MASTER THESIS

Triangular Arbitrage as an Interaction among Foreign Exchange Rates

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2001

論　文　要　旨
（和　文）

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（論文題目）
外国為替相場における相互作用としての三角裁定

（内容の要旨）
近年、外国為替相場や株価など自由市場における価格の変動や、個人や企業の所得の分布、金融資産のリスク管理、インフレーションの構造など、これまでは物理とは無縁だとされていた事象を、統計物理学の手法を用いて解明しようとする研究が盛んにおこなわれています。

現在までに、自由市場における価格の変動の記述を目的とするモデルは多く提案されています。しかし、いずれのモデルも単一の市場価格について議論しており、複数の価格間の相互作用を考慮に入れた価格変動のモデルは皆無であると言えます。

本研究の動機は以下の三つです。

- 自由市場における価格間に相互作用は存在するのか
- 存在するならば、その相互作用の原因はなにか
- その相互作用を定式化できるか

我々は、外国為替市場における円ドル、ドルユーロ、円ユーロの三つのレートを解析し、これらの疑問に対する答えを得ました。

まず、外国為替レート間には明らかに相互作用が存在することを示します。上記三つのレートの積（レート積）の確率分布は鋭いピークと幅広い裾野（いわゆるファットテール）を持ちますが、それぞれのレートの確率分布にはそれらの特徴がありません。これは三つのレート間に相関があることの証明になります。

次に、その相互作用は三角裁定取引という経済活動に起因していることを述べます。三角裁定取引とは、三つの為替レートを利用して利益をあげる取引で、三つのレートの積をある一定の値に収束させるはたらきがあると考えられます。

最後に、その三角裁定取引の効果を相互作用として取り入れた新たなモデルを提案します。このモデルは相互作用の関数を線形近似しており、レート積の分布をよく再現します。三角裁定取引がレート積をある一定値に収束させようとする効果と、各々のレートが独自に激しく変動する効果とが競合する結果、レート積の分布が鋭いピークと幅広い裾野を持つことが、このモデルを通してわかります。

Academic Year of 2001, Submitted on February 22, 2002

Graduate College of Science and Engineering, Aoyama Gakuin University

Master Thesis

Triangular arbitrage as an interaction among foreign exchange rates

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Supervisor: Naomichi Hatano

In recent years, many studies analyze economic and financial phenomena by the methods developed in statistical physics: for example, modeling of financial markets as stochastic processes, scaling properties of the distribution of fluctuations of financial market indices, dynamic numerical models of stock market price and so on.

Many statistical physical models of price change have been introduced so far. They discuss, however, only the change of one price. They did not consider any interaction among multiple prices.

This thesis has three motivations as follows:

- Does an interaction exist between the prices of the markets?
- If it does, what's the origin of the interaction?
- Is it possible to quantify the interaction?

We found the answers to these questions, analyzing three foreign exchange rates: the yen-dollar rate, the dollar-euro rate and the yen-euro rate.

First, we show that there are in fact triangular arbitrage opportunities in the spot foreign exchange markets, analyzing the time dependence of the three rates. The probability distribution of product of the three rates (the rate product) has a sharp peak and fat tails, while those of the three rates do not. It means that there is a correlation among the three rates.

Next, we claim that the interaction is caused by a financial activity called triangular arbitrage transaction. It has been argued that the triangular arbitrage makes the rate product converge to a certain value.

Finally, we introduce a model with the interaction caused by the triangular arbitrage. The model explains the real data well. The interaction makes the rate product converge to a certain value. On the other hand, random fluctuation of each rate makes the rate product fluctuate widely. As a result of the competition between the above effects, the distribution of the rate product has a sharp peak and fat tails.

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Abstract

In recent years, many studies analyze economic and financial phenomena by the methods developed in statistical physics: for example, modeling of financial markets as stochastic processes, scaling properties of the distribution of fluctuations of financial market indices, dynamic numerical models of stock market price and so on.

Many statistical physical models of price change have been introduced so far. They discuss, however, only the change of one price. They did not consider any interaction among multiple prices.

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Contents

1 Introduction

The triangular arbitrage is a financial activity that takes advantage of the three exchange rates among three currencies. Suppose that we exchange one US dollar to some amount of Japanese yen, exchange the amount of Japanese yen to some amount of euro, and finally exchange the amount of euro back to US dollar; then how much US dollar do we have? There are opportunities that we have more than one US dollar. The triangular arbitrage transaction is the trade that takes this type of opportunities. It has been argued that the triangular arbitrage makes the product of the three exchange rates converge to a certain value [1]. In other words, the triangular arbitrage is a form of interaction among currencies.

The purpose of this thesis[1] is to show that there is in fact triangular arbitrage opportunities in foreign exchange markets and they generate an interaction among foreign exchange rates. We first analyze real data in Sec. 2, showing that the product of three foreign exchange rates has a narrow distribution with fat tails. In order to explain the behavior, we propose in Sec. 3 a model of the time evolution of exchange rates with an interaction. The simulation explain the real data well. In Appendix A, details of the prices provided by information company are discussed. A few stochastic models of price change are reviewed in Appendix B and the Lévy and truncated Lévy distributions are reviewed in Appendix C.

2 Observation of the Triangular Arbitrage Opportunity in High-Frequency Data

In the present thesis, we analyze actual data of the yen-dollar rate, the yen-euro rate and the dollar-euro rate, taken from January 25 1999 to March 12 1999 except for weekends. We show in this section that there are actually triangular arbitrage opportunities and that the three exchange rates correlate strongly.

[1]The content of this thesis will be published in Physica A.
Y. Aiba, N. Hatano, H. Takayasu, K. Marumo, T. Shimizu, 'Triangular arbitrage as an interaction among foreign exchange rates,' *Physica A*, to be published.

2.1 Existence of Triangular Arbitrage Opportunities

In order to quantify the triangular arbitrage opportunities, we define the quantity

$$\mu(t) = \prod_{i=1}^{3} r_i(t) \ , \tag{1}$$

where $r_i(t)$ denotes each exchange rate at time t. We refer to this quantity as the rate product. There is a triangular arbitrage opportunity whenever the rate product is greater than unity.

To be more precise, there are two types of the rate product. One is based on the arbitrage transaction in the direction of dollar to yen to euro to dollar. The other is based on the transaction in the opposite direction of dollar to euro to yen to dollar. Since these two values show similar behavior, we focus on the first type of $\mu(t)$ hereafter. Thus, we specifically define each exchange rate as

$$r_1(t) \equiv \frac{1}{\text{yen-dollar ask } (t)} \tag{2}$$

$$r_2(t) \equiv \frac{1}{\text{dollar-euro ask } (t)} \tag{3}$$

$$r_3(t) \equiv \text{yen-euro bid } (t). \tag{4}$$

(Note the difference between

$$1 \quad [dollar] \quad \rightarrow \quad \text{yen-dollar bid} \quad [yen] \tag{5}$$

and

$$1 \quad [yen] \quad \rightarrow \quad \frac{1}{\text{yen-dollar ask}} \quad [dollar].) \tag{6}$$

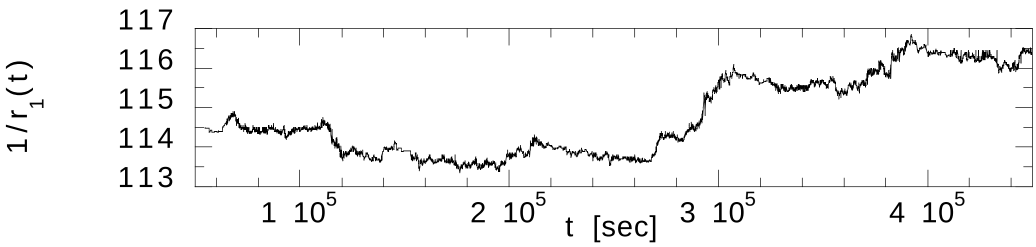
We assume here that an arbitrager can transact instantly at the bid and the ask prices provided by information companies and hence we use the prices at the same time to calculate the rate product. (See Appendix A for details.)

Figure 1(a)-(c) shows the actual changes of the three rates: the yen-euro ask, the dollar-euro ask and the yen-euro bid. Figure 1(d) shows the behavior of the rate product $\mu(t)$. We can see that the rate product μ fluctuates around the average

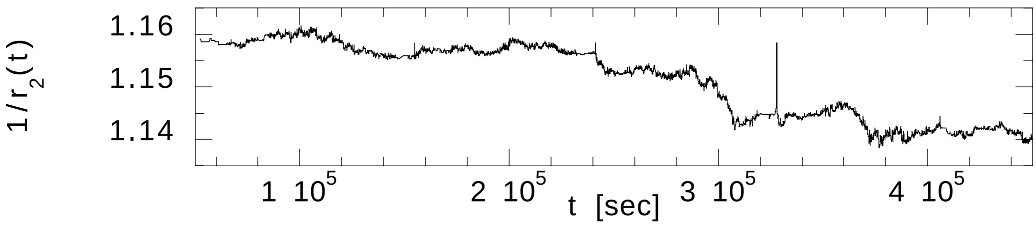
$$m \equiv \langle \mu(t) \rangle \simeq 0.99998. \tag{7}$$

(The average is less than unity because of the spread; the spread is the difference between the ask and the bid prices and is usually of the order of 0.05% of the prices.) The probability density function of the rate product μ (Fig. 2) has a sharp peak and fat tails while those of the three rates (Fig. 3) do not. It means that the fluctuations of the exchange rates have correlation that makes the rate product converge to the average m.

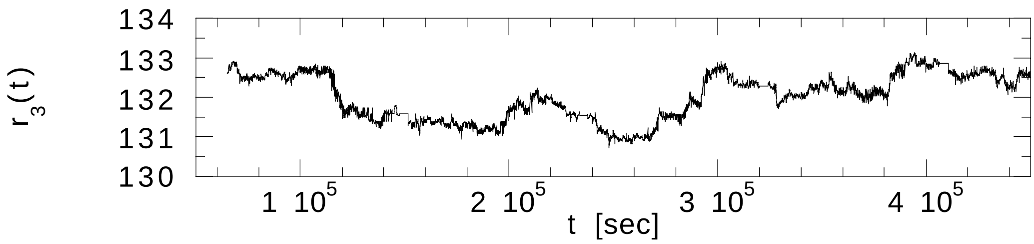
(a)



(b)



(c)



(d)

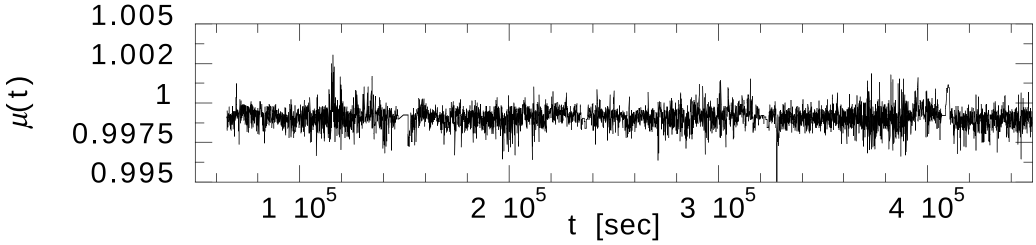
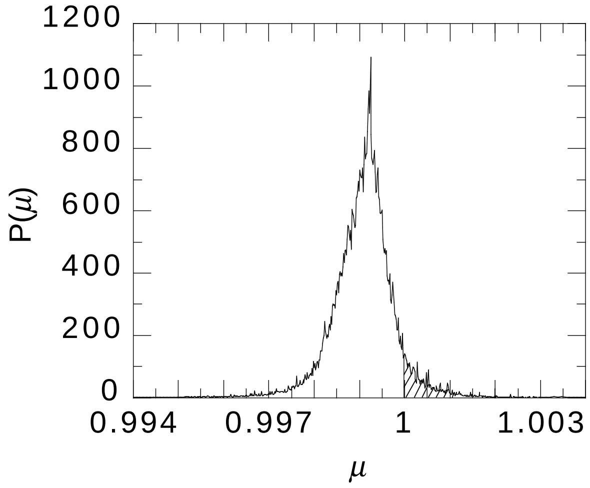


Figure 1: The time dependence of (a) the yen-dollar ask $1/r_1$, (b) the dollar-euro ask $1/r_2$, (c) the yen-euro bid r_3 and (d) the rate product μ. The horizontal axis denotes the seconds from 00:00:00, January 25 1999.

(a)



(b)

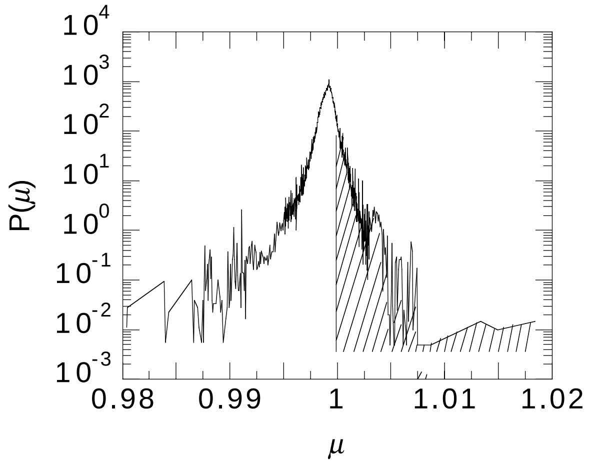
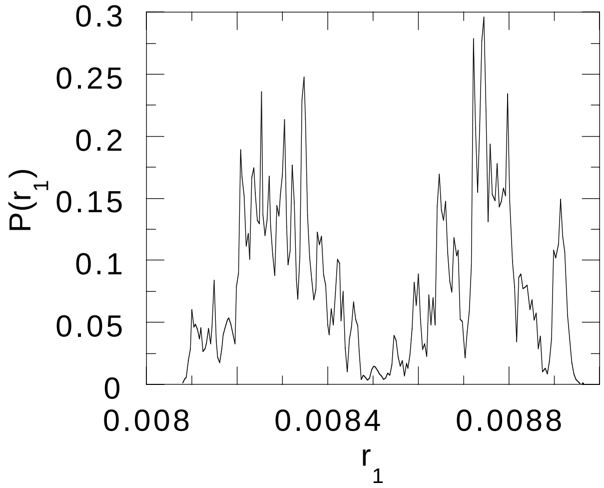
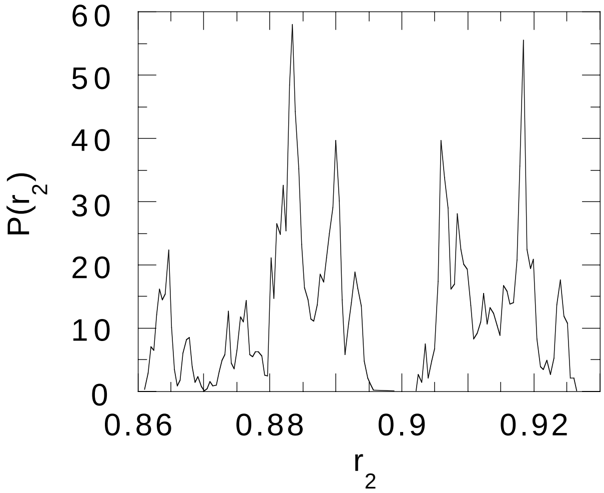


Figure 2: The probability density function of the rate product μ. (b) is a semi-logarithmic plot of (a). The shaded area represents triangular arbitrage opportunities. The data were taken from January 25 1999 to March 12 1999.

(a)


0.3
0.25
0.2
0.15
0.1
0.05
0
0.008
0.0084
0.0088
P(r_1)
r_1

(b)


60
50
40
30
20
10
0
0.86
0.88
0.9
0.92
P(r_2)
r_2

(c)

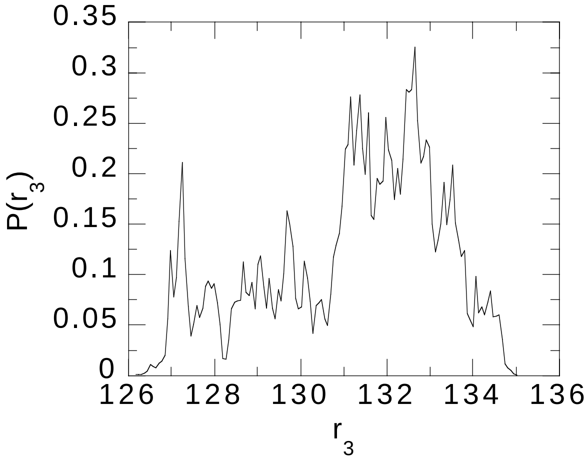


Figure 3: The probability density function of the three rates: (a) the reciprocal of the yen-dollar ask, r_1, (b) the reciprocal of the dollar-euro ask, r_2 and (c) the yen-euro bid r_3. The data were taken from January 25 1999 to March 12 1999.

2.2 Feasibility of the Triangular Arbitrage Transaction

We discuss here the feasibility of the triangular arbitrage transaction. We analyze the duration of the triangular arbitrage opportunities and calculate whether an arbitrager can make profit or not.

The shaded area in Fig. 2 represents triangular arbitrage opportunities. We can see that the rate product is grater than unity for about 6.4% of the time. It means that triangular arbitrage opportunities exist about ninety minutes a day. The ninety minutes, however, include the cases where the rate product μ is greater than unity very briefly. The triangular arbitrage transaction is not feasible in these cases.

In order to quantify the feasibility, we analyze the duration of the triangular arbitrage opportunities. Figure 4 shows the cumulative distributions of the duration τ_+ of the situation $\mu > 1$ and τ_- of $\mu < 1$. It is interesting that the distribution of τ_+ shows a power-law behavior while the distribution of τ_- dose not. This difference may suggest that the triangular arbitrage transaction is carried out indeed.

In order to confirm the feasibility of the triangular arbitrage, we simulate the triangular arbitrage transaction using our time series data. We assume that it takes an arbitrager $T_{\rm rec}$[sec] to recognize triangular arbitrage opportunities and $T_{\rm exe}$[sec] to execute a triangular arbitrage transaction; see Fig.

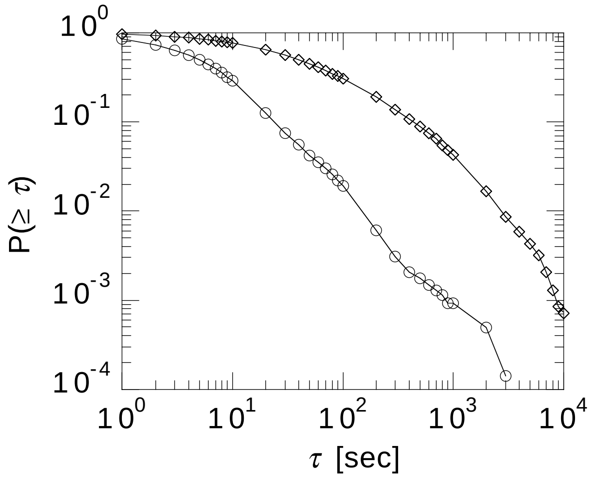


Figure 4: The cumulative distributions of τ_+ ($\circ$) and τ_- ($\diamond$). The distribution of τ_+ shows a power-low behavior. The data were taken from January 25 1999 to March 12 1999.

5. We also assume that the arbitrager transacts whenever the arbitrager recognizes the opportunities. Figure 6 shows how much profit the arbitrager can make from one US dollar (or Japanese yen or euro) in a day. We can see that the arbitrager can make profit if it takes the arbitrager a few seconds to recognize the triangular arbitrage opportunities and to execute the triangular arbitrage transaction.

3 Modeling

We here introduce a new model that takes account of the effect of the triangular arbitrage transaction as an interaction among the three rates. Many models of price change have been introduced so far: for example, the Lévy-stable non-Gaussian model [2]; the truncated Lévy flight [3]; the ARCH/GARCH processes [4, 5]. (See Appendix B for reviews of these models.) They discuss, however, only the change of one price. They did not consider an interaction among multiple prices. As we discussed in Sec. 2, however, the triangular arbitrage opportunity exists in the market and is presumed to affect price fluctuations in the way the rate product tends to converge to a certain value.

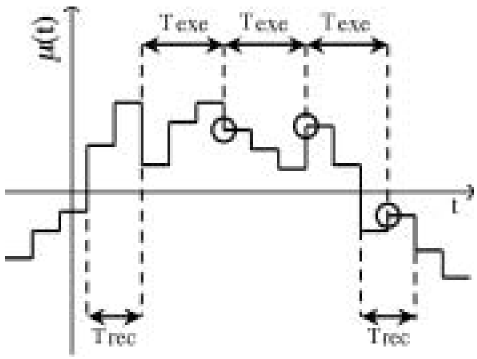


Figure 5: A conceptual figure of the profit calculation. We assume that it takes an arbitrager $T_{\rm rec}$[sec] to recognize triangular arbitrage opportunities and $T_{\rm exe}$[sec] to execute a triangular arbitrage transaction. The circles ($\circ$) indicate the instances where triangular arbitrage transactions are carried out.

3.1 Basic Time Evolution

The basic equation of our model is a time-evolution equation of the logarithm of each rate:

$$\ln r_i(t + \Delta t) = \ln r_i(t) + f_i(t) + g(\nu(t)), \quad (i = 1, 2, 3) \tag{8}$$

where ν is the logarithm of the rate product

$$\nu(t) \equiv \ln \mu(t) = \sum_{i=1}^{3} \ln r_i(t). \tag{9}$$

Just as μ fluctuates around $m = \langle \mu \rangle \simeq 0.99998$, the logarithm rate product ν fluctuates around

$$\epsilon \equiv \langle \ln \mu \rangle \simeq -0.00091 \tag{10}$$

(Fig. 7(a)). In this model, we focus on the logarithm of the rate-change ratio $\ln(r_i(t + \Delta t)/r_i(t))$, because the relative change is presumably more essential than the absolute change. We assumed in Eq. (8) that the change of the logarithm of each rate is given by an independent fluctuation $f_i(t)$ and an attractive interaction $g(\nu)$. The triangular arbitrage is presumed to make the logarithm rate product ν converge to the average ϵ; thus, the interaction function $g(\nu)$ should be negative for ν greater than ϵ and positive for ν less than ϵ:

$$g(\nu) \begin{cases} < 0 & , \quad \text{for } \nu > \epsilon \\ > 0 & , \quad \text{for } \nu < \epsilon. \end{cases} \tag{11}$$

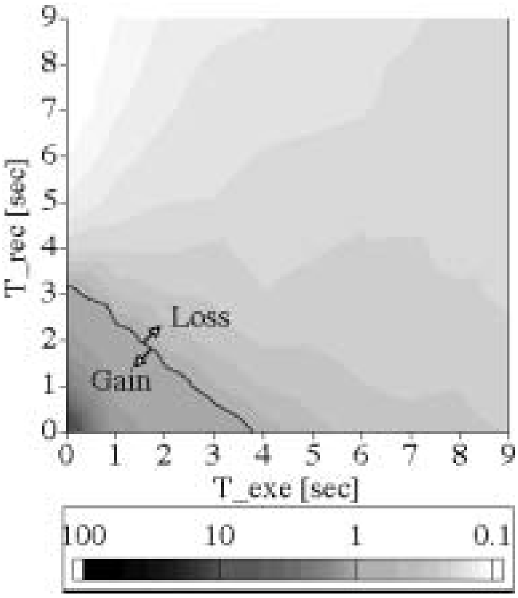
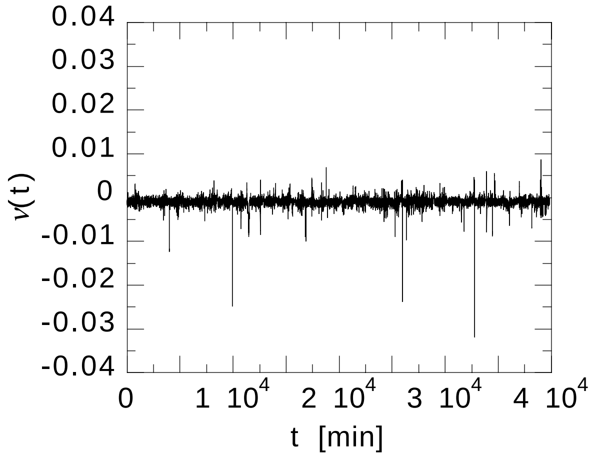
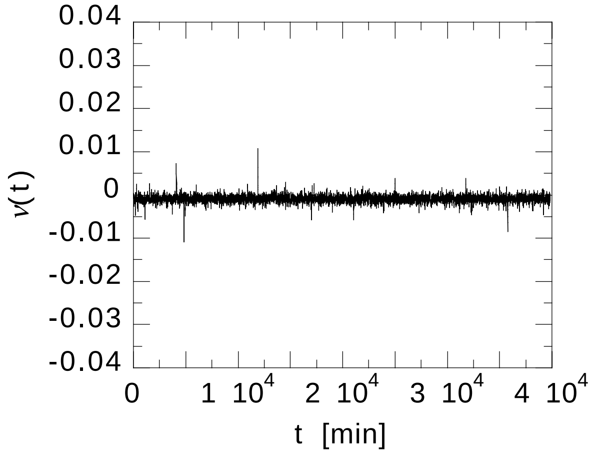


Figure 6: A phase diagram of the profit that an arbitrager can make from one US dollar (or Japanese yen or euro) in a day under the assumption shown in Fig. 5.

(a)


0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
v(t)
0
1 10^4
2 10^4
3 10^4
4 10^4
t [min]

(b)


0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
v(t)
0
1 10^4
2 10^4
3 10^4
4 10^4
t [min]

(c)

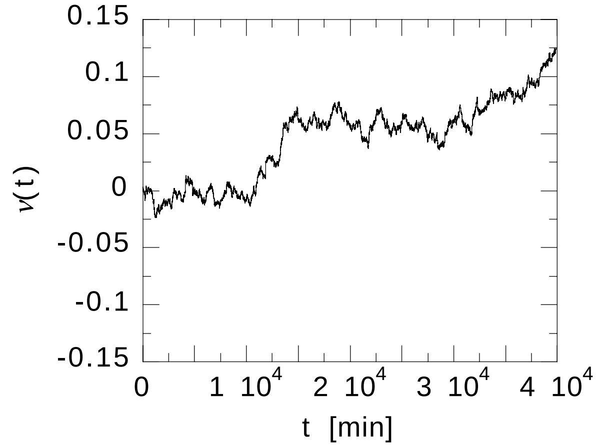


Figure 7: The time dependence of $\nu(t[\text{min}])$ of (a) the real data, (b) the simulation data with the interaction and (c) without the interaction. In (b), ν fluctuates around ϵ like the real data.

As a linear approximation, we define $g(\nu)$ as

$$g(\nu) \equiv -a(\nu - \epsilon) \tag{12}$$

where a is a positive constant which specifies the interaction strength.

The time-evolution equation of ν is given by summing Eq. (8) over all i:

$$\nu(t + \Delta t) - \epsilon = (1 - 3a)(\nu(t) - \epsilon) + F(t), \tag{13}$$

where

$$F(t) \equiv \sum_{i=1}^{3} f_i(t). \tag{14}$$

This is our basic time-evolution equation of the logarithm rate product.

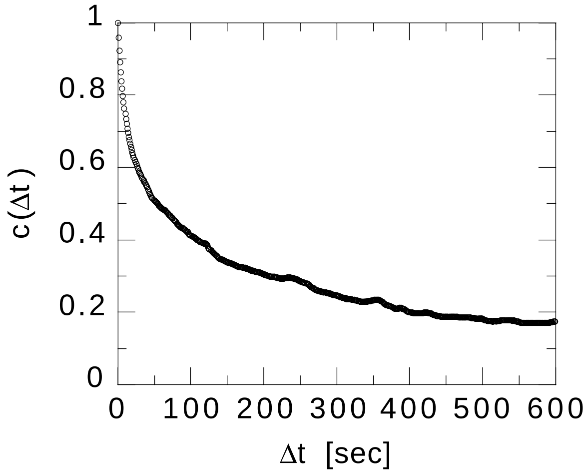
3.2 Estimation of Parameters

The interaction strength a is related to the auto-correlation function of ν as follows:

$$1 - 3a = c(\Delta t) \equiv \frac{\langle \nu(t + \Delta t)\nu(t)\rangle - \langle \nu(t)\rangle^2}{\langle \nu^2(t)\rangle - \langle \nu(t)\rangle^2}. \tag{15}$$

Using Eq. (15), we can estimate $a(\Delta t)$ from the real data series as a function of the time step Δt. The auto-correlation function $c(\Delta t)$ is shown in Fig.

(a)



(b)

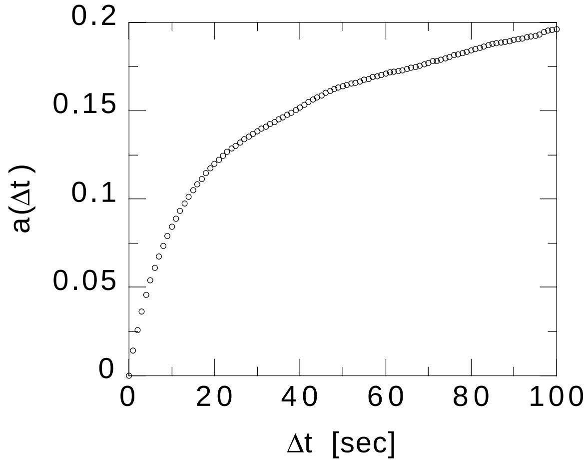


Figure 8: (a) The auto-correlation function of ν, $c(\Delta t)$. (b) The time-step dependence of $a(\Delta t)$.

rate	α	γ	l
r_1 (1/yen-dollar ask)	1.8	7.61×10^{-7}	1.38×10^{-2}
r_2 (1/dollar-euro ask)	1.7	4.06×10^{-7}	3.81×10^{-2}
r_3 (yen-euro bid)	1.8	6.97×10^{-7}	7.58×10^{-2}

Table 1: The estimates of the parameters.

8(a). The estimate of $a(\Delta t)$ is shown in Fig. 8(b). Hereafter, we fix the time step at $\Delta t = 60$[sec] and hence use

$$a(1[\text{min}]) = 0.17 \pm 0.02 \tag{16}$$

for our simulation.

On the other hand, the fluctuation of foreign exchange rates is known to be a fat-tail noise [6, 7]. Here we take $f_i(t)$ as the truncated Lévy process [3, 8]:

$$P_{\mathrm{T}}(f; \alpha, \gamma, l) = q P_{\mathrm{L}}(f; \alpha, \gamma)\Theta(l - |f|), \tag{17}$$

where q is the normalization constant, $\Theta(x)$ represents the step function and $P_{\mathrm{L}}(x; \alpha, \gamma)$ is the symmetric Lévy distribution of index α and scale factor γ:

$$P_{\mathrm{L}}(x; \alpha, \gamma) = \frac{1}{\pi}\int_0^\infty e^{-\gamma|k|^\alpha}\cos(kx)\mathrm{d}k \quad 0 < \alpha < 2. \tag{18}$$

(See Appendix C for reviews of the Lévy and the truncated Lévy distributions.) We determine the parameters α, γ and l by using the following relations for $1 < \alpha < 2$ [6, 9]:

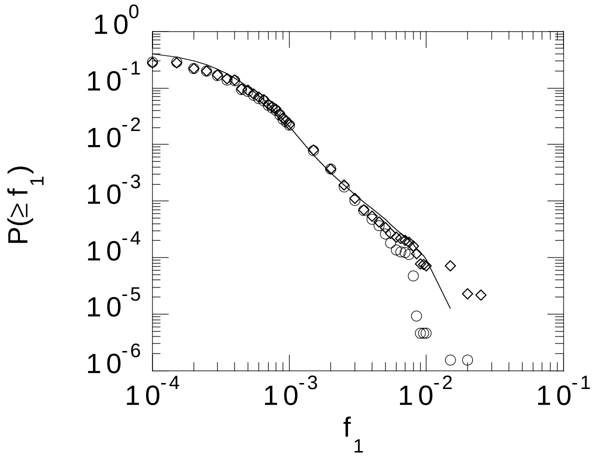
$$c_2 = \frac{\alpha(\alpha-1)\gamma}{|\cos(\pi\alpha/2)|} l^{2-\alpha}, \tag{19}$$

$$\kappa = \frac{(3-\alpha)(2-\alpha)|\cos(\pi\alpha/2)|}{\alpha(\alpha-1)\gamma} l^{\alpha}, \tag{20}$$

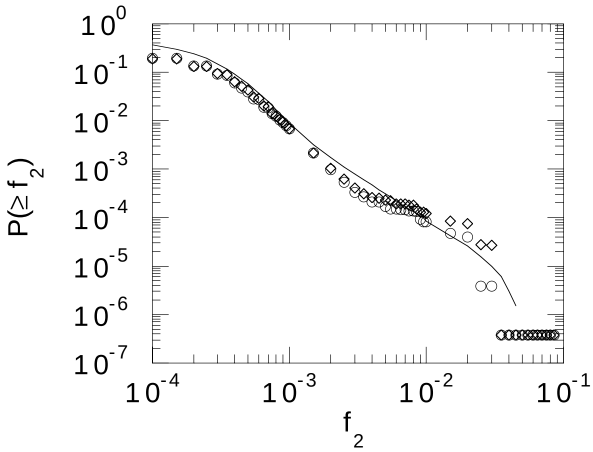
where c_n denotes the nth cumulant and κ is the kurtosis $\kappa = c_4/{c_2}^2$. The estimates are shown in Table 1. The generated noises with the estimated parameters are compared to the actual data in Fig. 9.

We simulated the time evolution (13) with the parameters given in Eqs. (10), (16) and Table 1. The probability density function of the results (Fig. 7(b)) is compared to that of the real data (Fig. 7(a)) with $\Delta t = 1$[min] in Fig. 10. The fluctuation of the simulation data is consistent with that of the real data. In particular, we see good agreement around $\nu \simeq \epsilon$ as a result of the linear approximation of the interaction function. Figure 7(c) shows $\nu(t)$ of the simulation without the interaction, i.e. $a = 0$. The quantity ν fluctuates freely, which is inconsistent with the real data.

(a)


P(≥ f1)
f1
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-4
10^-3
10^-2
10^-1

(b)


P(≥ f2)
f2
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-4
10^-3
10^-2
10^-1

(c)

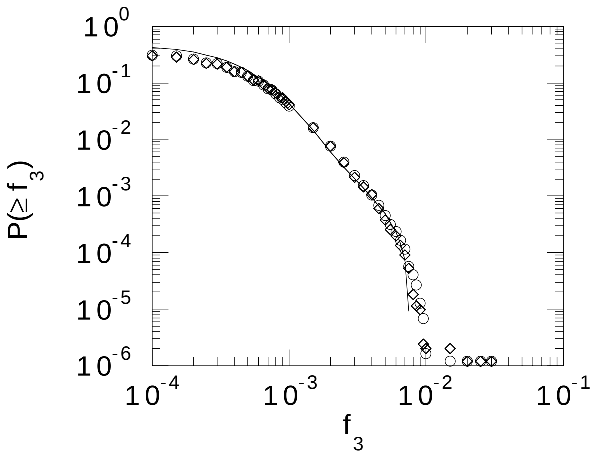


Figure 9: The cumulative distributions of the one-minute changes $|\ln r_i(t + 1[\mathrm{min}]) - \ln r_i(t)|$ ($\circ$ represents upward movements and $\diamond$ represents downward movements) and the generated noise f_i (—): (a) the yen-dollar ask and f_1, (b) the dollar-euro ask and f_2, and (c) yen-euro bid and f_3. The real data were taken from January 25 1999 to March 12 1999.

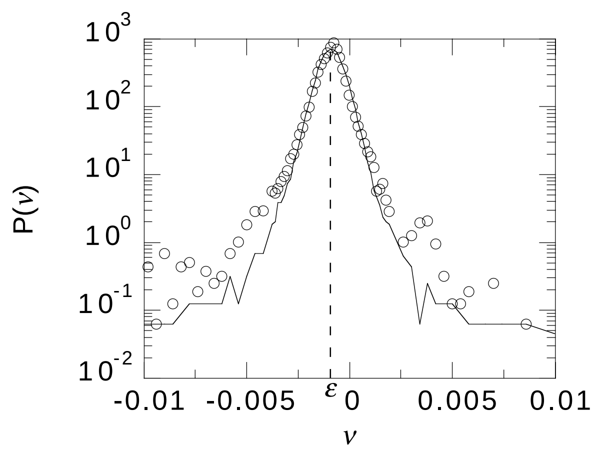


Figure 10: The probability density function of ν. The circle ($\circ$) denotes the real data and the solid line denotes our simulation data with the interaction. The simulation data fit the real data well.

3.3 Analytical approach

We can solve the time-evolution equation (13) analytically in some cases. Let us define

$$\omega(t) \equiv \nu(t) - \epsilon \tag{21}$$

and

$$\left\{ \begin{array}{lllllll} \omega & = & \omega(t), & \omega' & = & \omega(t+\Delta t), \\ F & = & F(t), & F' & = & F(t+\Delta t), \\ c & = & c(\Delta t). \end{array} \right. \tag{22}$$

Equation (13) is then reduced to

$$\omega' = c\omega + F \tag{23}$$

Assume that the probability of ω having a value in $\omega \sim \omega + \mathrm{d}\omega$ is $P_\omega(\omega)$. The joint probability of ω' having a value in $\omega' \sim \omega' + \mathrm{d}\omega'$ and F' having a value in $F' \sim F' + \mathrm{d}F'$ is given by

$$\begin{aligned} P_{\omega',F'}(\omega', F')\mathrm{d}\omega'\mathrm{d}F' &= P_\omega(\omega)P_F(F')\mathrm{d}\omega\mathrm{d}F' \\ &= \frac{1}{c}P_\omega(\frac{\omega' - F'}{c})P_F(F')\mathrm{d}\omega'\mathrm{d}F', \end{aligned} \tag{24}$$

where $P_F(F)$ is the probability of F having a value in $F \sim F + \mathrm{d}F$. The probability density function of ω' is thus given by

$$P_{\omega'}(\omega') = \frac{1}{c}\int P_\omega(\frac{\omega' - F'}{c})P_F(F')\mathrm{d}F'. \tag{25}$$

The characteristic function of ω' is the Fourier transform

$$\tilde{P}_{\omega'}(k) = \int P_{\omega'}(\omega')e^{i\omega' k}\mathrm{d}\omega' = \tilde{P}_\omega(ck)\tilde{P}_F(k), \tag{26}$$

where $\tilde{P}_\omega$ and $\tilde{P}_F$ are the Fourier transforms of P_ω and P_F, respectively. Then we obtain

$$\begin{aligned} \tilde{P}_{\omega(t)}(k) &= \tilde{P}_{\omega(t-\Delta t)}(ck)\tilde{P}_{F(t-\Delta t)}(k) \\ &= \tilde{P}_{\omega(0)}(c^N k)\prod_{n=0}^{N}\tilde{P}_F(c^n k) \\ &= \tilde{P}_{\omega(0)}(c^N k)\prod_{i=1}^{3}\prod_{n=0}^{N}\tilde{P}_{f_i}(c^n k) \\ &= \prod_{i=1}^{3}\prod_{n=0}^{N}\tilde{P}_{f_i}(c^n k), \end{aligned} \tag{27}$$

where $t = N\Delta t$. We here assumed $P_{\omega(0)}(\omega) = \delta(\omega)$ and hence $\tilde{P}_{\omega(0)}(k) = 1$.

The above argument shows the essential reason of the sharp peak and fat tails in Fig. 2. If we had $c = 1$, or $a = 0$ (without the interaction), the noise F at every time step would accumulate in ω and the probability density function of $\omega = \nu - \epsilon$ would be Gaussian due to the central limit theorem. If we have $c < 1$, or $a > 0$ (with the interaction), the noise at the past time steps decay as c^n. The largest contribution to ω comes from the noise one time step before, which is a fat-tail noise [3].

As a special case, if the noises $f_i(t)$ obey a Lévy distribution of the same index α and the same scale factor γ, namely if

$$\tilde{P}_{f_i}(k) = e^{-\gamma|k|^\alpha} \quad \text{for all } i, \tag{28}$$

the distribution of ω is also a Lévy distribution of the same index α and a different scale factor γ' given by

$$\gamma' = \frac{3}{1 - \{c(\Delta t)\}^\alpha}\gamma. \tag{29}$$

4 Summary

We first showed that triangular arbitrage opportunities exist in the foreign exchange market. The rate product μ fluctuates around a value m. Next, we introduced a model including the interaction caused by the triangular arbitrage transaction. We showed that the interaction is the reason of the sharp peak and the fat-tail property of the distribution of the logarithm rate product ν. Finally we showed that our model is solvable analytically in some cases.

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A The Prices Provided by Information Company

A typical example of the bid and the ask prices provided by information company is shown in Table 2. For the analysis in the present thesis, we extracted the bid price and the ask price out of these data except for obvious errors. We assumed that the price maintains the previous value till the next change. In this way, we constructed the time series of the bid and the ask prices, and calculated the rate product.

B Stochastic Models of Price Dynamics

B.1 Lévy-stable non-Gaussian Model

It was Mandelbrot [2, 3] in 1963 who first proposed the model that takes into account explicitly the leptokurtosis, which had been empirically found in the probability density function of a price change. Mandelbrot modeled the logarithm of the cotton prices as a stochastic process with Lévy stable non-Gaussian increments. The most interesting properties of the Lévy stable non-Gaussian processes are:

- their stability (i.e., their self-similarity);
- their relation to the limit theorem — they are attractors in the probability space.

Let us define $Y(t)$ as a price at time t. Mandelbrot's Lévy stable hypothesis implies that $\ln Y(t)$ undergoes a discontinuous time evolution and that $S(t) \equiv \ln Y(t+1) - \ln Y(t)$ is characterized by a non-Gaussian scaling and by a distribution with infinite second and higher moments. Since 1963, many papers have been devoted to considering the important problem of the finiteness or infiniteness of the variance of $S(t)$.

B.2 ARCH processes

The ARCH processes were introduced by Engle in 1982 [4]. ARCH models have been applied to several different areas of economics. Examples include (i) means and variances of inflation in the UK, (ii) stock returns, (iii) interest rates, and (iv) foreign exchange rates.

A stochastic process with auto regressive conditional heteroskedasticity, namely a stochastic process with 'nonconstant variances conditional in the past, but constant unconditional variances' [4] is an ARCH(p) process defined by the equation

$$\sigma_t^2 = \eta_0 + \eta_1 x_{t-1}^2 + \cdots + \eta_p x_{t-p}^2, \tag{30}$$

where $\eta_0, \eta_1, ..., \eta_p$ are positive variables and x_t is a random variable with zero mean and variance σ_t^2, characterized by a conditional probability density function $f_t(x)$.

Type	Time	Price
Bid Price	6:23:16	120.3500
Ask Price	6:23:16	120.4000
Last Trade	6:23:16	120.3750
Bid Price	6:23:16	120.3100
Ask Price	6:23:16	120.4100
Last Trade	6:23:16	120.3600
Bid Price	6:23:19	120.3400
Ask Price	6:23:19	120.3900
Last Trade	6:23:19	120.3650
Bid Price	6:23:20	120.3300
Ask Price	6:23:20	120.3800
Last Trade	6:23:20	120.3550
Bid Price	6:23:22	120.3200
Ask Price	6:23:22	120.3700
Last Trade	6:23:22	120.3450
Bid Price	6:23:35	120.3300
Last Trade	6:23:35	120.3500
Bid Price	6:23:46	120.3500
Ask Price	6:23:46	120.4500
Last Trade	6:23:46	120.4000

Table 2: An example of the real data of the yen-dollar rate provided by information company. The type 'Bid Price' means that someone in the market wanted to buy dollar (sell yen) at the 'Price' at the 'Time.' The type 'Ask Price' means that someone in the market wanted to sell dollar (buy yen) at the 'Price' at the 'Time.' The type 'Last Trade' means that the transaction was carried out at the 'Price' at the 'Time' in the market.

As the simplest case, the ARCH(1) process with a Gaussian conditional probability density function is defined by

$$\sigma_t^2 = \eta_0 + \eta_1 x_{t-1}^2. \tag{31}$$

The variance is given by

$$\sigma^2 = \frac{\eta_0}{1-\eta_1}. \tag{32}$$

The kurtosis of the ARCH(1) process is

$$\kappa \equiv \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} = 3 + \frac{6\eta_1^2}{1-3\eta_1^2}, \tag{33}$$

which is finite for

$$0 \le \eta_1 < \frac{1}{\sqrt{3}} \tag{34}$$

Hence, by varying η_0 and η_1, it is possible to obtain stochastic processes with the same unconditional variance but with different values of the kurtosis.

B.3 GARCH processes

In many applications using the linear ARCH(p) model, a large value of p is required. This usually poses some problems in the optimal determination of the $p+1$ parameters $\eta_0, \eta_1, ..., \eta_p$, which best describe the time evolution of a given economic time series. The overcoming of this difficulty leads to the introduction of generalized ARCH processes, called GARCH(p, q) processes, which was introduced by Bollerslev in 1986 [5].

This stochastic process defined by the equation

$$\sigma_t^2 = \eta_0 + \eta_1 x_{t-1}^2 + \cdots + \eta_p x_{t-p}^2 \xi_1 \sigma_{t-1}^2 + \cdots + \xi_p \sigma_{t-p}^2, \tag{35}$$

where $\eta_0, \eta_1, ..., \eta_p, \xi_1, ..., \xi_p$ are control parameters and x_t is a random variable with zero mean and variance σ_t^2, characterized by a conditional probability density function $f_t(x)$.

The simplest GARCH process is the GARCH(1,1) process with a Gaussian conditional probability density function. It can be shown [3] that the variance is

$$\sigma^2 = \frac{\eta_0}{1-\eta_1-\xi_1}, \tag{36}$$

and the kurtosis of the GARCH(1,1) process is

$$\kappa \equiv \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} = 3 + \frac{6\eta_1^2}{1-3\eta_1^2-\eta_1\xi_1-\xi_1^2}. \tag{37}$$

C Lévy and Truncated Lévy Distributions

The Lévy distribution is a family of probability distributions studied by P. Lévy. A stable Lévy distribution is defined by its characteristic function [10]

$$\tilde{P}_{\mathrm{L}}(z;\alpha,\beta,\gamma,m) = \exp\left\{-\gamma|z|^{\alpha}\left[1+i\beta\mathrm{sign}(t)\tan\left(\frac{\pi\alpha}{2}\right)\right]+imz\right\}. \quad (38)$$

The parameter β is a skewness parameter which characterize asymmetry of the distribution; i.e. $\beta = 0$ gives a symmetric distribution. The parameter α is the index of the distribution which gives the exponent of the asymptotic power-law tail. The parameter γ is a scale factor characterizing the width of the distribution, and m gives the peak position. It is sufficient to use the case $\beta = m = 0$ for our purpose.

The truncated Lévy distribution can be defined by its characteristic function

$$\tilde{P}_{\mathrm{T}}(z;\alpha,\gamma,l) = \exp\left\{-\gamma\frac{(z^2+l^{-2})^{\alpha/2}\cos(\alpha\arctan(l|z|))-l^{-\alpha}}{cos\left(\frac{\pi\alpha}{2}\right)}\right\}. \quad (39)$$

This distribution reduces to a Lévy distribution for $l \to \infty$ and to a Gaussian for $\alpha = 2$;

$$\tilde{P}_{\mathrm{T}}(z;\alpha,\gamma,l) \to \begin{cases} \exp(-\gamma|z|^{\alpha}) & \text{for} \quad l\to\infty, \\ \exp(-\gamma|z|^{2}) & \text{for} \quad \alpha = 2. \end{cases} \quad (40)$$

Notice that one could also use a hard-cutoff truncation scheme, such as Eq. (17) [3, 9].

References

[1] I. Moosa, 'Triangular Arbitrage in the Spot and Forward Foreign Exchange Markets,' *Quantitative Finance* **1** 387-390 (2001).

[2] B. B. Mandelbrot, 'The Variation of Certain Speculative Prices,' *J. Business* **36**, 394-419 (1963).

[3] R. N. Mantegna, H. E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambridge, 1999) pp.64-67.

[4] R. F. Engle, 'Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation,' *Econometrica* **50**, 987-1002 (1982).

[5] T. Bollerslev, 'Generalized Autoregressive Conditional Heteroskedasticity,' *J. Econometrics* **31**, 307-327 (1986).

[6] J. P. Bouchaud, M. Potters, *Theory of Financial Risks: From Statistical Physics to Risk Management* (Cambridge University Press, Cambridge, 2000) p.34 and pp.56-63.

[7] H. Takayasu, M. Takayasu, M. P. Okazaki, K. Marumo and T. Shimizu in *Fractal Properties in Economics,* Paradigms of Complexity, World Scientific, ed. Miroslav M. Novak, (2000) pp.243-258

[8] R. N. Mantegna, 'Fast, Accurate Algorithm for Numerical Simulation of Lévy Stable Stochastic Processes,' *Phys. Rev.* E **49**, 4677-4683 (1994).

[9] J. Voit, *The Statistical Mechanics of Financial Markets* (Springer, Berlin, 2001) pp.101-103.

[10] P. Lévy, *Calcul des probabilités* (Gauthier-Villars, Paris, 1925).