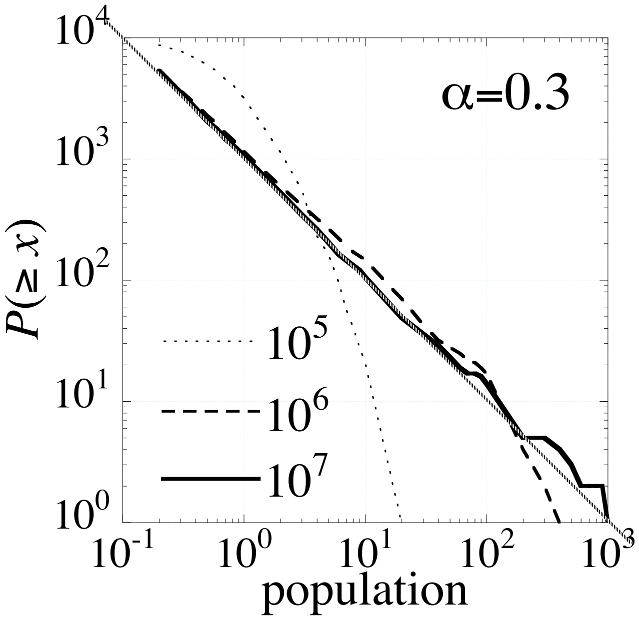
1. Choose an entity i randomly from $1 \leq i \leq N$.

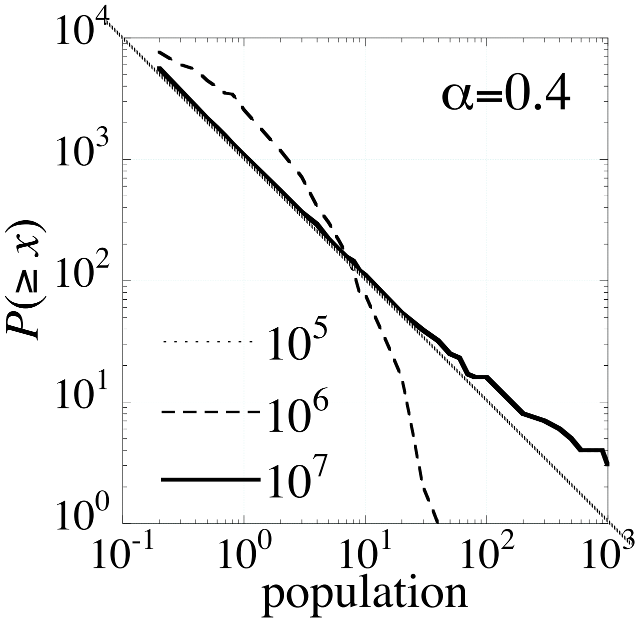


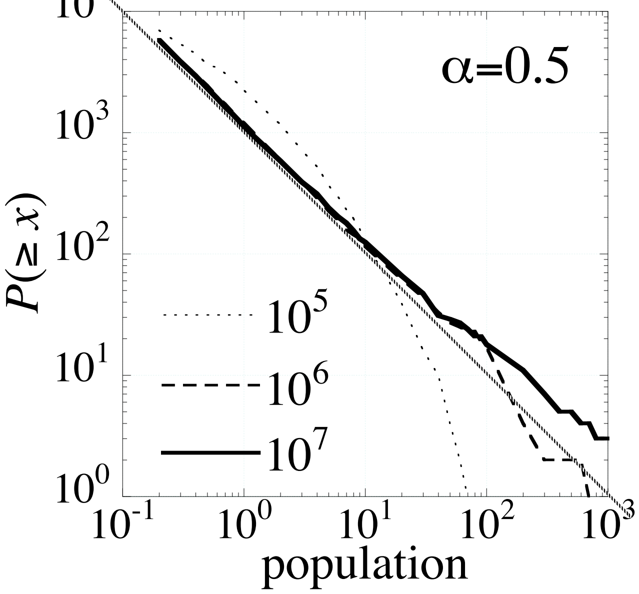
α=0.1
$P(\geq x)$
population
10^5
10^6
10^7

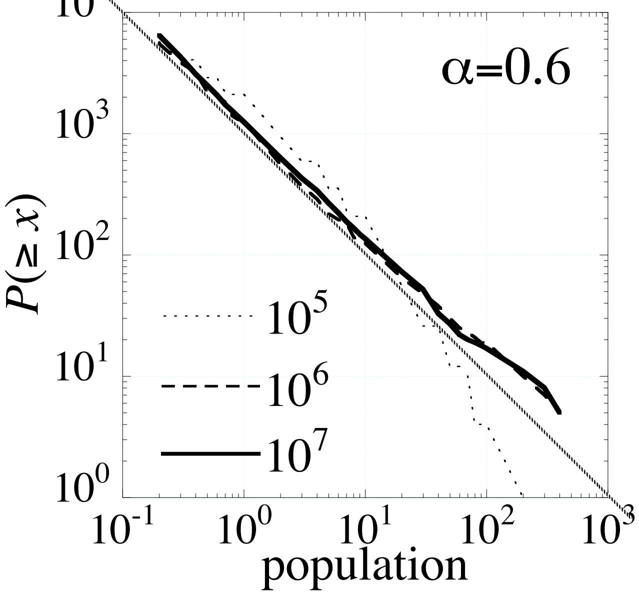


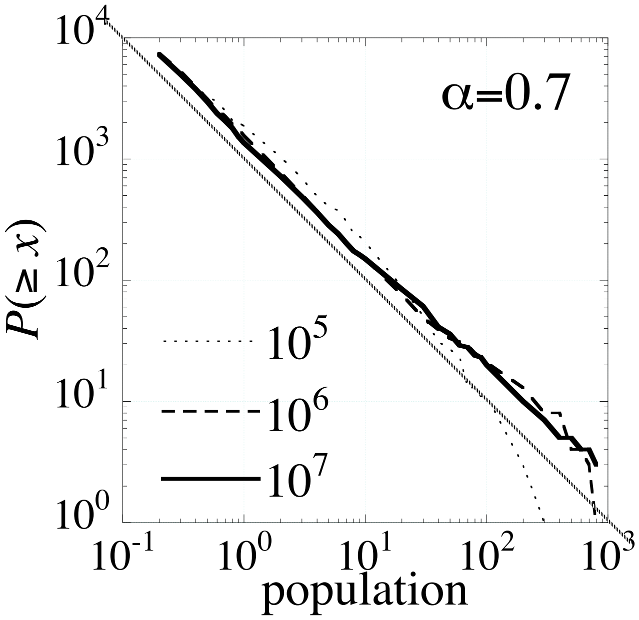
α=0.2
$P(\geq x)$
population
10^5
10^6
10^7


α=0.3
P(≥ x)
population
10^5
10^6
10^7

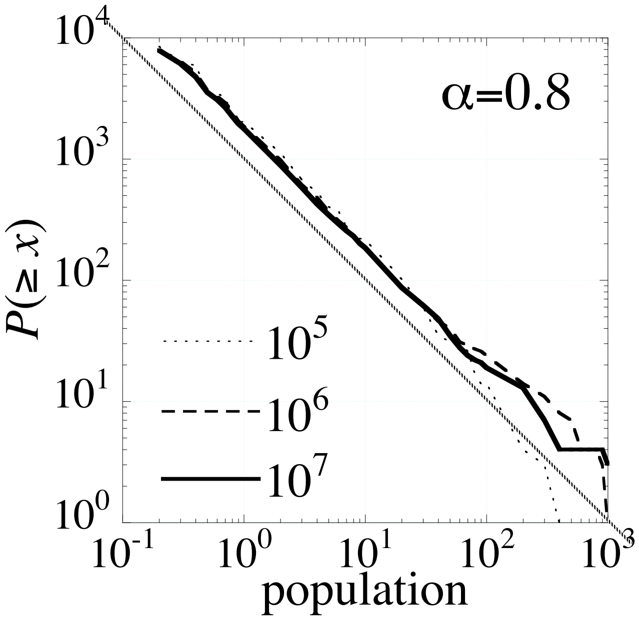

α=0.4
P(≥ x)
population
10^5
10^6
10^7


α=0.5
$P(\geq x)$
population
10^5
10^6
10^7


α=0.6
$P(\geq x)$
population
10^5
10^6
10^7



α=0.7
$P(\geq x)$
10^5
10^6
10^7
population



α=0.8
$P(\geq x)$
10^5
10^6
10^7
population

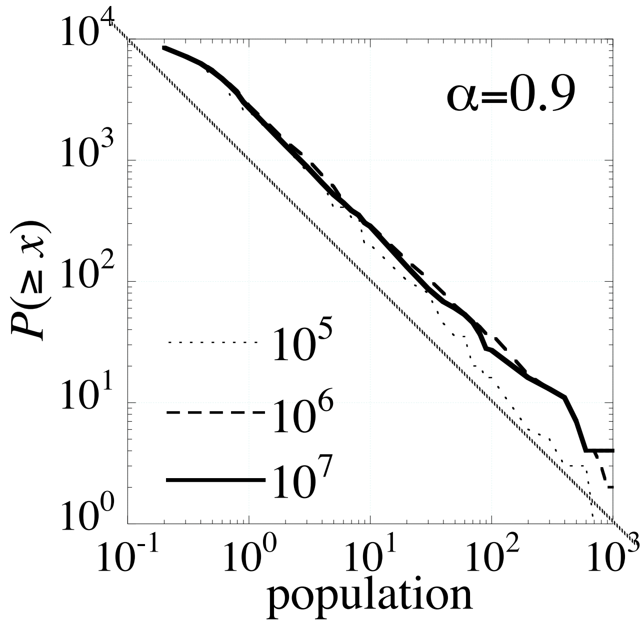


Figure 1: The results of the simulation for various values of α. The number of steps is $T = 10^5$ (dotted line), 10^6 (dashed line) and 10^7 (solid line). The straight line indicates the power law with the exponent $b = 1$, namely Zipf's law.

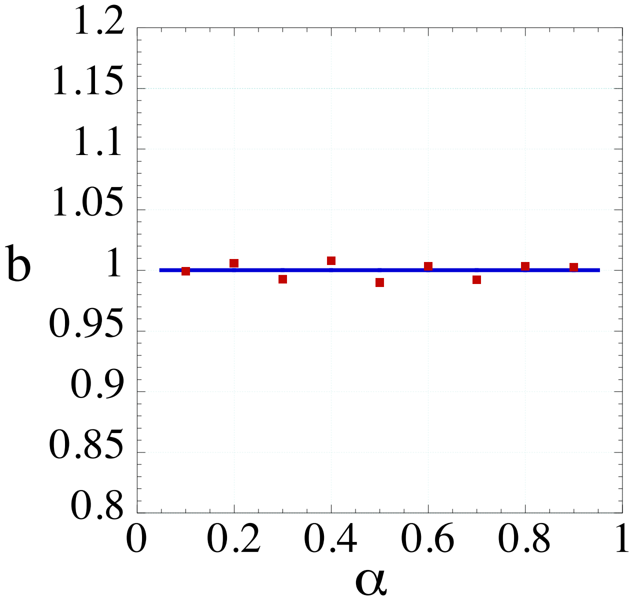


Figure 2: The power-law exponent b against the parameter α. The straight line shows the solution Eq. (31), while the squares indicate estimates from the results in Fig. 1.

2. For $i > 1$, we move the amount αx_{i-1} from the $(i-1)$th entity to the ith entity, where α is a constant parameter with $0 < \alpha < 1$. In other words, the following takes place:

$$x_{i-1} \rightarrow x_{i-1} - \alpha x_{i-1}, \tag{10}$$

$$x_i \rightarrow x_i + \alpha x_{i-1}. \tag{11}$$

   For $i = 1$, we increase all values x_i by the same amount $x_1\alpha/N$.

3. Rearrange the entities in the descending order of the values x_i.

An intuitive explanation of this modified model is that money flows from one company to a smaller company and the smaller company delivers products to the larger company. Figure 3 shows the results of the simulation of the above procedures for various values of α. (Here the number of the companies is $N = 10^4$.) The cumulative distribution function again converges to $P(\geq x) \propto x^{-1}$ in all cases. The operation for $i = 1$ in the step 2 corresponds to the boundary condition $x_i \geq x_{\text{l.b.}}$ of the population-distribution model.

The value of the lower bound $x_{\text{l.b.}}$ does not need to be fixed. We modify the population-distribution model and make $x_{\text{l.b.}}$ a changeable parameter. When the operation $x_i \rightarrow x_i - \alpha x_i < x_{\text{l.b.}}$ is chosen in the step 2 of the first model, we produce a random number S $(0 < S < 1)$. For

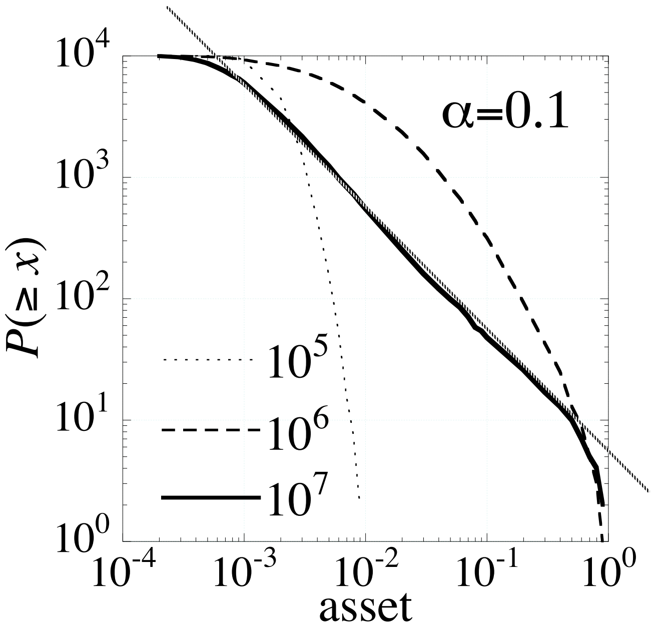
$$e^{-(x_{\text{l.b.}} - x_i)} > S, \tag{12}$$

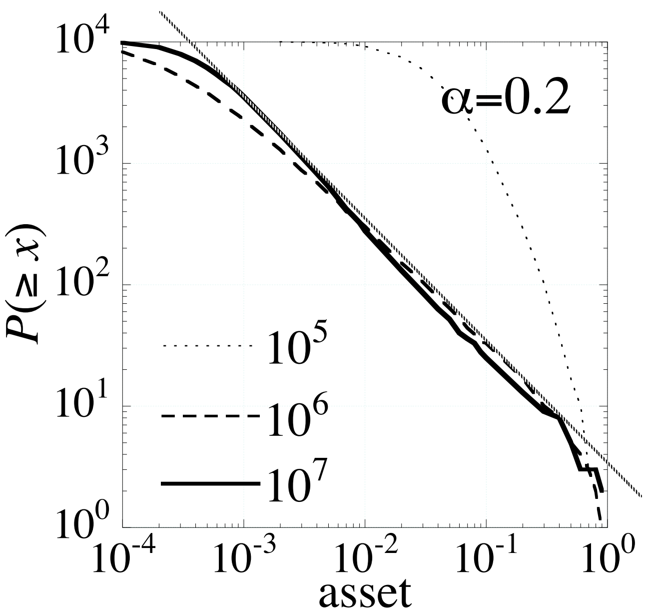
this operation is cancelled, but for

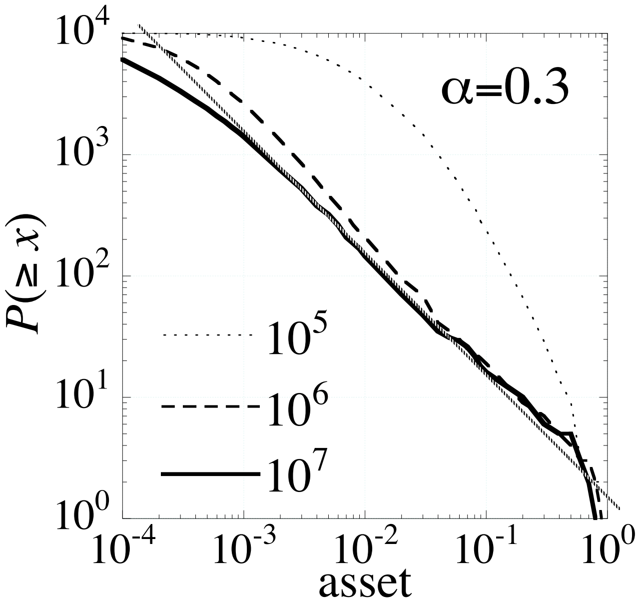
$$e^{-(x_{\text{l.b.}} - x_i)} < S, \tag{13}$$

x_i becomes the new value of $x_{\text{l.b.}}$. A broad change that exceeds the lower bound $x_{\text{l.b.}}$ is allowed with a small probability. Figure 4 shows the results of the modified model for $\alpha = 0.3$ and $N = 10^4$. Compare it with Fig. 1; we can see the distribution in Fig. 4 behaves in the form of the Zipf's law, while shifting to the negative direction.

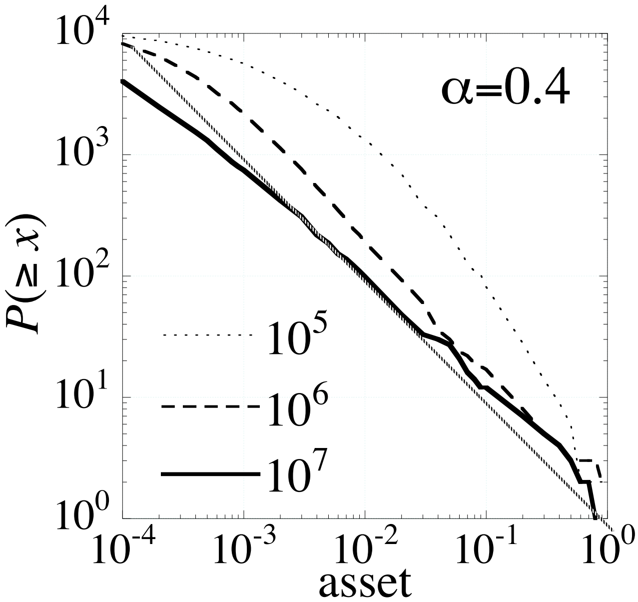
The value α does not need to be fixed as well. We can modify the population-distribution model and make α as a changeable parameter; We choose α randomly from $0 < \alpha < 1$ every time between the steps 1 and 2 in the population-distribution model. In other words, the value of α changes at each time step. Figure 5 shows the results of the latter modified model for $N = 10^4$. Compare it with Fig. 1; we can see the distribution in Fig. 5 behaves in the form of the Zipf's law. The results in Figs. 4 and 5 show that Zipf's law of our model holds out against a wide variety of modification.


α=0.1
10^4
10^3
10^2
10^1
10^0
$P(\geq x)$
10^5
10^6
10^7
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
asset

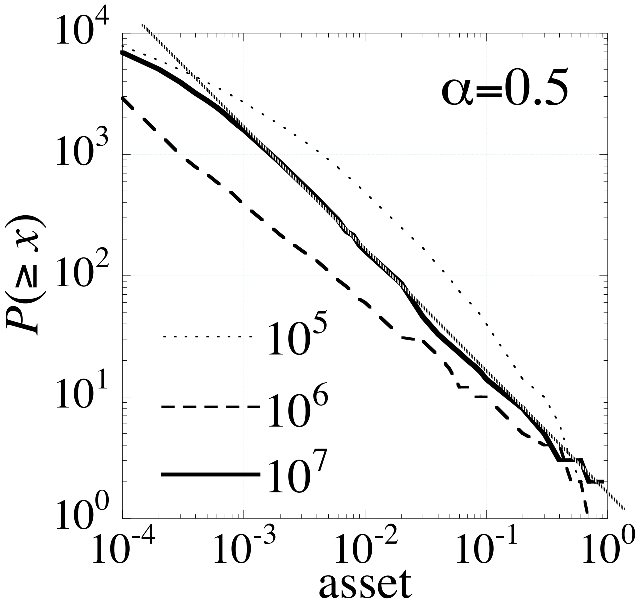

α=0.2
10^4
10^3
10^2
10^1
10^0
$P(\geq x)$
10^5
10^6
10^7
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
asset



α=0.3
10^4
10^3
10^2
10^1
10^0
$P(\geq x)$
10^5
10^6
10^7
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
asset



α=0.4
10^4
10^3
10^2
10^1
10^0
$P(\geq x)$
10^5
10^6
10^7
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
asset


α=0.5
$P(\geq x)$
asset
10^5
10^6
10^7
10^0
10^1
10^2
10^3
10^4
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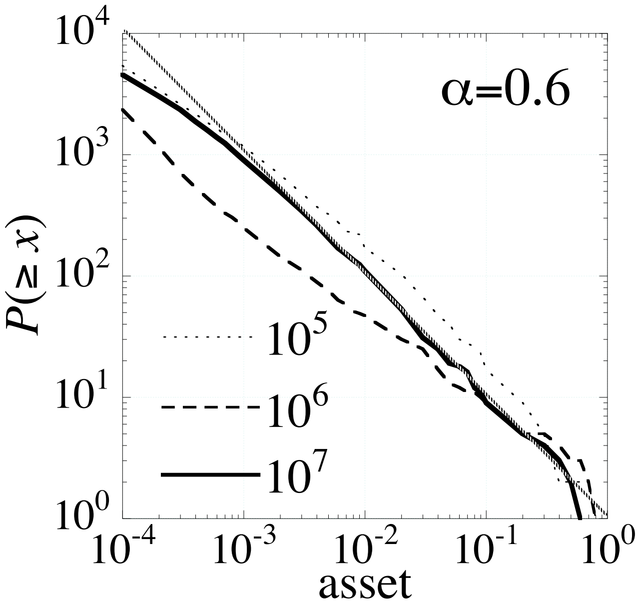
10^{-3}
10^{-2}
10^{-1}
10^0


α=0.6
$P(\geq x)$
asset
10^5
10^6
10^7
10^0
10^1
10^2
10^3
10^4
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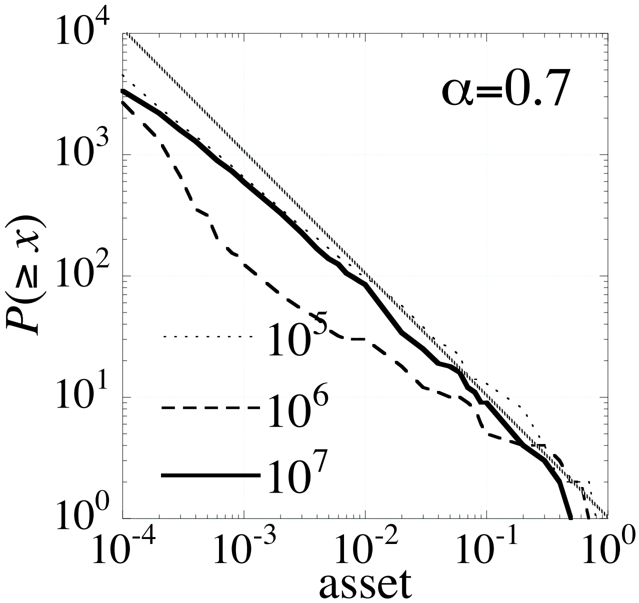
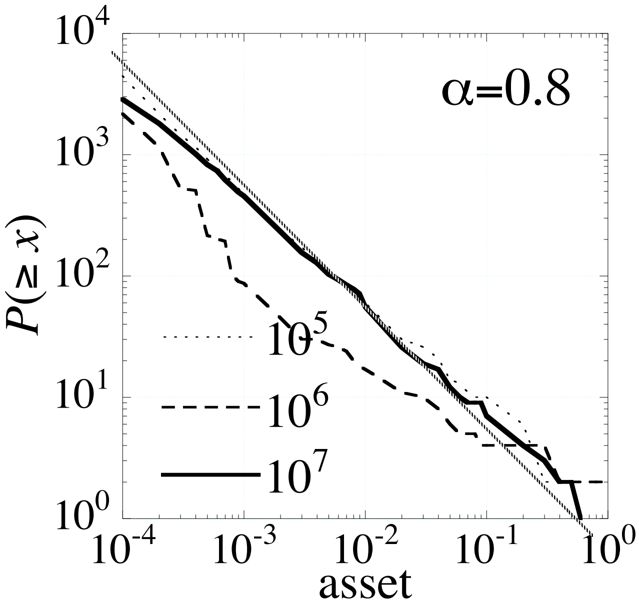
10^{-2}
10^{-1}
10^0



α=0.7
$P(\geq x)$
asset
10^5
10^6
10^7



α=0.8
$P(\geq x)$
asset
10^5
10^6
10^7

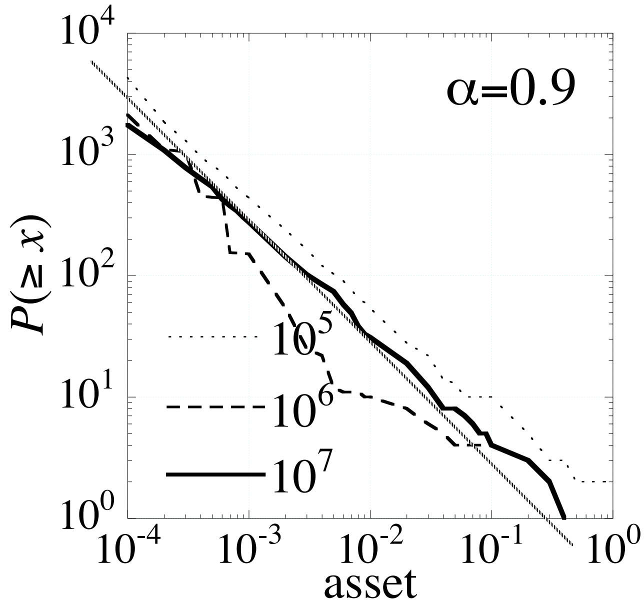


Figure 3: The results of the simulation of the asset distribution for various values of α. The number of steps is $T = 10^5$ (dotted line), 10^6 (dashed line) and 10^7 (solid line). We normalized the results by the asset of the highest entity. The straight line indicates the power law with the exponent $b = 1$, namely Zipf's law.

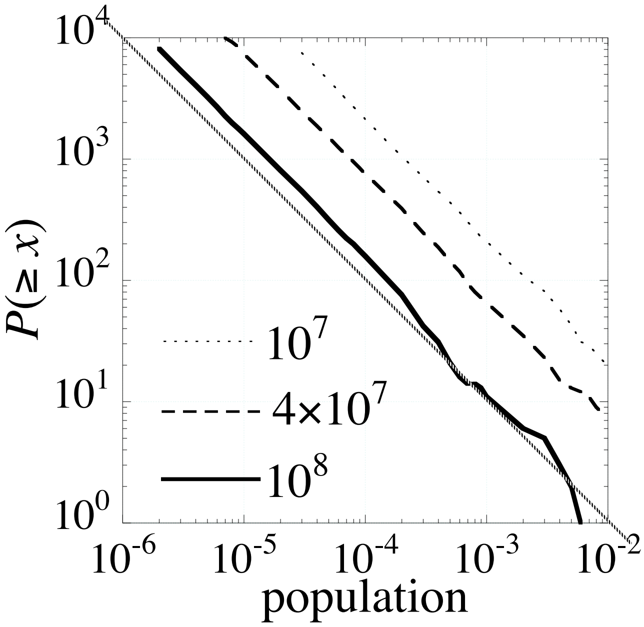


Figure 4: The results of the simulation for $\alpha = 0.3$ with a broad lower bound. The number of steps is $T = 10^7$ (dotted line), 4×10^7 (dashed line) and 10^8 (solid line). The straight line indicates the power law with the exponent $b = 1$, namely Zipf's law.

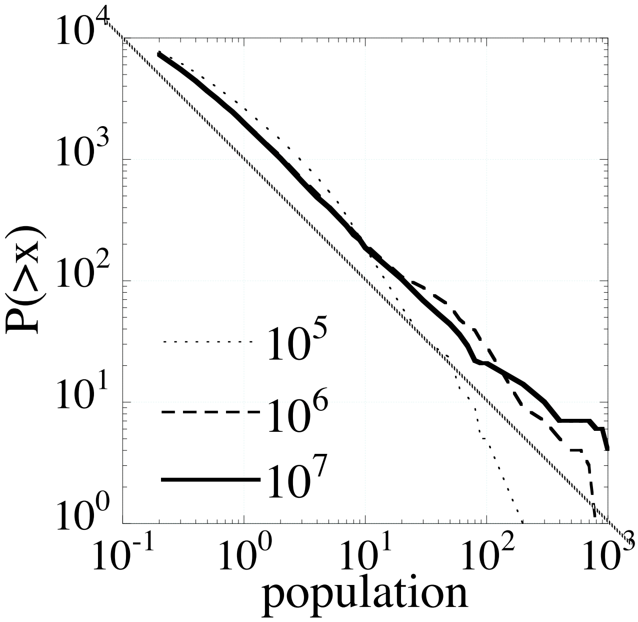


Figure 5: The results of the simulation with randomly chosen α ($0 < \alpha < 1$). The number of steps is $T = 10^5$ (dotted line), 10^6 (dashed line) and 10^7 (solid line). The straight line indicates the power law with the exponent $b = 1$, namely Zipf's law.

We see from the above variations that the essence of the model is the increase and the decrease proportional to the size of each entity; this is the key to the universality of Zipf's law. On the basis of the simpleness and the robustness of our model, we believe that Zipf's law in many fields of science has the same origin of "proportional change."

3 Continuum Limit and Universality

In this section we give a theoretical explanation as to why the model in the previous section reproduces Zipf's law. Let us consider the limit $N \to \infty$. Hence we regard the set of variables $\{x_i\}$ as a continuous variable x. We consider in this limit the time evolution of the probability distribution function $p(x)$. We note that Eq. (9) gives probability flows in the directions

$$x \to x \pm \alpha x = (1 \pm \alpha)x. \tag{14}$$

The time-evolution equation of the probability distribution function $p(x)$ is therefore given by the following:

$$\frac{\partial}{\partial t} p(x,t) = -\gamma p(x,t) + \frac{1}{2}\frac{\gamma}{1-\alpha} p\left(\frac{x}{1-\alpha}, t\right) + \frac{1}{2}\frac{\gamma}{1+\alpha} p\left(\frac{x}{1+\alpha}, t\right), \tag{15}$$

where $p(x,t)$ is the probability distribution of x at time t and γ is a constant determining the unit of time. The first term on the right-hand side represents a flow from the point x to the points $(1 \pm \alpha)x$, while the second and third terms represents flows from $x/(1 \pm \alpha)$ into x.

The coefficients $1/(1 \pm \alpha)$ in front of p in Eq. (15) are necessary in order to satisfy the probability conservation. To see this, we integrate Eq. (15) over x as

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^\infty p(x,t)dx &= -\gamma \int_0^\infty p(x,t)dx + \frac{1}{2}\frac{\gamma}{1-\alpha} \int_0^\infty p\left(\frac{x}{1-\alpha}, t\right) dx \\ &+ \frac{1}{2}\frac{\gamma}{1+\alpha} \int_0^\infty p\left(\frac{x}{1+\alpha}, t\right) dx. \end{aligned} \tag{16}$$

Changing the variables to be

$$x' = \frac{x}{1-\alpha}, \qquad x'' = \frac{x}{1+\alpha}, \tag{17}$$

we can rewrite Eq. (16) as

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^\infty p(x,t)dx &= -\gamma \int_0^\infty p(x,t)dx + \frac{\gamma}{2} \int_0^\infty p(x',t)dx' \\ &+ \frac{\gamma}{2} \int_0^\infty p(x'',t)dx'' = 0. \end{aligned}$$

Thus the total probability is conserved. (In the present case, the probability would be conserved without the coefficients $1/(1 \pm \alpha)$: However, they are necessary when Eq. (15) is generalized later; see Eq. (35).)

Let us see Eq. (15) in the logarithmic scale; we change the variable to be $\xi = \log x$. The probability distribution function is transformed to

$$p(x,t)dx = p(e^\xi, t)e^\xi d\xi = \tilde{p}(\xi, t)d\xi. \tag{18}$$

In other words, we define

$$\tilde{p}(\xi, t) = p(x,t)x = p(e^\xi, t)e^\xi. \tag{19}$$

By rewriting p in terms of $\tilde{p}$ as

$$\begin{aligned} p(x,t) &= e^{-\xi}\tilde{p}(\xi,t) = \frac{1}{x}\tilde{p}(\log x, t), & (20)\\ p\left(\frac{x}{1 \mp \alpha}, t\right) &= \frac{1 \mp \alpha}{x}\tilde{p}\left(\log \frac{x}{1 \mp \alpha}, t\right) \\ &= \frac{1 \mp \alpha}{x}\tilde{p}(\xi - \log(1 \mp \alpha), t), & (21) \end{aligned}$$

we transform the evolution equation (15) to

$$\frac{\partial}{\partial t}\tilde{p}(\xi, t) = -\gamma\tilde{p}(\xi, t) + \frac{\gamma}{2}\tilde{p}(\xi + \beta_-, t) + \frac{\gamma}{2}\tilde{p}(\xi - \beta_+, t), \tag{22}$$

where

$$\beta_\pm \equiv \pm\log(1 \pm \alpha) > 0. \tag{23}$$

We can regard the evolution equation (22) as a random walk from ξ to $\xi \pm \beta_\pm$, that is, a random walk with asymmetrically fixed step sizes.

Let us consider the meaning of the asymmetric random walk. We carry out the Taylor expansion of the second and third terms on the right-hand side of Eq. (22). Assuming $O(\beta_\pm^3) \simeq 0$, we can rewrite Eq. (22) as

$$\begin{aligned} \tfrac{\partial}{\partial t}\tilde{p}(\xi,t) = -2\gamma\tilde{p}(\xi,t) \quad &+\gamma\left(\tilde{p}(\xi,t) + \tfrac{\partial}{\partial \xi}\tilde{p}(\xi,t)\beta_- + \tfrac{\partial^2}{\partial \xi^2}\tilde{p}(\xi,t)\tfrac{\beta_-^2}{2}\right) \\ &+\gamma\left(\tilde{p}(\xi,t) - \tfrac{\partial}{\partial \xi}\tilde{p}(\xi,t)\beta_+ + \tfrac{\partial^2}{\partial \xi^2}\tilde{p}(\xi,t)\tfrac{\beta_+^2}{2}\right). \end{aligned} \tag{24}$$

By rearranging the right-hand side of Eq.(24), we have

$$\frac{\partial}{\partial t}\tilde{p}(\xi,t) = \gamma\left(\frac{\beta_+^2}{2} + \frac{\beta_-^2}{2}\right)\frac{\partial^2}{\partial \xi^2}\tilde{p}(\xi,t) + \gamma(-\beta_+ + \beta_-)\frac{\partial}{\partial \xi}\tilde{p}(\xi,t). \tag{25}$$

The first and second terms on the right-hand side of Eq. (25) correspond to the diffusion and advection terms, respectively. Consequently we can regard the equation of the asymmetric random walk, Eq. (22), as an advection-diffusion equation (25) approximately. For $0 < \alpha < 1$, the advection coefficient is

$$-\beta_+ + \beta_- = -\log(1+\alpha) - \log(1-\alpha) = \log\frac{1}{1-\alpha^2} > 0. \tag{26}$$

It suggests that the probability distribution function in the logarithmic scale $\tilde{p}(\xi, t)$ moves to the negative direction while diffusing.

Let us now concentrate on finding stationary solutions $\tilde{p}_s(\xi)$ of Eq. (22):

$$-2\gamma\tilde{p}_s(\xi) + \gamma\tilde{p}_s(\xi+\beta_-) + \gamma\tilde{p}_s(\xi-\beta_+) = 0. \tag{27}$$

We assume a solution of the form

$$\tilde{p}_s \propto e^{-b\xi}. \tag{28}$$

(See Appendix for the validity of this assumption). By plugging Eq. (28) into Eq. (27), we have

$$-2\gamma e^{-b\xi} + \gamma e^{-b\xi}e^{-b\beta_-} + \gamma e^{-b\xi}e^{b\beta_+} = \gamma e^{-b\xi}(-2 + e^{-b\beta_-} + e^{b\beta_+}) = 0, \tag{29}$$

or

$$e^{b\beta_+} + e^{-b\beta_-} = 2. \tag{30}$$

Rewriting Eq. (30) using Eq. (23), we obtain the following equation for b:

$$f(b) \equiv (1+\alpha)^b + (1-\alpha)^b = 2. \tag{31}$$

Figure 6 shows $f(b)$ for various values of α. Though the form of $f(b)$ changes with α, the two solutions of Eq. (31), $b = 0$ and 1, are stable. Obviously $b = 0$ is a solution of Eq. (31). The solution $b = 0$ means that the stationary solution $\tilde{p}_s$ is a uniform distribution:

$$\tilde{p}_s \propto e^{-b\xi} = \text{const.} \tag{32}$$

As we suggested in Eq. (25), our model can be regarded as diffusion with advection in the logarithmic scale. We hence expect that the solution $b = 0$, or a uniform distribution appears only in the case of periodic boundary conditions. For our model with the lower bound $x_{\text{l.b.}}$, the solution $b = 1$ survives:

$$\tilde{p}_s \propto e^{-\xi} \tag{33}$$

According to the argument at the end of Sec. 1, the solution (33) means $P(\geq x) \propto x^{-1}$. This is in good agreement with our simulation data in Fig. 1. The present argument indicates that a random walk with asymmetric step sizes in the logarithmic scale is the essence of the universality of Zipf's law.

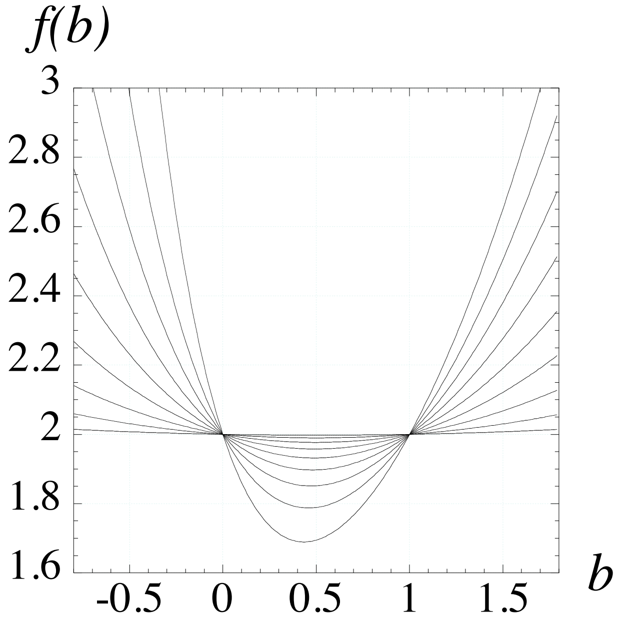


Figure 6: The function $f(b)$ defined in Eq. (31) for various values of α. The curvature of $f(b)$ increases as α becomes larger.

4 Extension of the model

So far, we have focused on Zipf's law ($b = 1$). However, various power-law exponents ($b \neq 1$) have been reported in many phenomena including economics [7, 8, 9] and family names [10]. In this section, we extend our model in order to reproduce power-law distributions with other exponents.

The extension is basically modification of the asymmetry of the random walk. We introduce two types of extension. In both extensions, the initial and the boundary conditions are the same as in the model in Sec. 2. Extension A of the model evolves as follows:

1. Choose an entity i randomly from $1 \leq i \leq N$.

2. Add the amount $\alpha_+ x_i$ or subtract the amount $\alpha_- x_i$ randomly from the chosen entity with the probability 1/2.

$$x_i \to x_i \pm \alpha_\pm x_i. \tag{34}$$

Here the parameters α_+ and α_- are made different; α_+ is the growth parameter and α_- is the anti-growth parameter. We restrict ourselves to the case $0 < \alpha_+ < \alpha_- < 1$; otherwise, we can have a meaningless solution $b < 0$ (See Eq. (38) below).

Figure 7 shows results of the simulation of the above procedures for various values of α and $N = 10^5$. We can still see the power-law distribution, but the power-law exponents b changed from unity. The power-law exponent b takes various values in the modified model while it was fixed to $b = 1$ for any values of α in Sec. 2.

We can easily modify the argument in Sec. 3 so as to analyze the above modified model, Extension A. Instead of Eq. (15) we now write the time-evolution equation of $p(x)$ as

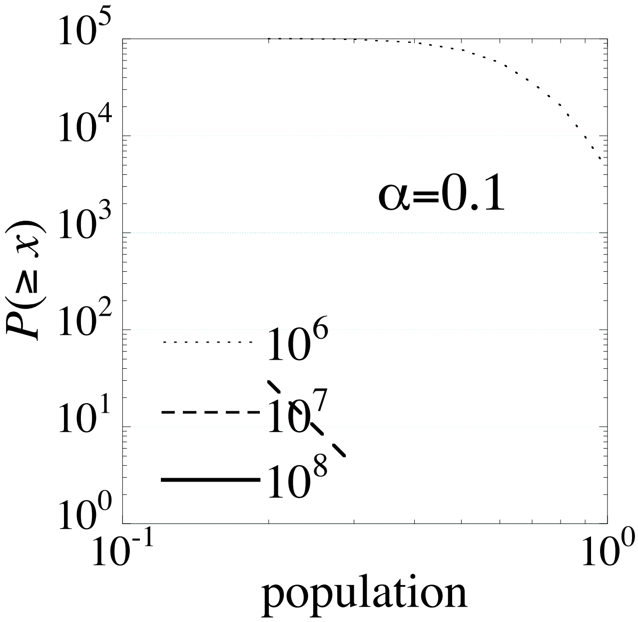
$$\frac{\partial}{\partial t} p(x,t) = -\gamma p(x,t) + \frac{1}{2}\frac{\gamma}{1-\alpha_-} p\left(\frac{x}{1-\alpha_-}, t\right) + \frac{1}{2}\frac{\gamma}{1+\alpha_+} p\left(\frac{x}{1+\alpha_+}, t\right). \tag{35}$$

Note that the probability is still conserved. By rewriting p in terms of $\tilde{p}$, we transform the evolution equation (35) to

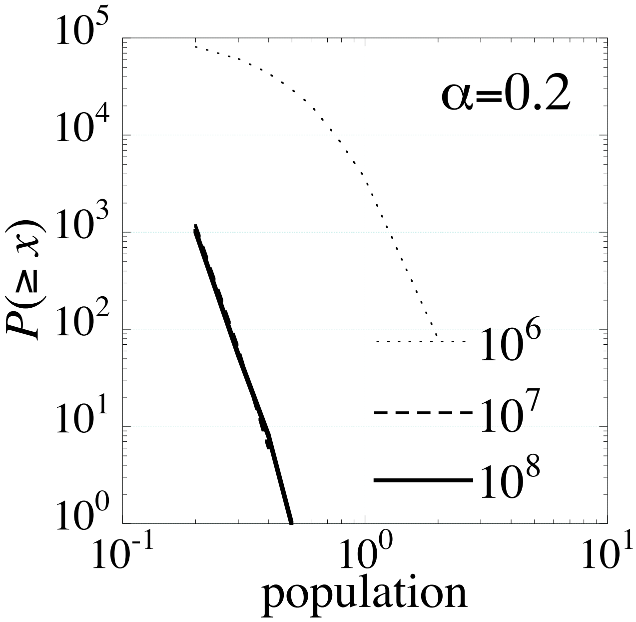
$$\frac{\partial}{\partial t}\tilde{p}(\xi,t) = -\gamma\tilde{p}(\xi,t) + \frac{\gamma}{2}\tilde{p}(\xi+\beta_-,t) - \frac{\gamma}{2}\tilde{p}(\xi-\beta_+,t), \tag{36}$$

where

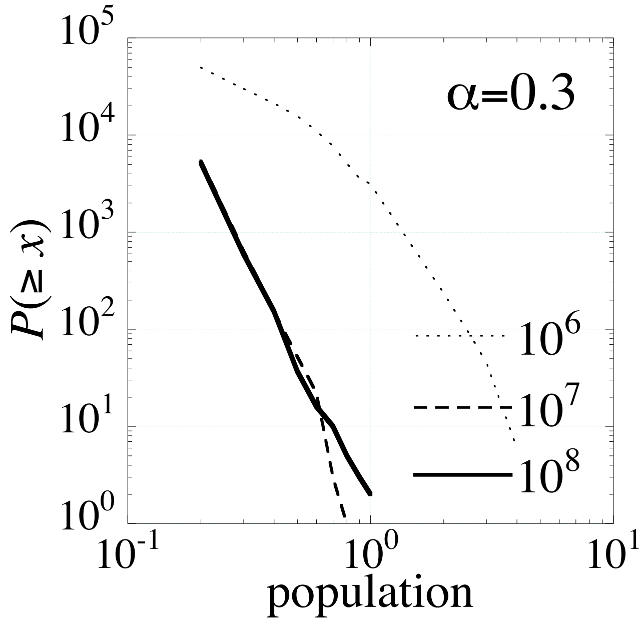
$$\beta_\pm \equiv \pm\log(1 \pm \alpha_\pm) > 0. \tag{37}$$



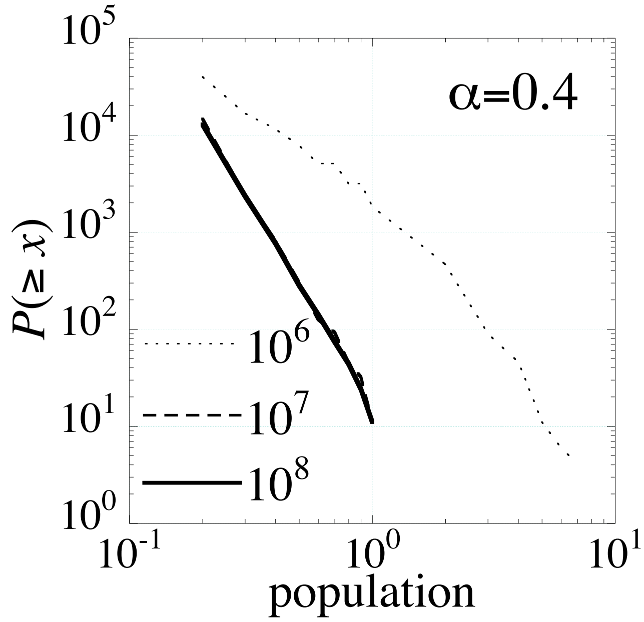
10^5
10^4
10^3
10^2
10^1
10^0
$P(\geq x)$
α=0.1
10^6
10^7
10^8
10^{-1}
10^0
population



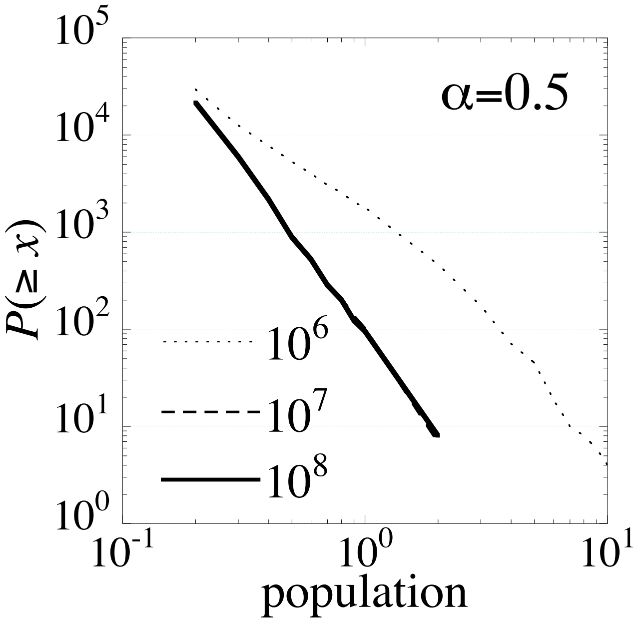
10^5
10^4
10^3
10^2
10^1
10^0
$P(\geq x)$
α=0.2
10^6
10^7
10^8
10^{-1}
10^0
10^1
population



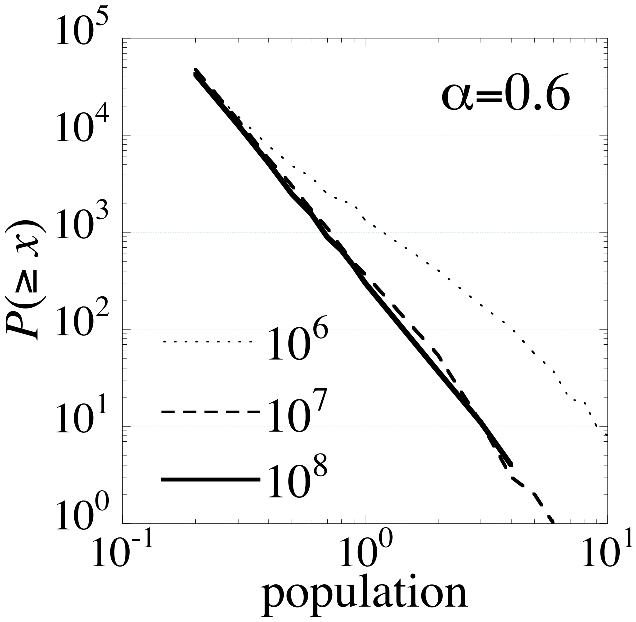
α=0.3
10^5
10^4
10^3
10^2
10^1
10^0
$P(\geq x)$
10^6
10^7
10^8
10^{-1}
10^0
10^1
population



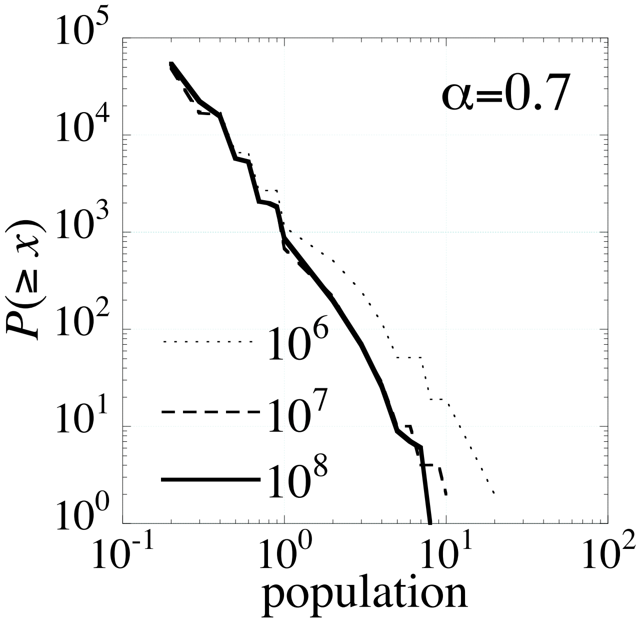
α=0.4
10^5
10^4
10^3
10^2
10^1
10^0
$P(\geq x)$
10^6
10^7
10^8
10^{-1}
10^0
10^1
population

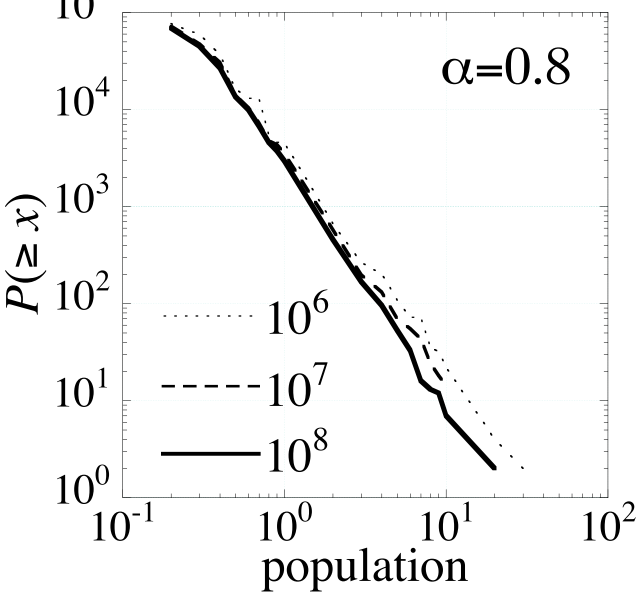


α=0.5
10^5
10^4
10^3
10^2
10^1
10^0
$P(\geq x)$
10^6
10^7
10^8
10^{-1}
10^0
10^1
population



α=0.6
10^5
10^4
10^3
10^2
10^1
10^0
$P(\geq x)$
10^6
10^7
10^8
10^{-1}
10^0
10^1
population


α=0.7
$P(\geq x)$
10^5
10^4
10^3
10^2
10^1
10^0
10^6
10^7
10^8
10^{-1}
10^0
10^1
10^2
population


α=0.8
$P(\geq x)$
10^5
10^4
10^3
10^2
10^1
10^0
10^6
10^7
10^8
10^{-1}
10^0
10^1
10^2
population

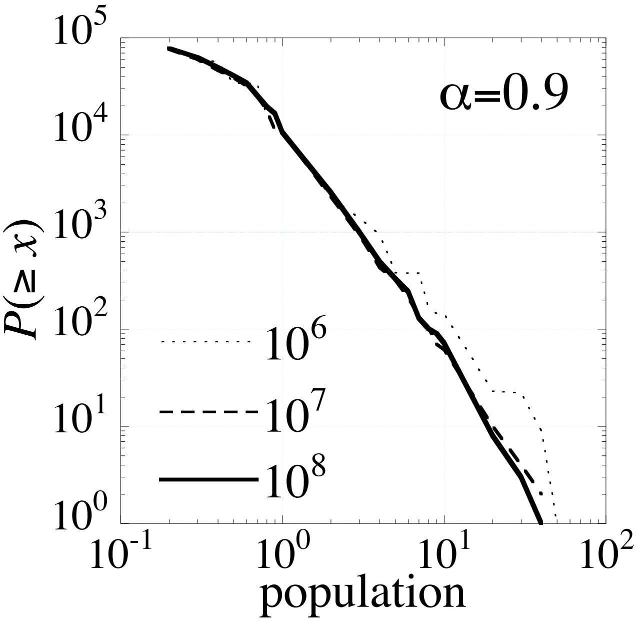


Figure 7: The results of the simulation of Extension A for various values of α. The number of steps is $T = 10^6$ (dotted line), 10^7 (dashed line) and 10^8 (solid line).

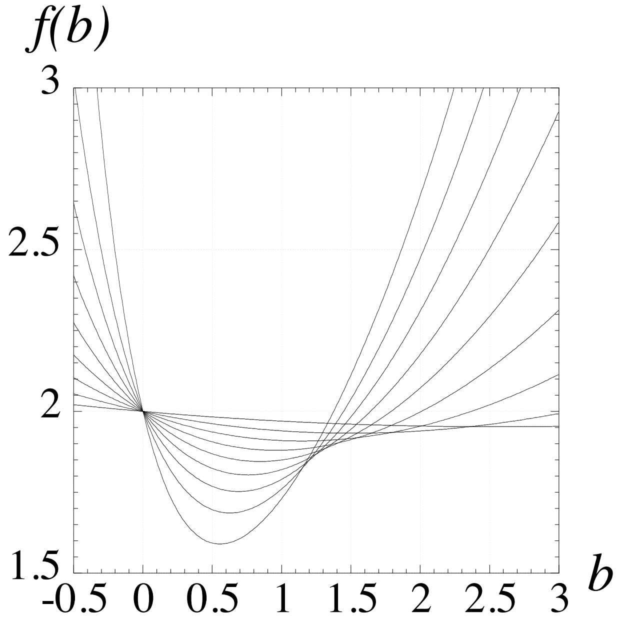


Figure 8: The function $f(b)$ defined in Eq. (38) b for various values of $\alpha_\pm$ with the ratio $\alpha_+/\alpha_- = 0.7$ fixed. The curvature of $f(b)$ increases as α becomes larger.

Note the difference between Eq. (23) and Eq. (37). The equation for the exponent b now reads

$$f(b) = (1 + \alpha_+)^b + (1 - \alpha_-)^b = 2. \tag{38}$$

Figure 8 shows $f(b)$ for various values of $\alpha_\pm$ with $\alpha_+/\alpha_- = 0.7$. Though the solution $b = 0$ is still stable, the other solution depends on $\alpha_\pm$. Owing to the asymmetry between α_+ and α_-, the relevant solution of Eq. (38) now depends on α_+ and α_-. Figure 9 shows that the simulation results of Extension A (Fig. 7) are in good agreement with the solution of Eq. (38).

Now we introduce Extension B of the model that also reproduces power-law exponents $b \neq 1$. Extension B of the model evolves as follows:

1. Choose an entity i randomly from $1 \leq i \leq N$.

2. Add or subtract randomly the amount αx_i from the chosen entity with

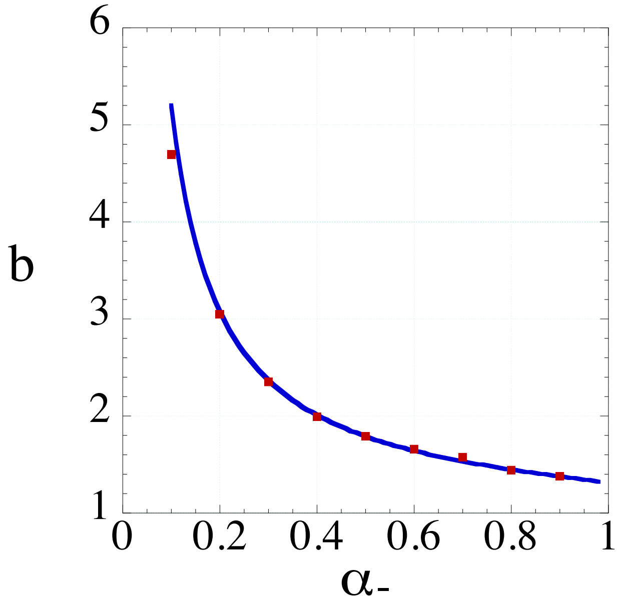
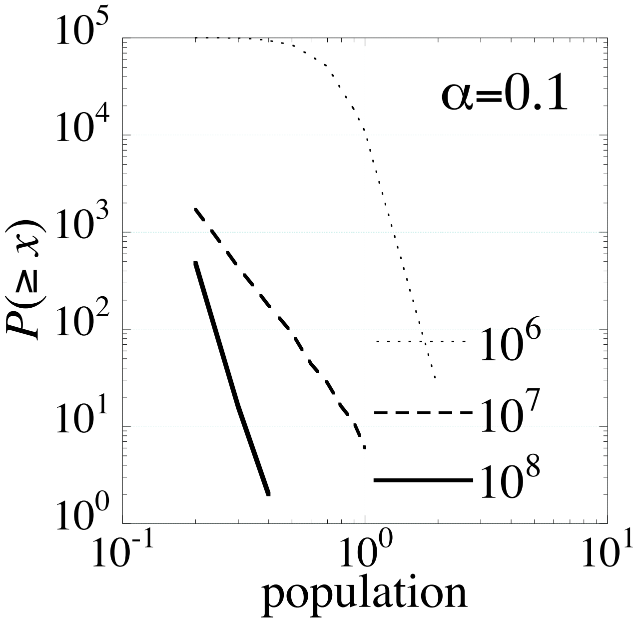
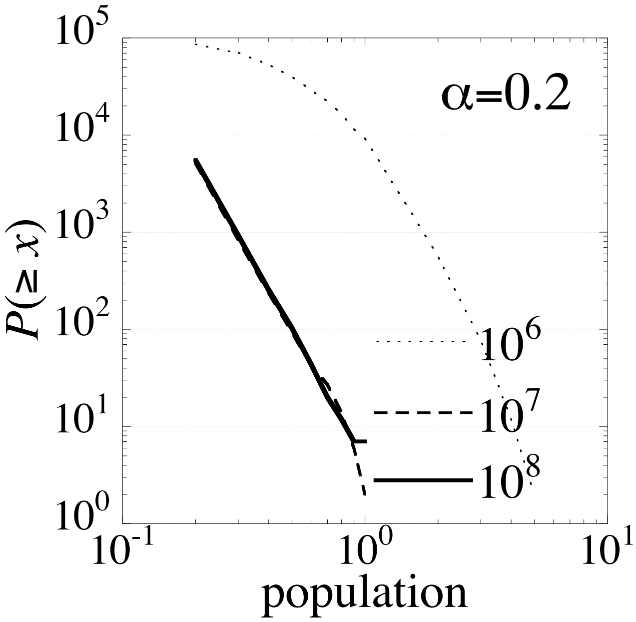
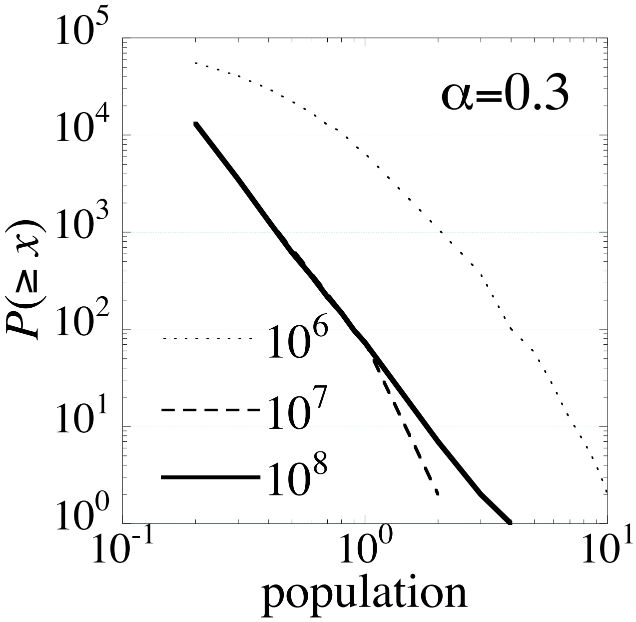
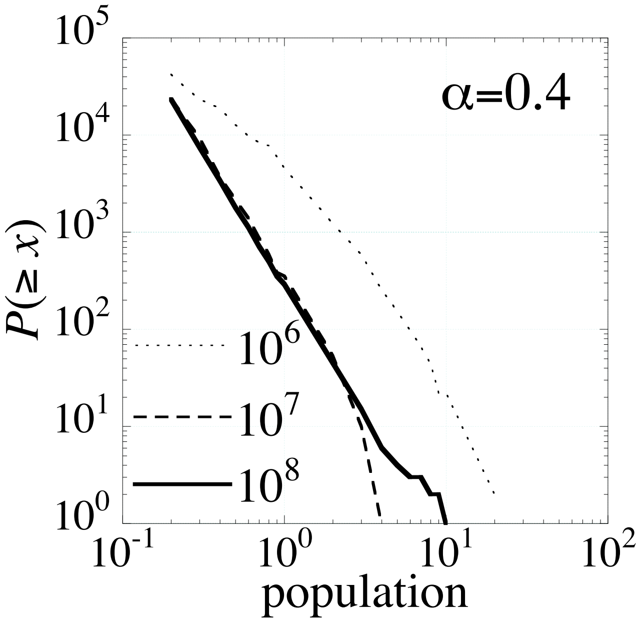


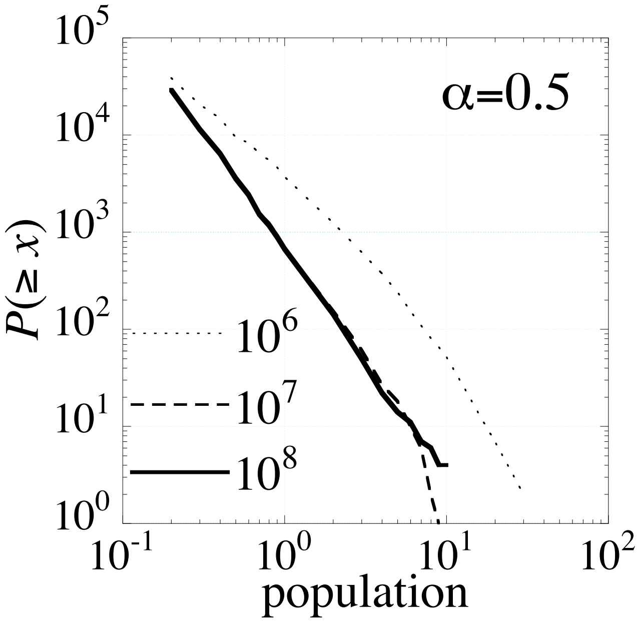
Figure 9: The power-law exponent b against the parameter α_- while fixing $\alpha_+/\alpha_- = 0.7$. The solid line shows the solution of Eq. (38), while the squares are the simulation results.


α=0.1
P(≥ x)
population
10^6
10^7
10^8

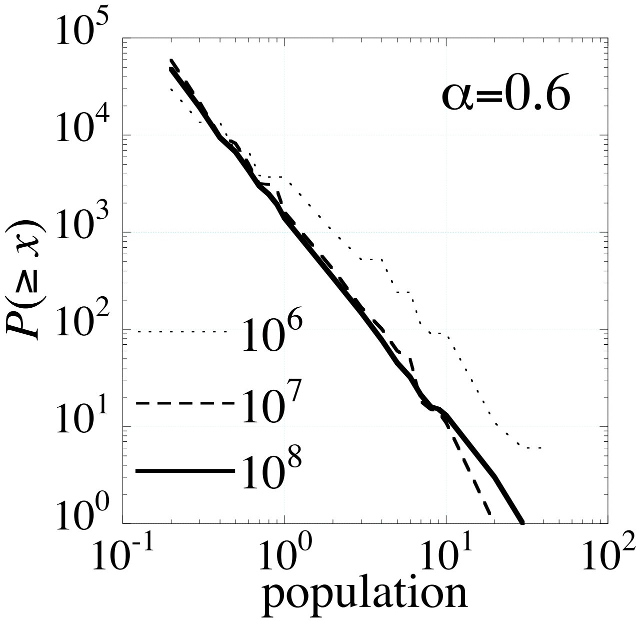

α=0.2
P(≥ x)
population
10^6
10^7
10^8


α=0.3
$P(\geq x)$
population
10^6
10^7
10^8

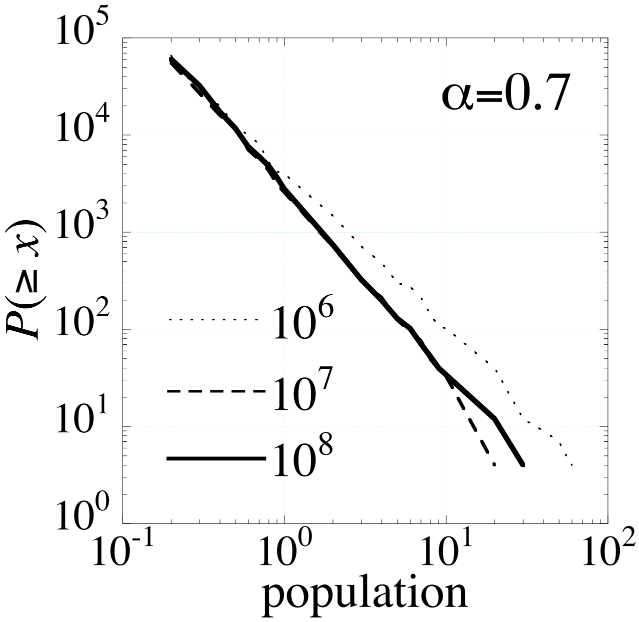

α=0.4
$P(\geq x)$
population
10^6
10^7
10^8

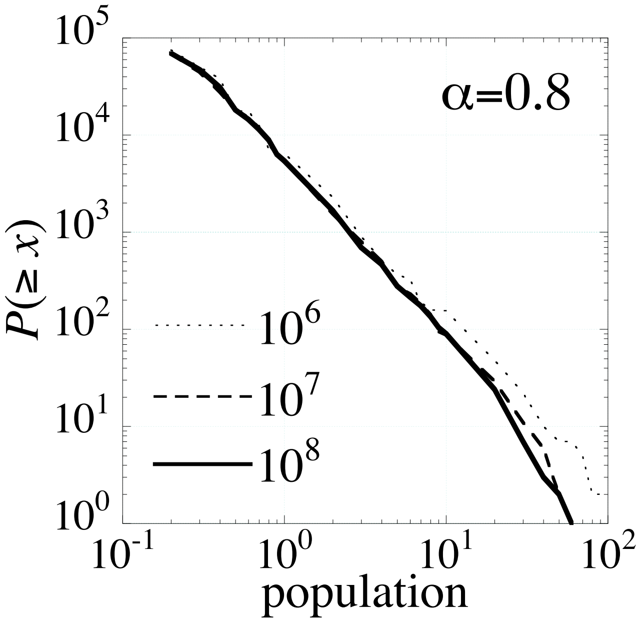


α=0.5
$P(\geq x)$
population
10^6
10^7
10^8



α=0.6
$P(\geq x)$
population
10^6
10^7
10^8


α=0.7
10^6
10^7
10^8
$P(\geq x)$
population
10^{-1} 10^0 10^1 10^2
10^0 10^1 10^2 10^3 10^4 10^5


α=0.8
10^6
10^7
10^8
$P(\geq x)$
population
10^{-1} 10^0 10^1 10^2
10^0 10^1 10^2 10^3 10^4 10^5

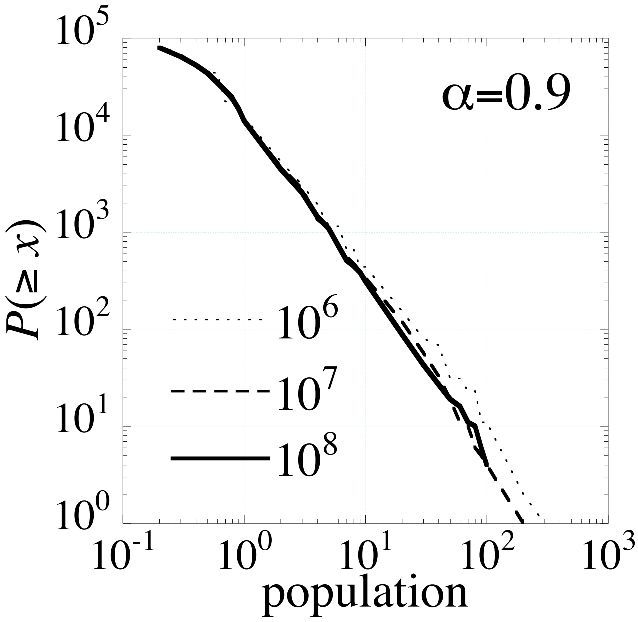


Figure 10: The results of the simulation of Extension B for various values of α with $\rho_+ = 1/3$ and $\rho_- = 2/3$. The number of steps is $T = 10^6$ (dotted line), 10^7 (dashed line), 10^8(solid line).

the probabilities ρ_+ and ρ_-, respectively ($\rho_- + \rho_- = 1$). In other words,

$$x_i \to \begin{cases} x_i + \alpha x_i & \text{with probability } \rho_+. \\ x_i - \alpha x_i & \text{with prabability } \rho_-. \end{cases} \tag{39}$$

Figure 10 shows the results of the simulation of Extension B. We again see power laws with $b \neq 1$. The time evolution equation of $p(x)$ now reads

$$\frac{\partial}{\partial t} p(x,t) = -\gamma p(x,t) + \rho_- \frac{\gamma}{1-\alpha} p(\frac{x}{1-\alpha}, t) + \rho_+ \frac{\gamma}{1+\alpha} p(\frac{x}{1+\alpha}, t). \tag{40}$$

By rewriting p in terms of $\tilde{p}$, we transform the evolution equation (40) to

$$\frac{\partial}{\partial t} \tilde{p}(\xi, t) = -\gamma \tilde{p}(\xi, t) + \rho_- \gamma \tilde{p}(\xi + \beta_-, t) + \rho_+ \gamma \tilde{p}(\xi - \beta_+, t), \tag{41}$$

where

$$\beta_\pm \equiv \pm \log(1 \pm \alpha) > 0. \tag{42}$$

The equation for the exponent b now reads

$$f(b) = 2\rho_-(1-\alpha)^b + 2\rho_+(1+\alpha)^b = 2 \tag{43}$$

Figure 11 shows $f(b)$ for various values of α. Though the solution $b = 0$ is still stable, the other solution depends on α. Owing to the asymmetry between ρ_+ and ρ_-, the relevant solution of Eq. (43) depends on α, ρ_+ and ρ_-. Figure 12 shows that the simulation results of Extension B (Fig. 10) are in good agreement with the solution of Eq. (43).

5 Summary

In the present thesis, we explained the origin of the universality of Zipf's law theoretically and demonstrated it numerically. We claim that the essence of Zipf's law is a size change of each entity proportional to its size. The simulation results of our simple and generic model indicate that the present explanation of Zipf's law is applicable to various phenomena both in natural and social sciences.

We also extended the model slightly and reproduced power laws other than Zipf's law. We revealed that some asymmetry between the increase and the decrease causes the deviation from Zipf's law. Equations (38) and (43) may be effective in estimating parameters in real power-law phenomena.

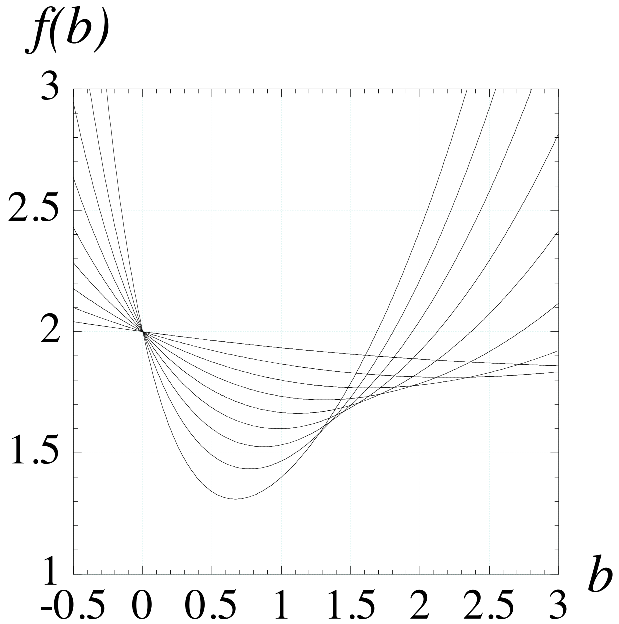


Figure 11: The function $f(b)$ defined in Eq. (43) for various values of α with $\rho_+ = 1/3$ and $\rho_- = 2/3$ fixed. The curvature of $f(b)$ increase as α becomes larger.

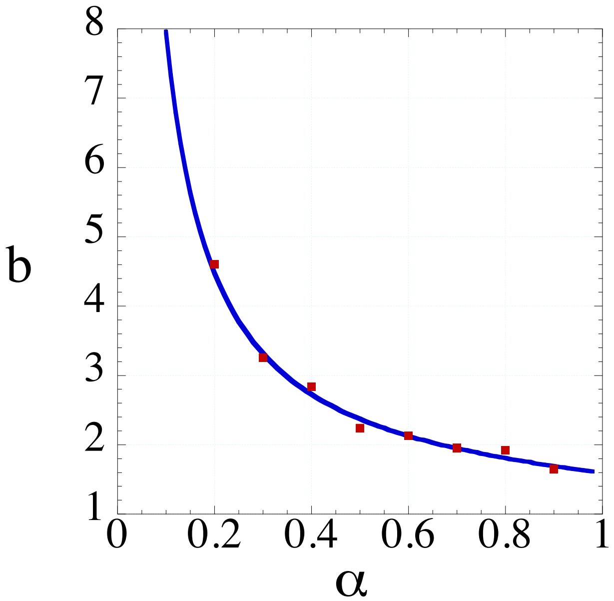


Figure 12: The power-law exponent b against the parameter α while fixing $\rho_+/\rho_- = 1/2$. The solid line shows the solution of Eq. (43), while the squares are the simulation results.

Acknowledgments

I am grateful to Prof.N. Hatano for his great help. I would also like to thank Y. Aiba and T. Murai for their helpful comments.

A Exponential Solutions

In this appendix, we discuss the validity of the assumption (28) for the solution of Eq. (27). Carrying out the Fourier transform of Eq. (27), we obtain

$$-2\tilde{\tilde{p}}_s(k) + e^{ik\beta_+}\tilde{\tilde{p}}_s(k) + e^{-ik\beta_-}\tilde{\tilde{p}}_s(k) = 0, \tag{44}$$

or

$$e^{ik\beta_+} + e^{-ik\beta_-} = 2, \tag{45}$$

where

$$\tilde{p}_s(\xi) = \int \tilde{\tilde{p}}_s(k) e^{ik\xi} dk. \tag{46}$$

By substituting Eq. (23), we can write Eq. (45) in the form

$$(1-\alpha)^{-ik} + (1+\alpha)^{-ik} = 2. \tag{47}$$

Let us look for solutions k in the complex plane by writing

$$k = k_r + ib, \tag{48}$$

where k_r and b are real numbers. If we have solutions with $k_r = 0$ only, the assumption Eq. (28) is justified. Substituting Eq. (48) for k in Eq. (47), we have

$$(1-\alpha)^b e^{ik_r\beta_+} + (1+\alpha)^b e^{-ik_r\beta_-} = 2. \tag{49}$$

To satisfy Eq. (49), we need the phase factors to be real:

$$e^{ik_r\beta_+} = e^{-ik_r\beta_-} = 1 \tag{50}$$

or

$$-e^{ik_r\beta_+} = e^{-ik_r\beta_-} = 1. \tag{51}$$

In the case of Eq. (50), we have

$$k_r = \frac{2m\pi}{\beta_+} = \frac{2n\pi}{\beta_-}, \tag{52}$$

while in the case of Eq. (51) we have

$$k_r = \frac{(2m+1)\pi}{\beta_+} = \frac{2n\pi}{\beta_-}, \tag{53}$$

where m and n are integers. In order for the solutions (52) and (53) to exist, the ratio of the step sizes β_+ and β_- must be a rational number. It means that the random walker has a possibility of returning exactly to the starting point. We consider that such cases are quite exceptional; If we take a general value of α, the ratio β_+/β_- is an irrational number in general. Hence we neglect the solutions (52) and (53) and assume $k_r = 0$. Therefore the remaining solutions have the form $k = ib$. We can thus justify the assumption (28).

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